The Outer Model Programme

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presenting joint work with Sy Friedman and Philipp Lücke

February 13, 2013

L-like models

L-like

Vaguely speaking, for a model of set theory to be L-like means that it satisfies properties of Gödel's constructible universe of sets L.

The most canonical L-like model is of course L itself. L is an inner model of the universe of sets V, in the sense that $\mathrm{Ord} \subseteq L$ and $L \subseteq V$. L is defined by induction over the ordinals:

Definition of L

- $L_0 = \emptyset$,
- $L_{\alpha+1} = \{x \subseteq L_\alpha \mid x \text{ is definable over } (L_\alpha, \in)$ by a first-order formula using parameters from $L_\alpha\}$,
- $L_\gamma = \bigcup_{\alpha<\gamma} L_\alpha$ if γ is a limit ordinal.
- $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$.

L is the smallest inner model of set theory.

L-like principles

L-like principles

- GCH - $\forall\kappa\ 2^\kappa = \kappa^+$.
- $\diamondsuit$.
- $\square_\kappa$ for various κ, global $\square$.
- Lightface definable wellorders $\longrightarrow$ there is a lightface definable wellorder $\leq$ of L such that for every limit ordinal α, $\leq\restriction L_\alpha$ is a wellorder of and lightface definable over L_α.
- Condensation.

Gödel's Condensation Lemma (Gödel, 1939)

If $M \prec (L_\alpha, \in)$, then for some $\bar{\alpha} \leq \alpha$, $M \cong (L_{\bar{\alpha}}, \in)$.

Realizing this was the crucial step in Kurt Gödel's proof that the GCH holds in L.

Some Questions

L does not allow for larger large cardinals. Can L-like principles coexist with larger large cardinals?

What is the relationship between L-like principles - can some fail while others hold? In the presence of large cardinals?

The first question has been attacked by the *Inner Model Programme* for a long time. For example a cardinal κ is measurable if there exists a (definable) nontrivial elementary embedding $j\colon (V, \in) \to (M, \in)$ with *critical point* κ for some (definable) $M \subseteq V$. Dana Scott has shown (in 1961) that no cardinal can be measurable in L. For κ to be measurable is equivalent to the existence of a κ-complete, non-principal ultrafilter on κ. Let U be such an ultrafilter. Similar to L one can now construct a *canonical inner model* $L[U]$ for a measurable cardinal. This model is very L-like.

Limitations / The Outer Model Programme

Similarly, canonical inner models can be constructed for even larger large cardinals, like strong or Woodin cardinals. Set theorists have been trying for a long time to obtain such canonical inner models for large cardinals even larger than Woodin cardinals - an example being the central property of a supercompact cardinal - but no progress has been made in recent years. The Outer Model Programme attacks the first question from a completely different direction - namely by obtaining L-like properties in *forcing extensions* of the universe of set theory.

The Outer Model Programme

Basic Idea: Starting from a model of ZFC with large cardinals, obtain L-like properties in a forcing extension and preserve large cardinals.

Advantage: We can deal with "arbitrary large" large cardinals.

Some sample results for ω-superstrongs

A large cardinal property at the "edge of known inconsistency" is that of ω-superstrong cardinals:

Definition (ω-superstrong)

κ is ω-superstrong if there is an elementary embedding $j\colon (V,\in) \to (M,\in)$ with critical point κ for some $M \subseteq V$ with $V_{j^\omega(\kappa)} \subseteq M$.

By the famous Kunen inconsistency result, such j with $V_{j^\omega(\kappa)+1} \subseteq M$ is inconsistent.

Theorem (Friedman, 2007)

- $\mathrm{Con}(\omega\text{-}superstrong) \to \mathrm{Con}(GCH + \omega\text{-}superstrong)$
- $\mathrm{Con}(\omega\text{-}superstrong) \to \mathrm{Con}(\diamondsuit + \omega\text{-}superstrong)$
- $\mathrm{Con}(\omega\text{-}superstrong) \to \mathrm{Con}(def.\ wo. + \omega\text{-}superstrong)$

Square

There are situations where L-like principles and large cardinals are incompatible, an example is given by Jensen's $\square$ principle:

Limitations for $\square$

- If κ is subcompact, $\square_\kappa$ fails. (Jensen)
- If κ is supercompact, $\square_\lambda$ fails for every $\lambda \geq \kappa$. (Solovay)

There are positive results about forcing $\square_\kappa$ when κ is not subcompact.

Together with Sy Friedman and Philipp Lücke, I have been working on a specific instance of the second question, related to lightface definable wellorders.

Theorem (Aspero - Friedman, 2009)

Assume GCH. Then there is a cofinality-preserving forcing which introduces a lightface definable wellorder of H_{κ^+} for every regular uncountable κ, preserving the GCH. Moreover all inaccessibles, all instances of supercompactness and many other large cardinal properties are preserved.

What about the non-GCH case?

Theorem (F-H-L)

Assume SCH. There is a class forcing P with the following properties:

- *P preserves all inaccessibles and all supercompacts.*
- *Whenever κ is inaccessible, P introduces a lightface definable wellorder of H_{κ^+}.*
- *P is cofinality-preserving and preserves the continuum function.*

Back to the first question - Condensation

Together with Sy Friedman, I have been working on the problem of obtaining some form of Condensation in L-like outer models. In contrast to the other L-like principles considered so far, we first have to clarify what 'Condensation' is supposed to be when taken out of the context of L:

Models of the form L[A]

To define our desired Condensation property, we will assume that we are in a model of the form V=L[A] where A is a class sized predicate.

If $M \prec (L_\alpha[A], \in, A)$, we say that M *condenses* if for some $\bar{\alpha} \leq \alpha$, $M \cong (L_{\bar{\alpha}}[A], \in, A)$.

Generalized Condensation Principles

Local Club Condensation (Friedman)

Assume $V = L[A]$. If α has uncountable cardinality κ and $\mathcal{A} = (L_\alpha[A], \in, A, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma : \omega \leq \gamma < \kappa \rangle$ of condensing substructures of $\mathcal{A}$ whose domains B_γ have union $L_\alpha[A]$, each B_γ has cardinality card γ and contains γ as a subset.

For a desired application we were working on, we had to consider an additional property which is easily seen to follow from Condensation in L (but not from Local Club Condensation):

Acceptability

Assume V=L[A]. For any ordinals $\gamma \geq \delta$, if there is a new subset of δ in $L_{\gamma+1}[A]$, then

$$H^{L_{\gamma+1}[A]}(\delta) = L_{\gamma+1}[A].$$

Our L-like model

Theorem (Friedman, H)

Starting with a model containing an ω-superstrong cardinal, we can force to obtain a generic extension of the form $L[A]$ such that A witnesses both Local Club Condensation and Acceptability.

PFA

L-like inner models are very useful to determine the consistency strength of set theoretic principles. Using our L-like outer model, we were able to obtain a *quasi lower bound result* for the consistency strength of a (large fragment of) the Proper Forcing Axiom (PFA).

The Proper Forcing Axiom (PFA) is a significant strengthening of Martin's Axiom (for $\aleph_1$) that has many applications in set theory but also outside of set theory. While Martin's Axiom can be obtained by forcing over any model of ZFC (and thus is equiconsistent with ZFC alone), PFA has much higher consistency strength.

A consistency upper bound is given by the following classic theorem:

Theorem (Baumgartner, 1984)

If there is a supercompact cardinal, then PFA holds in a proper forcing extension of the universe.

A result of Neeman

Theorem (Neeman)

Assume V is a proper (forcing) extension of a fine structural inner model M and satisfies (a certain fragment of) PFA. Then there is a Σ^2_1-indescribable gap $[\kappa, \kappa^+)$ in M.

Σ^2_1-indescribable gaps $[\kappa, \kappa^+)$ are just slightly larger than subcompacts - they are subcompact limits of subcompacts (and a little more). The problem with this theorem is that no fine structural inner models even with subcompacts are currently known to exist.

Our L-like model is not fine structural, but luckily, Neeman's proof can be slightly adapted to work for our L-like model and we get the following:

Theorem (Friedman, H)

Assume V is a proper (forcing) extension of an L-like model M and satisfies (a certain fragment of) PFA. Then there is a Σ^2_1-indescribable gap $[\kappa, \kappa^+)$ in M.

Our quasi lower bound result

By basically taking the contraposition of the last theorem, we obtain the following:

Theorem (Friedman, H)

It is consistent that there is a model with a proper class of subcompacts but no proper (forcing) extension satisfies (a certain fragment of) PFA.

Rephrasing the above, we might say:

A proper class of subcompacts is a quasi lower bound for (a certain fragment of) PFA with respect to proper (forcing) extensions.

Thank you.