PFA AND CLASS FORCING

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Abstract. Assuming the existence of a supercompact cardinal, we construct a model of ZFC that may contain many local large cardinals of high consistency strength, a class generic extension but no set generic extension of which satisfies the proper forcing axiom PFA.

1. Introduction

We will show the following.

Theorem 1.1. *Assume* PFA *holds in* V*. There is a (class-)generic extension* W *such that* PFA *holds in a further class-generic extension of* W *which collapses a proper class of cardinals, but* PFA *holds in no eventually cardinal-preserving extension of* W *and thus in particular in no set-generic extension of* W*.*

An immediate corollary is the following:

Corollary 1.2. *Assuming the consistency of a supercompact cardinal, there is a model of* ZFC *a class-generic extension of which satisfies* PFA*, but no set-generic extension of which does.*

Proof. Starting from a model with a supercompact cardinal, obtain a forcing extension V satisfying PFA. Now Apply Theorem 1.1 to this model. $\square$

With a little more work, we will also show the following.

Theorem 1.3. *Assuming the consistency of a supercompact cardinal and a proper class of subcompact (or ω-superstrong) cardinals, there is a model of* ZFC *with a proper class of subcompact (or ω-superstrong) cardinals a class-generic extension of which satisfies* PFA*, but no eventually cardinal-preserving extension of which and thus in particular no set-generic extension of which does.*

This has the following corollaries, which are particular instances of the more general (folklore) rule that the existence of *local* large cardinals[1] can never suffice to ensure that PFA holds in a set forcing extension of the universe.

Corollary 1.4. *Assuming the consistency of a proper class of subcompact cardinals, it is consistent to have a universe with a proper class of subcompact cardinals no set forcing extension of which satisfies* PFA*.*

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[1] We say that a large cardinal property is *local* if it is preserved by sufficiently distributive forcing.

Corollary 1.5. *Assuming the consistency of a proper class of ω-superstrong cardinals, it is consistent to have a universe with a proper class of ω-superstrong cardinals no set forcing extension of which satisfies* PFA*.*

While Corollary 1.5 deals with a large cardinal close to the edge of known inconsistency, Corollary 1.4 is relevant to (in the sense of sheding additional light on) the following result of Sy Friedman and the author:

Theorem 1.6 (Friedman, Holy)**.** [FH, Theorem 1] *Assuming the consistency of a proper class of subcompact cardinals, it is consistent that there is a proper class of subcompact cardinals, but* PFA *restricted to posets which are* $(2^{\aleph_0})^+$*-linked holds in no proper extension*[2] *of the universe.*

While the proof of Theorem 1.6 given in [FH] is highly intricate, the short proof of Corollary 1.4 shows that the statement made by Theorem 1.6 about the full PFA and about set forcing (or, in fact, eventually cardinal-preserving) extensions can be obtained by much simpler means.[3]

2. Prerequisites

We will make use of the following: [4]

Theorem 2.1. [KY04] PFA *is preserved by* $<\omega_2$*-closed forcing.*

For the rest of this section, we will present definitions and results from Sections 6.1 and 6.2 of [CFM01] that will be our basic tools in Section 3.

Definition 2.2. Let κ be an uncountable cardinal. A $\square_\kappa$-sequence is a sequence $\langle C_\alpha \mid \alpha < \kappa^+, \lim(\alpha)\rangle$ such that for all $\alpha < \kappa^+$

(i) C_α is closed and unbounded in α.
(ii) If $\operatorname{cof}(\alpha) < \kappa$, then $\operatorname{otp}(C_\alpha) < \kappa$.
(iii) For all $\beta \in \lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$.

Definition 2.3. Let κ be an uncountable cardinal. We define a forcing $A(\kappa)$ which adds a generic $\square_\kappa$-sequence and thus forces $\square_\kappa$. $p \in A(\kappa)$ iff

(i) p is a function with $\operatorname{dom}(p) = \{\beta \le \alpha \mid \lim(\beta)\}$ for some limit ordinal $\alpha < \kappa^+$.
(ii) For all $\beta \in \operatorname{dom}(p)$, $p(\beta)$ is club in β and $\operatorname{otp}(p(\beta)) \le \kappa$.
(iii) If $\operatorname{cof}(\beta) < \kappa$ then $\operatorname{otp}(p(\beta)) < \kappa$.
(iv) For all $\beta \in \operatorname{dom}(p)$ and $\gamma \in \lim(p(\beta))$, $p(\gamma) = p(\beta) \cap \gamma$.

If $p, q \in A(\kappa)$ then $q \le p$ iff q end-extends p.

The following is well-known and easily verified.

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[2] A proper extension of the universe is an extension of the universe which preserves the stationarity of S for every stationary $S \subseteq [\gamma]^{\aleph_0}$ for all γ.

[3] This is however not supposed to undermine the importance of Theorem 1.6, it just says that the statement it makes about the full PFA is mostly interesting in the case when the proper extension in question collapses a proper class of cardinals and that, and that's of course the most important point here, Theorem 1.6 is important for in fact it proves a (highly non-trivial) statement about a *small fragment* of PFA (which is by no means provable using the methods of the present note).

[4] This strengthens a result from [Lar00], where it is (essentially) shown that PFA is preserved by $<\omega_2$-directed closed forcing. This weaker result would actually be sufficient for our present purposes.

Lemma 2.4. *Whenever $p \in A(\kappa)$ and $\xi < \kappa^+$ then there is $q \leq p$ in $A(\kappa)$ such that* $\max(\mathrm{dom}(p)) \geq \xi$. $\square$

Definition 2.5. Let P be a partial ordering, let α be an ordinal. The game $G_\alpha(P)$ is played as follows: Players I and II take turns to choose elements of a decreasing sequence $\langle p_\beta \mid \beta < \alpha \rangle$ of conditions in P, with I playing at at stage 0 and at all odd stages and II playing at all even stages except stage 0, but including all limit stages. If the play reaches an even stage $\gamma < \alpha$ where player II cannot move then player I wins, otherwise player II wins. The poset P is *$<\alpha$-strategically closed* iff player II has a winning strategy for the game $G_\alpha(P)$.

Lemma 2.6. [CFM01, Lemma 6.7] *$A(\kappa)$ is $<(\kappa+1)$-strategically closed.*

Definition 2.7. Let $\vec{C} = \langle C_\alpha \mid \alpha < \kappa^+ \rangle$ be a $\square_\kappa$-sequence in V. Let $\mathrm{W} \supseteq \mathrm{V}$. We say that $C \in \mathrm{W}$ *threads* $\vec{C}$ if C is club in κ^+ and $\forall \alpha \in \lim(C)\ C \cap \alpha = C_\alpha$. It is clear that no such C can exist in V or any extension of V in which κ^+ is a cardinal because every initial segment of C can have order type at most κ.

Definition 2.8. Let κ be an uncountable cardinal and let $\vec{C} = \langle C_\alpha \mid \alpha < \kappa^+ \rangle$ be a $\square_\kappa$-sequence. Let $\gamma \leq \kappa$ be a regular cardinal. We define a forcing $T_\gamma(\vec{C})$ which threads $\vec{C}$. $c \in T_\gamma(\vec{C})$ iff

(i) c is a closed bounded subset of κ^+.
(ii) $\mathrm{otp}\,(c) < \gamma$.
(iii) $\forall \beta \in \lim(c)\ c \cap \beta = C_\beta$.

If $c, d \in T_\gamma(\vec{C})$, then $d \leq c$ iff d end-extends c.

Clearly, $T_\gamma(\vec{C})$ adds a club $C \subseteq \kappa^+$ which threads $\vec{C}$. It is not clear though in general what the order type of C will be and whether the forcing satisfies some amount of distributivity. We will not actually use the following lemma from [CFM01], but its proof will be varied and made use of in section 3.

Lemma 2.9. [CFM01, Lemma 6.9] *Let κ be an uncountable cardinal and let $\gamma \leq \kappa$ be a regular cardinal. Let $\vec{C}$ denote the generic $\square_\kappa$ sequence added by forcing with $A(\kappa)$.*

(i) $A(\kappa) * T_\gamma(\vec{C})$ has a $<\gamma$-closed dense subset.
(ii) $T_\gamma(\vec{C})$ adds a generic club which threads $\vec{C}$ and has order-type γ. $(\kappa^+)^{\mathrm{V}}$ has cofinality γ in $\mathrm{V}^{A(\kappa) * T_\gamma(\vec{C})}$.

It follows that $T_\gamma(\vec{C})$ is $<\gamma$-distributive in $\mathrm{V}^{A(\kappa)}$.

3. The Main Theorem

Proof of Theorem 1.1. Assume PFA holds in V. Let S be a proper class of regular cardinals such that for every $\lambda \in S$, $\sup(\{2^\nu \mid \nu \in S \cap \lambda\}) < \lambda$ and such that $\min(S) \geq \omega_2$. Let P be the class sized iteration with Easton support which at stage λ forces with $A(\lambda)$ to add a $\square_\lambda$-sequence $\vec{C}_\lambda$ if $\lambda \in S$ and is trivial at stage λ otherwise. Let $\dot{Q}$ denote a P-name for the class sized Easton support product which at stage λ forces with $T_\lambda(\vec{C}_\lambda)$ if $\lambda \in S$ and is trivial at stage λ otherwise. Let G be P-generic over V and let H be $\dot{Q}^G$-generic over $\mathrm{W} = \mathrm{V}[G]$. Clearly no eventually cardinal-preserving extension of W satisfies PFA since the latter refutes $\square_\kappa$ for all $\kappa \geq \omega_2$, $\square_\kappa$ holds in W for every $\kappa \in S$, and the validity of $\square_\kappa$ can not be destroyed in extensions which preserve κ^+ as a cardinal. We will thus establish Theorem 1.1 by showing that forcing with $P*\dot{Q}$ preserves ZFC and that PFA holds in $\mathrm{V}[G*H]$. We will do the latter by showing that $P*\dot{Q}$ has a dense subset of conditions which is $<\gamma_{\min S}$-closed and by applying Theorem 2.1.

Claim 1. *If $\kappa \in S$, $P_{<\kappa}$ has a dense subset of size less than κ.*

Proof. By induction on κ. If $\kappa = \min S$, $P_{<\kappa}$ is trivial and thus the claim holds for κ. If $\kappa \in S$, a condition in $P_\kappa^{\mathrm{V}^{P_{<\kappa}}} = A(\kappa)^{\mathrm{V}^{P_{<\kappa}}}$ can be identified with a subset of κ in $\mathrm{V}^{P_{<\kappa}}$ and thus if $P_{<\kappa}$ has a dense subset of size less than κ inductively (which will be the case if $\max(S\cap\kappa)$ exists), we use a standard nice names argument to obtain that $P_{<\kappa+1}$ has a dense subset of size at most $2^\kappa < \min(S\setminus(\kappa+1))$. As P is trivial in the interval $(\kappa, \min(S\setminus(\kappa+1)))$, the same is true for $P_{<\min(S\setminus(\kappa+1))}$. Finally, if $\kappa \in S$ is such that $\operatorname{otp}(S\cap\kappa)$ is a limit ordinal, $P_{<\kappa}$ has a dense subset of size $\leq \sup(S\cap\kappa)^+ < \kappa$. $\square$

Claim 2. *$P[\kappa,\infty)$ is $<(\kappa+1)$-strategically closed.*

Proof. For $\lambda \geq \kappa$, $A_\lambda^{\mathrm{V}^{P_{<\lambda}}}$ is $<(\kappa+1)$-strategically closed (see Lemma 2.6), which implies the statement of the claim. $\square$

It now follows by standard arguments that P is cofinality-preserving. Let

$$D_0 = \{(p,q) \in P*\dot{Q} \mid \forall\kappa\ q(\kappa) \text{ is a } P_{<\kappa}\text{-name}\}.$$

Claim 3. *D_0 is dense in $P*\dot{Q}$.*

Proof. Let $(p,q) \in P*\dot{Q}$ be given. For every $\kappa \in S$, $q(\kappa)$ is a P-name for a bounded subset of κ^+ and may therefore be identified with a subset of κ. For any particular $\kappa \in S$, using Claim 1 and Claim 2, we may extend p to p' such that p' forces that $q(\kappa)$ has a $P_{<\kappa}$-name while keeping $p'\restriction\kappa = p\restriction\kappa$ by a standard reduction argument. Replace $q(\kappa)$ by that name to obtain q' from q. Then $(p',q') \leq (p,q)$ and $q(\kappa)$ is a $P_{<\kappa}$-name. We may iterate this process over $\operatorname{supp}(q)$ to obtain $(p^*,q^*) \leq (p,q)$ such that for all $\kappa \in S$, $q^*(\kappa)$ is a $P_{<\kappa}$-name. $\square$

Let

$$D_1 = \{(p,q) \in D_0 \mid \forall\kappa\ p\restriction\kappa \Vdash \max(\operatorname{dom}(p(\kappa))) = \max(q(\kappa))\}.$$

Claim 4. *D_1 is dense in D_0 and hence in $P*\dot{Q}$.*

Proof. Let $(p,q) \in D_0$ be given. Fix some $\kappa \in (\operatorname{dom}(p)\cup\operatorname{dom}(q))$. Since both $p(\kappa)$ and $q(\kappa)$ are $P_{<\kappa}$-names and $P_{<\kappa}$ has a dense subset of size less than κ, it follows that for every $\kappa \in (\operatorname{dom}(p)\cup\operatorname{dom}(q))$,

$$\exists\xi_\kappa < \kappa^+\ \mathbf{1}_{P_{<\kappa}} \Vdash \max(\operatorname{dom}(p(\kappa))), \max(q(\kappa)) < \xi_\kappa.$$

We may assume that each ξ_κ is a limit ordinal and choose $p' \leq p$ by choosing $p'(\kappa)(\xi_\kappa)$ cofinal in ξ_κ to end-extend $q(\kappa)$ and such that $p'(\kappa) \in A(\kappa)$. Let, for every such κ, $q'(\kappa) = q(\kappa)\cup\{\xi_\kappa\}$. Then $(p',q') \in P*\dot{Q}$, $(p',q') \leq (p,q)$ and in fact $(p',q') \in D_1$. $\square$

In fact, the above proof shows that the following subset of D_1 is dense in D_0 and hence in $P*\dot{Q}$:

$$D_2 = \{(p,q) \in D_0 \mid \forall\kappa\ \mathbf{1}_{P_{<\kappa}} \text{ decides } \max(\operatorname{dom}(p(\kappa))) = \max(q(\kappa))\}.$$

Claim 5. *D_2 is $<\gamma_{\min(S)}$ closed.*

Proof. Fix a decreasing sequence $\langle (p^\delta, q^\delta) \mid \delta < \beta \rangle$ of conditions in D_1 for some $\beta < \gamma_{\min(S)}$. We want to construct a lower bound (p, q) for this sequence. Let $\alpha^\delta(\kappa) = \max(\operatorname{dom}(p^\delta(\kappa)))$ for each $\delta < \beta$ and each $\kappa \in \operatorname{dom}(p^\delta)$. Let $\alpha(\kappa) = \sup_{\delta<\beta} \alpha^\delta(\kappa)$ for all $\kappa \in \bigcup_{\delta<\beta} \operatorname{dom}(p^\delta)$. Define $q(\kappa) = \bigcup_{\delta<\beta} q^\delta(\kappa) \cup \{\alpha(\kappa)\}$ and let p be such that for every $\kappa \in \bigcup_{\delta<\beta} \operatorname{dom}(p^\delta)$,

- $p(\kappa) \restriction \alpha(\kappa) = \bigcup_{\delta<\beta} p^\delta(\kappa)$,
- $\max(\operatorname{dom}(p(\kappa))) = \alpha(\kappa)$ and
- $p(\kappa)(\alpha(\kappa)) = q(\kappa) \cap \alpha(\kappa)$.

It is now easy to see that $(p, q) \le (p^\delta, q^\delta)$ for every $\delta < \beta$. □

To finish the proof of Theorem 1.1, it only remains to show that forcing with $P * \dot{Q}$ preserves ZFC. P itself clearly preserves ZFC by standard arguments (see [Fri00, Section 2.2]), being a reverse Easton iteration of increasingly distributive forcings. Given $t = (p, q) \in D_2$ and a cardinal η, let $l_\eta((p, q)) = (p \restriction \eta, q \restriction \eta)$ and let $u_\eta((p, q)) = (p \restriction [\eta, \infty), q \restriction [\eta, \infty))$. Let $l_\eta(D_2) = \{l_\eta(t) \mid t \in D_2\}$.

Claim 6. *$u_\eta(P * \dot{Q})$ is $<\gamma_\eta$-closed.*

Proof. Similar to the proof of Claim 5. □

Claim 7. *Let $\kappa \in S$ and $\gamma = \min(S \setminus (\kappa + 1))$. Every κ-sequence of elements of* V *in* $\mathrm{V}^{P*\dot{Q}}$ *is in* $\mathrm{V}^{P*\dot{Q}_{<\gamma}}$*. Hence $(P * \dot{Q})/(P * \dot{Q}_{<\gamma})$ is $<\kappa$-distributive.*

Proof. $l_\gamma(D_2)$ has a dense subset of size $2^\kappa < \gamma$ by the proof of Claim 1. Now given a $P * \dot{Q}$-name $\dot{x}$ for a sequence of length κ of elements of V and $t \in D_2$, consider every possible value of $l_\eta(s)$ for $s \in D_2$. Consecutively for each $i < \kappa$, for each such value extend $u_\eta(t)$ such that together with $l_\eta(s)$ it decides $\dot{x}(i)$ if possible. Let $t^* \le u_\eta(t)$ be the final condition obtained by this process after $2^\kappa \cdot \kappa$-many steps. We made sure that $\dot{x}$ is forced by $l_\eta(t)$ together with $u_\eta(t^*)$ (this is the condition r such that $l_\eta(r) = l_\eta(t)$ and $u_\eta(r) = t^*$), which is a condition stronger than t, to have a $P * \dot{Q}_{<\gamma}$-name, as desired. □

Claim 8. *$P * \dot{Q}$ preserves* ZFC*.*

Proof. By Claim 7 together with the arguments of [Fri00, Section 2.2]. □

□

Proof of Theorem 1.3. Start with a model with a supercompact cardinal θ and a proper class X of subcompact (or ω-superstrong) cardinals above (in the case of ω-superstrongs, also choose a witnessing ultrapower embedding j_κ whenever κ is ω-superstrong and make sure that whenever $\kappa \in X$, ${j_\kappa}^\omega(\kappa) < \min(X \setminus (\kappa + 1))$. Force PFA using Baumgartner's iteration of size θ that collapses θ to become ω_2, to obtain a model V which preserves the subcompactness (or ω-superstrength) of all cardinals in X (their witnessing embeddings may all be lifted for the iteration is *small*) and satisfies PFA. Now choose S as in the proof of Theorem 1.1, but additionally make sure that (in the case of treating subcompacts) $S \cap X = \emptyset$ and that S is bounded below every element of X or (in the case of treating ω-superstrongs) that S contains no cardinals in $[\kappa, {j_\kappa}^\omega(\kappa)]$ for every $\kappa \in X$ and that S is bounded below every $\kappa \in X$. This will make sure that forcing with P, as defined in the proof of

Theorem 1.1, preserves the subcompactness (or ω-superstrength) of all cardinals in X.[5] The resulting model V^P will be as desired - V^Q will satisfy PFA while no eventually cardinal-preserving extension of V^P will do so, by the same argument as in the proof of Theorem 1.1. $\square$

Proof of Corollary 1.4 and 1.5. Almost like the proof of Theorem 1.3 - start with a model with a proper class of subcompacts (or ω-superstrongs). Now (after omitting the step where we forced PFA) continue by forcing with P over that model, as in the proof of Theorem 1.3. Then V^P will be as desired in each case. $\square$

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[5] The preservation statement is immediate for subcompacts and requires an application of the proof of [Fri06, Lemma 3] for ω-superstrongs.