A GENERALIZATION OF THE NOTION OF BOUNDED DEGREE FOR INFINITE GRAPHS

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Abstract. We consider graphs whose vertices are ordinals (*ordinal graphs*). For those graphs, we generalize the notion of vertex degree to the notion of splitting-type – the order-type of the set of neighbours of a vertex. We use this to define a proper hierarchy of collections of ordinal graphs, investigate various natural strong isomorphism relations between ordinal graphs, which preserve splitting-types, and investigate resulting notions of graph complexity.

1. Introduction

In this note, a *graph* is a pair $G = \langle |G|, E(G)\rangle$, where we call $|G|$ the *domain* or *vertex set* of G and $E(G)$ the *edge relation* of G, a binary relation on $|G|$. If $x, y \in |G|$, we sometimes write $G(x, y)$ instead of $\langle x, y\rangle \in E(G)$. We will always assume that $E(G)$ is symmetric and irreflexive, i.e. our graphs are undirected simple graphs. Moreover all our graphs will be *ordinal graphs*, i.e. graphs G such that $|G| \subseteq \mathrm{On}$, which also means that we can canonically consider them to be (injectively) vertex-labeled graphs.

If $x, y \in |G|$ and $G(x, y)$, we call x and y *adjancent* and say that x and y are *neighbours* in G. We denote the set of neighbours of x in G by $N^G(x)$, and we let $N^G_*(x) = N^G(x) \cup \{x\}$. For $x \in |G|$, the *degree* of x in G is the cardinality of $N^G(x)$.

Graphs with bounded vertex degree are of central importance in graph theory and combinatorial set theory. The following classic result is of particular relevance for us here. A graph G is *locally finite* if every node in G has finite degree. More generally if κ is any cardinal, a graph G is locally of size less than κ if every node in G has degree less than κ. We say that a graph G can be weakly embedded into a graph H if there is an injective map from $|G|$ to $|H|$ such that for any $x, y \in |G|$, $G(x, y) \to H(x, y)$. We say G *can be embedded* into H if there is an injective map from $|G|$ to $|H|$ such that for any $x, y \in |G|$, $G(x, y) \leftrightarrow H(x, y)$. If $\mathcal{G}$ is a family of graphs we say that X is *(weakly) universal* for $\mathcal{G}$ if $X \in \mathcal{G}$ and every $G \in \mathcal{G}$ can be (weakly) embedded into X.

Richard Rado has shown that there is a universal countable graph, while Nicolaas Govert de Bruijn has shown that the class of all countable, locally finite graphs does not contain a weakly universal element (see [Rad64] for both results). The possible existence of universal objects has then been studied for many other classes of countable graphs.

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Assuming the GCH, it follows from classical results in model theory on the existence of saturated models (see [CK90]) that a universal graph of size κ exists for every uncountable cardinal κ (for a particular κ, $2^{<\kappa} = \kappa$ suffices), and its existence can fail when the GCH fails, for example at κ^+ after adding κ^{++}-many Cohen subsets of κ (this is due to Saharon Shelah, see [KS92] or [Džа15, Theorem 2.1] for a proof).

It seems obvious to ask whether de Bruijn's result has some kind of analogy in the uncountable case. However the notion of a graph G of size κ which is locally of size less than κ is not very interesting whenever κ is regular and uncountable by the following easy observation. A *path* in G is a sequence $S = \langle x_i \mid i \in I \rangle$ for some interval $I \subseteq \mathbb{Z}$ such that whenever $i, i+1$ are both in I, $G(i, i+1)$ holds. If I is finite, we say that S is a path from $x_{\min I}$ to $x_{\max I}$ in G. A graph G is *connected* if for any two distinct $x, y \in |G|$, there is a path from x to y in G.

Observation 1.1. *If G is a connected graph, κ is a regular uncountable cardinal and each node in G has degree less than κ, then $|G|$ is of size less than κ.*

Proof. Given $X \subseteq |G|$, let $N^G(X) = X \cup \bigcup \{N^G(x) \mid x \in X\}$. Given any $x \in G$, $G = \bigcup_{n<\omega} (N^G)^n(x)$, where we let $(N^G)^0(x) = \{x\}$ and $(N^G)^{i+1}(x) = N((N^G)^i(x))$ for $i < \omega$. The result follows since if $X \subseteq |G|$ has cardinality less than κ then, using regularity of κ, $N^G(X)$ has cardinality less than κ. □

Given two graphs G and H we say they are *disjoint* if $|G| \cap |H| = \emptyset$. Given a family of graphs $\langle G_i \mid i \in I \rangle$, we say that G is the *union* of $\langle G_i \mid i \in I \rangle$ if $|G| = \bigcup_{i \in I} |G_i|$ and $E(G) = \bigcup_{i \in I} E(G_i)$. It follows that if κ is regular and G is a graph in which each node has degree less than κ, then G is the union of a family of pairwise disjoint graphs of size less than κ.

If $\mathcal{G}$ is a family of graphs, the (weak) universality number of $\mathcal{G}$ is the least size of a family $\mathcal{H} \subseteq \mathcal{G}$ such that for every $G \in \mathcal{G}$ there is $H \in \mathcal{H}$ such that G can be (weakly) embedded into H. Hence there is a (weakly) universal object in the collection of graphs of size κ which are locally of size less than κ iff the (weak) universality number for graphs of size less than κ is at most κ. [1]

In this paper, we introduce a notion of degree for (infinite) ordinal graphs such that the class of graphs with bounded vertex degree possesses more interest also for graphs which are of size a regular uncountable cardinal. The basic idea is to consider the order-type rather than the cardinality of the set of neighbours of a vertex.

Definition 1.2. If G is an ordinal graph, we let

$$\operatorname{split}(G) = \sup\{\operatorname{ot}(N(x)) + 1 \mid x \in G\}$$

and

$$\operatorname{split}_*(G) = \sup\{\operatorname{ot}(N_*(x)) + 1 \mid x \in G\}.$$

We say that G is *$<\alpha$-splitting* iff $\operatorname{split}(G) \leq \alpha$ and say that G is *$<\alpha_*$-splitting* iff $\operatorname{split}_*(G) \leq \alpha$. We say that G is *α-splitting* iff G is $<(\alpha+1)$-splitting and say that G is *α_*-splitting* iff G is $<(\alpha+1)_*$-splitting.

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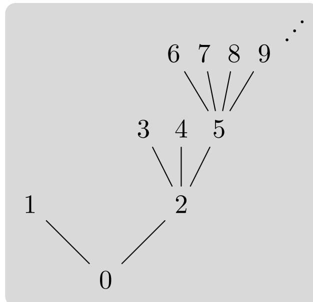
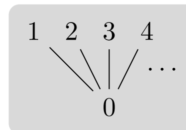
[1] The latter is trivially true for the weak universality number, and is a trivial consequence of $2^{<\kappa} = \kappa$ otherwise, however its negation is also known to be consistent due to a result of Shelah (see [KS92] or [Dža15, Theorem 2.1] for a proof).



$<\omega$-splitting



ω-splitting

Of course an ordinal graph is $<\omega$-splitting iff it is locally finite and more generally it is $<\kappa$-splitting iff the degree of each of its nodes is less than κ. In the light of Observation 1.1, we will thus work with graphs G of some size κ (i.e. $|G|$ of cardinality κ) with $\mathrm{split}_*(G) \in (\kappa, \kappa^+)$, where κ is an infinite cardinal. Our main interest will be the case when κ is regular and uncountable, however most of our results apply to all infinite cardinals, and we will explicity mention whenever we make any extra assumptions on κ. When we consider α-splitting (or α_*-splitting) graphs G and $\alpha \in [\kappa, \kappa^+)$ in the following, we will tacitly assume that $|G|$ has size κ (and we will usually only consider connected graphs). In order to tackle and make sense of our original question on universal graphs, we want to consider isomorphisms between ordinal graphs that preserve splitting types. A very strong demand on such maps would be to be order-preserving – this notion will be useful to us in places, however is too strong to produce interesting structural properties. On the other extreme, we could simply consider all isomorphisms that preserve the splitting type of a graph – however it is easily observed that this notion is too weak and such isomorphisms can almost completely destroy the order structure of a given graph. We want to introduce various natural forms of strong isomorphism relations between ordinal graphs that preserve splitting types and lie between the two extremes outlined above. If G and H are graphs, an isomorphism f from G to H is a bijection from $|G|$ to $|H|$ such that for any $x, y \in G$, $G(x, y) \leftrightarrow H(f(x), f(y))$.

Definition 1.3. Assume G, H are (isomorphic) ordinal graphs. We introduce a series of properties the definitions of which will follow a certain scheme. We say that G and H are N isomorphic if there is an isomorphism f from G to H such that $P(G, H, f)$. We abbreviate this property of G and H by $I(G, H)$ and call a witnessing isomorphism f an N preserving isomorphism. We carry the above definition out for the following triples $(N, P(G, H, f), I)$:

- $N =$ order, $I =$ oi and $P(G, H, f)$ states that f is order-preserving, i.e. for any $x, y \in G$, $x < y$ iff $f(x) < f(y)$.[2]
- $N =$ locally order, $I =$ loi and $P(G, H, f)$ states that for any $x \in G$, $f \upharpoonright N(x)$ is order-preserving, i.e. for any $y, z \in N(x)$, $y < z$ iff $f(y) < f(z)$.
- $N =$ locally star order, $I = \mathrm{l}_*\mathrm{oi}$ and $P(G, H, f)$ states that for any $x \in G$, $f \upharpoonright N_*(x)$ is order-preserving.
- $N =$ locally order type, $I =$ loti and $P(G, H, f)$ states that for any $x \in G$, $\mathrm{ot}\left(N^G(x)\right) = \mathrm{ot}\left(N^H(f(x))\right)$.

[2] x and y (as well as $f(x)$ and $f(y)$) are ordinals, $<$ denotes the usual ordering of ordinals.

- $N =$ locally star order type, $I = \mathrm{l_*oti}$ and $P(G, H, f)$ states that for any $x \in G$, $\mathrm{ot}\left(N_*^G(x)\right) = \mathrm{ot}\left(N_*^H(f(x))\right)$.
- $N =$ locally order type star, $I = \mathrm{loti}^*$ and $P(G, H, f)$ states that for any $x \in G$, $\mathrm{ot}\left(N^G(x) \cap x\right) = \mathrm{ot}\left(N^H(f(x)) \cap f(x)\right)$ and $\mathrm{ot}\left(N^G(x) \setminus x\right) = \mathrm{ot}\left(N^H(f(x)) \setminus f(x)\right)$.
- We let $\mathrm{OI} = \{\mathrm{oi}, \mathrm{loi}, \mathrm{l_*oi}, \mathrm{loti}, \mathrm{l_*oti}, \mathrm{loti}^*\}$ denote the collection of order isomorphism relations. If $I \in \mathrm{OI}$ and $I(G, H)$ as witnessed by the isomorphism f, we abbreviate this situation as $f \colon I(G, H)$. Note that each $I \in \mathrm{OI}$ canonically gives rise to an equivalence relation on ordinal graphs.

Lemma 1.4. *For ordinal graphs G, H, $oi(G, H) \to \mathrm{l_*oi}(G, H) \to \mathrm{loi}(G, H) \to \mathrm{loti}(G, H)$, $\mathrm{l_*oi}(G, H) \to \mathrm{loti}^*(G, H) \to \mathrm{l_*oti}(G, H)$ and $\mathrm{loti}^*(G, H) \to \mathrm{loti}(G, H)$. Moreover, no other implications between any of those properties are provable, i.e. the following diagram is* complete, *in the sense that the transitive hull of the displayed implications contains all possible implications.*

$$\begin{tikzcd} & oi \arrow[d] & & \\ & l_*oi \arrow[dl] \arrow[dr] & & \\ loi \arrow[dr] & & loti^* \arrow[dl] \arrow[dr] & \\ & loti & & l_*oti \end{tikzcd}$$

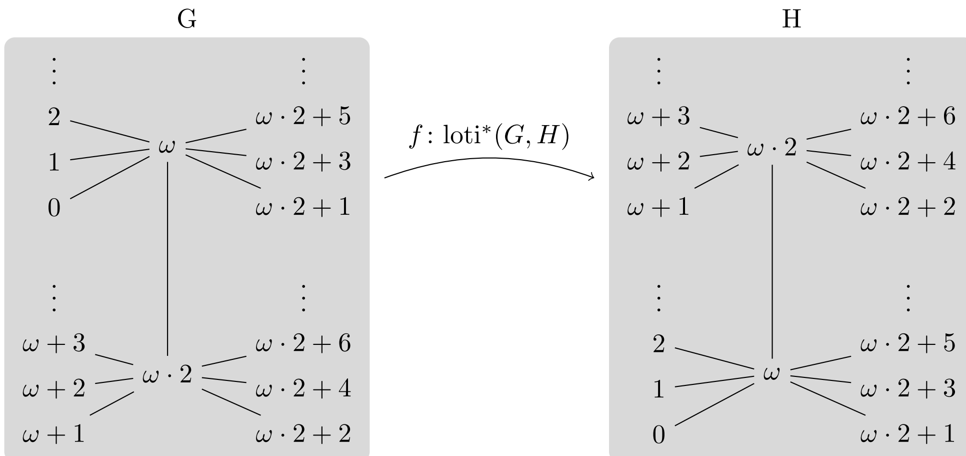
Proof. It is straightforward and very easy to see that the above implications all hold. Witnesses for no other implications to hold are given in the picture below, in the sense that for all relevant combinations of $I, J \in \mathrm{OI}$, we give examples of graphs G and H such that $I(G, H)$ holds while $J(G, H)$ fails. Note that the final two examples are necessarily provided by infinite graphs.

	G	H
$\mathrm{l_*oi}$, not oi	$0 — 2 — 3 — 1$	$1 — 2 — 3 — 0$
loi, not loti^*	$2 — 0 — 1$	$2 — 1 — 0$
loti^*, not loi	$2 — 0 — 1$	$1 — 0 — 2$
loti, not $\mathrm{l_*oti}$	0 $/ \mid \backslash \cdots$ $1 \quad 2 \quad 3$	ω $/ \mid \backslash \cdots$ $1 \quad 2 \quad 3$
$\mathrm{l_*oti}$, not loti	0 $/ \mid \backslash \cdots$ $\omega \quad 1 \quad 2$	ω $/ \mid \backslash \cdots$ $0 \quad 1 \quad 2$

□

Lemma 1.5. *If $f \colon \mathrm{l_*oi}(G, H)$ and $G(x, y)$, then $x < y$ iff $f(x) < f(y)$. This is not necessarily true for any of the weaker notions of isomorphism that we consider in this paper, i.e. for $I \in \{\mathrm{loi}, \mathrm{loti}^*, \mathrm{loti}, \mathrm{l_*oti}\}$.*

Proof. The only nontrivial case is when $I = \mathrm{loti}^*$. We provide an example of $f\colon \mathrm{loti}^*(G,H)$ with $\langle \omega, \omega\cdot 2\rangle \in E(G)$, but $f(\omega) = \omega\cdot 2 > \omega = f(\omega\cdot 2)$ in the picture below, where f maps nodes according to their position in the picture, so for example $f(0) = \omega + 1$.



$\square$

Observation 1.6. *For any graphs G and H, $f\colon \mathrm{l_*oi}(G,H)$ holds iff $f\colon \mathrm{loi}(G,H)$ and*

$$\forall x,y\ [(x,y)\in G \wedge x<y] \to f(x) < f(y).$$

Observation 1.7. *If G is α-splitting and $\mathrm{loti}(G,H)$ then H is α-splitting. If G is α_*-splitting and $\mathrm{l_*oti}(G,H)$ then H is α_*-splitting.*

The following lemma will be useful in Section 3.

Lemma 1.8. *Assume $\alpha \in [\kappa,\kappa^+)$, G is α_*-splitting, $f\colon \mathrm{loti}(G,H)$ and whenever $\mathrm{ot}\left(N^G(x)\right) = \alpha$, there is y so that $(x,y)\in G$, $\mathrm{ot}\left(N^G(y)\right) = \kappa$ and $N^G(x)\cap N^G(y)$ has cardinality κ. Then $f\colon \mathrm{l_*oti}(G,H)$.*

Proof. Assume for a contradiction that the assumptions of the lemma hold, however $f\colon \mathrm{l_*oti}(G,H)$ fails. This means that there is $x\in |G|$ with $\mathrm{ot}\left(N^G(x) = \alpha\right)$ and $\mathrm{ot}\left(N^H_*(f(x))\right) = \alpha + 1$. Since $\mathrm{ot}\left(N^H(f(x))\right) = \alpha$ using that $f\colon \mathrm{loti}(G,H)$, it follows that $f(x) = \max(N^H_*(f(x)))$. In particular thus, $f(x) > z$ for every $z \in N^H(f(x)) \cap N^H(f(y))$. But since this set has cardinality κ by assumption, is contained in $N^H(f(y))$ and since $(f(x),f(y))\in H$, it follows that $\mathrm{ot}\left(N^H(f(y))\right) > \kappa$, contradicting that $\mathrm{ot}\left(N^G(y)\right) = \kappa$ and $f\colon \mathrm{loti}(G,H)$, which imply $\mathrm{ot}\left(N^H(f(y))\right) = \kappa$. $\square$

G is a *subgraph* of H if $|G|\subseteq |H|$ and for any $x,y\in |G|$, $G(x,y)\to H(x,y)$. G is an *induced subgraph* of H if $|G|\subseteq|H|$ and for any $x,y\in|G|$, $G(x,y)\leftrightarrow H(x,y)$. If $I\in \mathrm{OI}$, we say that G (strongly) *I-embeds* into H if there is G' such that $I(G,G')$ and G' is a(n induced) subgraph of H. The next observation shows that the collections of α-splitting and α_*-splitting graphs for $\kappa<\alpha<\kappa^+$ form a proper hierarchy with respect to our order isomorphism relations.

Observation 1.9. *If $\kappa < \alpha < \beta < \kappa^+$, $\mathrm{split}(G) = \alpha$ and $\mathrm{split}(H) = \beta$, then H does not* loti-*embed into G. If $\mathrm{split}_*(G) = \alpha$ and $\mathrm{split}_*(H) = \beta$, then H does not* $\mathrm{l_*oti}$-*embed into G.*

Proof. If $\mathrm{split}(H) = \beta$ and $\alpha < \beta$, there is $x \in |H|$ with $\mathrm{ot}\left((N(x))\right)^H \geq \alpha$. If $f\colon \mathrm{loti}(H, H')$, then $\mathrm{ot}\left(N(f(x))\right)^{H'} \geq \alpha$ and thus H' cannot be a subgraph of G. The proof for $_*$-splitting is similar. □

If $\mathcal{C}$ is a collection of graphs we say that X is (strongly) *I-universal* for $\mathcal{C}$ if $X \in \mathcal{C}$ and every $G \in \mathcal{C}$ (strongly) I-embeds into X. Given an ordinal graph G, we let $||G||^I$ be $\min\{\alpha \in \mathrm{On} \mid \exists H\ I(G, H) \wedge |H| = \alpha\}$.

For any $I \in \mathrm{OI}$ and ordinal graphs G, $||G||^I$ provides a measure of complexity for G. Our main results will show that very restricted classes of ordinal graphs (we will mainly consider the smallest classes in our hierarchy - these are κ_*-splitting, κ-splitting and $(\kappa + 1)_*$-splitting graphs) often still contain objects of arbitrary complexity.

Theorem 1.10.
- *For any $\gamma < \kappa^+$, there is a $(\kappa + 1)_*$-splitting graph G such that $||G||^{\mathrm{l_*oti}} > \gamma$ and $||G||^{\mathrm{loti}} > \gamma$ (Theorem 3.1).*
- *For every $\gamma < \kappa^+$, there exists a κ_*-splitting graph G such that $||G||^{\mathrm{loi}} > \gamma$ (Theorem 3.4).*
- *Assume κ is a regular and uncountable cardinal. There are κ_*-splitting graphs G and G_* such that $||G||^{\mathrm{loti}} = ||G_*||^{\mathrm{l_*oti}} = \kappa \cdot 2$ (Theorem 3.5).*

An easy consequence is the non-existence of universal objects for these classes.

Corollary 1.11. *Assume $I \in \mathrm{OI}$ and $\kappa < \alpha < \kappa^+$. Then there is no I-universal α-splitting (or α_*-splitting) graph. In fact, no ordinal graph of size κ can I-embed every α-splitting (or α_*-splitting) graph. If $I \in \{\mathrm{loi}, \mathrm{l_*oi}, \mathrm{oi}\}$, this is true also if $\alpha = \kappa$.*

Proof. Assume X is an I-universal α-splitting (or α_*-splitting) graph. We may then assume that $|X| = \gamma$ for some $\gamma < \kappa^+$. Let G be α-splitting (or α_*-splitting) with $||G||^I > \gamma$, as provided by either Theorem 3.1 or Theorem 3.4. Such G cannot I-embed into X by Observation 1.9, contradicting supposed universality of X. □

2. Induced Order Isomorphisms

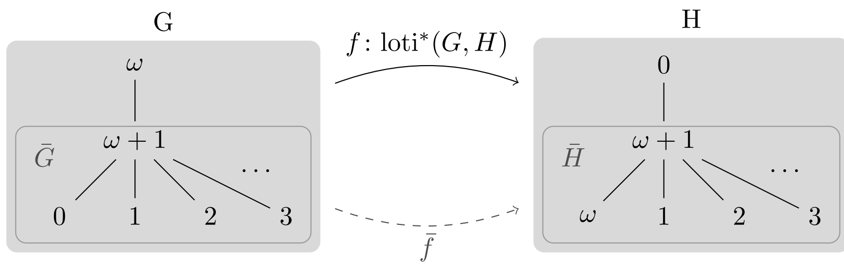
In this short section we treat the question for which $I \in \mathrm{OI}$, the restriction of a function f witnessing $I(G, H)$ to a subset of $|G|$ necessarily yields a witnessing function for $I(\bar{G}, \bar{H})$ for the corresponding induced subgraphs $\bar{G}, \bar{H}$ of G, H. We will obtain a positive answer for loi (and any of it's strengthenings) and a negative answer for loti^* (and its weakenings). These results will be relevant in Section 3.

Lemma 2.1. *Assume $I \in \{\mathrm{oi}, \mathrm{l_*oi}, \mathrm{loi}\}$. Assume $f\colon I(G, H)$ and $\bar{G}$ is an induced subgraph of G. Let $\bar{H}$ be the induced subgraph of H with vertex set $f[|\bar{G}|]$. Then $f \restriction |\bar{G}|\colon I(\bar{G}, \bar{H})$.*

Proof. Straightforward and very easy. □

Lemma 2.2. *There are graphs G and H and $f\colon \mathrm{loti}^*(G, H)$ such that for some induced subgraph $\bar{G}$ of G, if $\bar{H}$ is the induced subgraph of H with vertex set $f[|\bar{G}|]$, then $f \restriction |\bar{G}|\colon \mathrm{loti}(\bar{G}, \bar{H})$ and $f \restriction |\bar{G}|\colon \mathrm{l_*oti}(\bar{G}, \bar{H})$ are both not valid.*

Proof. We let G, H, $\bar{G}$ and f be as shown in the picture below.



f maps nodes according to their position in the picture, i.e. $f(\omega)=0$, $f(0)=\omega$ and f is the identity map otherwise. To see for example that $\bar{f}\colon \mathrm{loti}(\bar{G},\bar{H})$ is not valid, note that $\mathrm{ot}\left(N^{\bar{G}}(\omega+1)\right)=\omega$, while $\mathrm{ot}\left(N^{\bar{H}}(\omega+1)\right)=\omega+1$. □

3. Minimal Domains

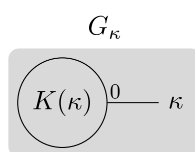
In this central section, we provide the proofs of our main results.

For any $X\subseteq \mathrm{On}$, let $K(X)$ denote the complete graph with vertex set X.

Theorem 3.1. *For any $\gamma<\kappa^+$, there is a $(\kappa+1)_*$-splitting graph G such that $||G||^{\mathrm{l_*oti}}>\gamma$ and $||G||^{\mathrm{loti}}>\gamma$. Moreover if $f\colon \mathrm{loti}(G,H)$, then H is $(\kappa+1)_*$-splitting (this trivially holds if $f\colon \mathrm{l_*oti}(G,H)$).*

Proof. We will construct G_γ with $||G_\gamma||^{\mathrm{l_*oti}}>\gamma$ and $||G_\gamma||^{\mathrm{loti}}>\gamma$ by induction on $\gamma<\kappa^+$. However we will verify a stronger property during the induction: For $\kappa\le\gamma<\kappa^+$, there is a $(\kappa+1)_*$-splitting graph G_γ and $x=x(G_\gamma)=\max(|G_\gamma|)$ which is adjancent only to 0 in G_γ, such that $f(x)\ge\gamma$ whenever $f\colon \mathrm{l_*oti}(G_\gamma,H)$ or $f\colon \mathrm{loti}(G_\gamma,H)$.

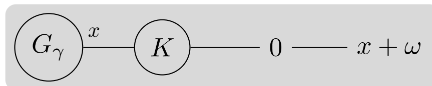
Case $\gamma=\kappa$: We let $G_\kappa=K(\kappa)$ together with one additional node that is κ, adjancent only to 0. Now $\mathrm{ot}\left(N_*^{G_\kappa}(0)\right)=\mathrm{ot}\left(N^{G_\kappa}(0)\right)=\kappa+1$ and for every $0<\alpha<\kappa$, $\mathrm{ot}\left(N_*^{G_\kappa}(\alpha)\right)=\mathrm{ot}\left(N^{G_\kappa}(\alpha)\right)=\kappa$. This implies that if $f\colon \mathrm{l_*oti}(G_\kappa,H)$ or $f\colon \mathrm{loti}(G_\kappa,H)$, then $f(\kappa)\ge\kappa$. Obviously, G_κ is $(\kappa+1)_*$-splitting. We will picture G_κ as follows.



We will now proceed to define G_γ for $\kappa<\gamma<\kappa^+$, and will only later show that those are as desired. In fact, we will only define G_γ for limit ordinals γ (which clearly suffices).

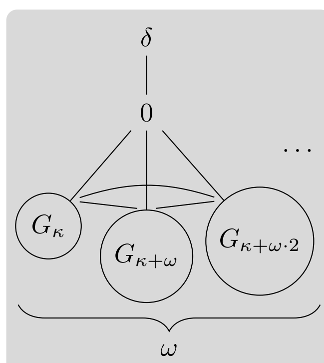
Given G_γ for a limit ordinal γ, with $x=x(G_\gamma)$, obtain $G^*_{\gamma+\omega}$ from G_γ by introducing ω-many additional nodes $\langle x+i\mid 1\le i<\omega\rangle$ adjancent to each other and all adjancent to x. Let $G^{**}_{\gamma+\omega}$ be an order isomorphic copy of $G^*_{\gamma+\omega}$ such that $0\notin|G^{**}_{\gamma+\omega}|$ and such that the corresponding order isomorphism f maps $x+i$ to itself for every $i<\omega$. Now we obtain $G_{\gamma+\omega}$ from $G^{**}_{\gamma+\omega}$ by introducing 2 additional nodes, that is 0 adjancent to $x+i$ for $1 \leq i < \omega$ and $x+\omega$ adjancent only to 0. Obviously, $G_{\gamma+\omega}$ is $(\kappa+1)_*$-splitting. We picture $G_{\gamma+\omega}$ as follows, with $K = K([x+1, x+\omega))$.

$G_{\gamma+\omega}$



Now assume γ is a limit of limit ordinals and we have constructed G_δ for limit ordinals $\delta < \gamma$. By replacing each G_δ with a suitable order isomorphic copy, we may assume that $|G_{\delta_0}| \cap |G_{\delta_1}| = \emptyset$ and $x(G_{\delta_0}) < x(G_{\delta_1})$ whenever $\delta_0 < \delta_1$ are both limit ordinals less than γ, and that $0 \notin |G_\delta|$ for any limit ordinal $\delta < \gamma$. Let $\langle \gamma_i \mid i < \mathrm{cof}(\gamma) \rangle$ be cofinal in γ with each γ_i a limit ordinal. We construct G_γ as the graph obtained as the union of G_{γ_i} for $i < \mathrm{cof}(\gamma)$, where we additionally connect $x(G_{\gamma_i})$ and $x(G_{\gamma_j})$ by an edge for any $i, j < \mathrm{cof}(\gamma)$ and add two additional nodes, that is 0 adjancent to $x(G_{\gamma_i})$ for $i < \mathrm{cof}(\gamma)$ and an ordinal δ greater than $x(G_{\gamma_i})$ for $i < \mathrm{cof}(\gamma)$ which is adjancent only to 0. That G_γ is $(\kappa+1)_*$-splitting follows since $\mathrm{ot}\left(\{x(G_{\gamma_i}) \mid i < \mathrm{cof}(\gamma)\}\right) \leq \kappa$. We provide a sample illustration for $G_{\kappa+\omega\cdot\omega}$ with $\gamma_i = \kappa + \omega \cdot i$ for $i < \omega$.

$G_{\kappa+\omega\cdot\omega}$



That the G_γ are as desired is a consequence of the following.

Claim 1. *Assume $\bar{G} = G_\gamma$ for some limit ordinal $\kappa \leq \gamma < \kappa^+$. If G^* is an ordinal graph with $|\bar{G}| \cap |G^*| = x(\bar{G})$, $G = \bar{G} \cup G^*$ and either $f\colon \mathrm{l}_*\mathrm{oti}(G, H)$ or $f\colon \mathrm{loti}(G, H)$, then $f(x(\bar{G})) = x(\bar{H}) \geq \gamma$, where $\bar{H}$ is the induced subgraph of H with vertex set $f[\bar{G}]$.*

Proof. By induction on γ. We will argue for both cases for f simultaneously, and use the notation $N^{\mathcal{G}}_{(*)}(x)$ for a graph $\mathcal{G}$ and some $x \in |\mathcal{G}|$, which denotes $N^{\mathcal{G}}_*(x)$ in the case when $f\colon \mathrm{l}_*\mathrm{oti}(G, H)$ and denotes $N^{\mathcal{G}}(x)$ in the case when $f\colon \mathrm{loti}(G, H)$. Assume $\gamma = \kappa$. The claim follows in this case from what we have shown in the second paragraph of the proof of the theorem, as this proof did not refer to $\mathrm{ot}\left(N_{(*)}(x(\bar{G}))\right)$, and for any other $y \in \bar{G}$, $\mathrm{ot}\left(N^{\bar{G}}_{(*)}(y)\right) = \mathrm{ot}\left(N^{G}_{(*)}(y)\right)$.

Now assume the claim holds for some limit ordinal γ, we want to show it holds for $\gamma + \omega$. Let $x = x(G_\gamma)$, where G_γ refers to the corresponding induced subgraph of $\bar{G} = G_{\gamma+\omega}$, which in turn is an induced subgraph of G. Then $f(x) \geq \gamma$ inductively and $\operatorname{ot}\left(N^H_{(*)}(f(x))\right) = \omega$. Thus we can find a strictly increasing sequence $\langle n_i \mid i < \omega\rangle$ of natural numbers with $n_0 = 0$ such that $\langle f(x+n_i) \mid i < \omega\rangle$ is a strictly increasing sequence of ordinals and for every $i < \omega$ there is $j < \omega$ such that $f(x+i) < f(x+n_j)$. Now let 0 be as in the construction of $G_{\gamma+\omega}$. Then $\operatorname{ot}\left(N^H_{(*)}(f(0)) = \omega + 1\right)$ and this implies that $f(x+\omega) \geq \gamma + \omega$.

Finally, assume that $\gamma > \kappa$ is a limit of limit ordinals and the claim holds for all $\bar{\gamma} < \gamma$. Let $\langle \gamma_i \mid i < \operatorname{cof}(\gamma)\rangle$ be cofinal in γ with each γ_i a limit ordinal. If G_γ is the disjoint union of G_{γ_i} for $i < \operatorname{cof}(\gamma)$ with the $x(G_{\gamma_i})$ connected by edges together with an additional node 0 adjancent to all $x(G_{\gamma_i})$ and another additional sufficiently large node δ adjancent only to 0, then it follows that $f(x(G_{\gamma_i})) = x(f[G_{\gamma_i}]) \geq \gamma_i$ inductively. Now since $\operatorname{ot}\left(N^G_{(*)}(0)\right) = \operatorname{cof}(\gamma) + 1$ and thus $\operatorname{ot}\left(N^{\bar{H}}_{(*)}(f(0))\right) = \operatorname{cof}(\gamma) + 1$, it follows that $f(\delta) = \max(\bar{H}) \geq \gamma$. □

The main statement of the theorem now follows (for any γ) by taking $G = G_\gamma$ in the statement of the claim. It remains to verify the second statement of the theorem for every G_γ. Using Lemma 1.8, it suffices to show that whenever $x \in |G_\gamma|$ is such that $\operatorname{ot}\left(N^{G_\gamma}(x)\right) = \kappa+1$, then there is y with $(x,y) \in G_\gamma$, $\operatorname{ot}\left(N^{G_\gamma}(y)\right) = \kappa$ and such that $N^{G_\gamma}(x) \cap N^{G_\gamma}(y)$ has cardinality κ. We will do so by induction on $\gamma < \kappa^+$.

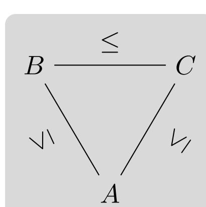
If $\gamma = \kappa$, the only node $x \in |G_\kappa|$ with $\operatorname{ot}\left(N^{G_\kappa}(x)\right) = \kappa+1$ is 0, and any $y \in (0,\kappa)$ is as required in this case.

At stage $\gamma+\omega$, note that the only nodes $x \in |G_{\gamma+\omega}|$ with $\operatorname{ot}\left(N^{G_{\gamma+\omega}}(x)\right) = \kappa+1$ are elements of $|G_\gamma| \setminus x(G_\gamma)$, allowing us to find appropriate y within G_γ inductively.

Assume now γ is a limit of limit ordinals and $x \in |G_\gamma|$ satisfies $\operatorname{ot}\left(N^{G_\gamma}(x)\right) = \kappa + 1$. If $x \in |G_{\gamma_i}| \setminus \{x(G_{\gamma_i})\}$ for some $i < \operatorname{cof}(\gamma)$, then we find the appropriate y within G_{γ_i} inductively. If $\operatorname{cof}(\gamma) < \kappa$, no other such x exist. Thus assume that $\operatorname{cof}(\gamma) = \kappa$. In this case, $x = 0$ (and no other x) also satisfies $\operatorname{ot}\left(N^{G_\gamma}(x)\right) = \kappa+1$, but here $y = x(G_{\gamma_0})$ is as required. □

Now we turn our attention to the most narrow class of graphs in our hierarchy, the κ_*-splitting graphs. We show that for such graphs G, $||G||^{\mathrm{loi}}$ can still be arbitrarily large. For this result, the following graphs will be a basic ingredient:

Definition 3.2. Given sets of ordinals A, B and C, with injective enumerations $A = \{a_i \mid i < \kappa\}$, $B = \{b_i \mid i < \kappa\}$ and $C = \{c_i \mid i < \kappa\}$, we denote by $\triangledown(A,B,C)$ the graph with vertex set $A \cup B \cup C$ in which a_i is connected to both b_j and c_j whenever $i \leq j$, and b_i is connected to c_j whenever $i \leq j$. We picture this graph as follows:



Lemma 3.3. *If $A = \{a_i \mid i < \kappa\}$, $B = \{b_i \mid i < \kappa\}$ is given by an increasing enumeration and $C = \{c_i \mid i < \kappa\}$ is given by an increasing enumeration (thus B and C both have order-type κ, while A may have any order-type less than κ^+), $\sup\{a_i \mid i < \kappa\} < b_0$ and $\sup B = \sup C$, then $G = \nabla(A, B, C)$ is κ_*-splitting and $\operatorname{ot}\left(N_*^G(c_i)\right) < \kappa$ for every $i < \kappa$.*

Proof. Note that our assumptions imply that $\operatorname{ot}(B \cup C) = \kappa$. Thus $\operatorname{ot}\left(N_*^G(a_i)\right) = \kappa$ for every $i < \kappa$. Similarly $\operatorname{ot}\left(N_*^G(b_i)\right) = \kappa$ for every $i < \kappa$. Finally, the cardinality of $N_*^G(c_i)$ is less than κ and thus we are done. □

Theorem 3.4. *For every $\alpha < \kappa^+$, there exists a κ_*-splitting graph G such that $||G||^{\mathrm{loi}} \geq \alpha$ and such that whenever $f\colon \mathrm{loi}(G, H)$, then H is κ_*-splitting.*

Remark: If some graph G is κ_*-splitting and $f\colon \mathrm{loi}(G, H)$, it is immediate that H is κ-splitting, however H will not be κ_*-splitting in general.

Proof. Let $\alpha < \kappa^+$. We want to find a graph G which is κ_*-splitting such that $||G||^{\mathrm{loi}} \geq \kappa \cdot \alpha$. Let $A_0 := \emptyset$ and $B_0 := \kappa$. Let $\{b_0^i \mid i < \kappa\}$ be an increasing enumeration of B_0. For $\gamma \in [1, \alpha)$, let $A_\gamma := \{\kappa \cdot \gamma + 2 \cdot i \mid i < \kappa\}$, let $B_\gamma := \{\kappa \cdot \gamma + 2 \cdot i + 1 \mid i < \kappa\}$ and let $A_\gamma = \{a_\gamma^i \mid i < \kappa\}$ and $B_\gamma = \{b_\gamma^i \mid i < \kappa\}$ be their increasing enumerations. We will perform an induction and construct graphs G_γ for $\gamma \leq \alpha$ such that $G = G_\alpha$ is as desired. However note that for different α we will construct different graphs $\langle G_\gamma \mid \gamma \leq \alpha\rangle$ along our inductive construction.

Fix a disjoint partition $\kappa = \bigcup_{i \leq \alpha} S_i$ with each S_i of cardinality κ. We will construct a sequence of graphs $\langle G_\gamma \mid \gamma \leq \alpha\rangle$ such that the following hold for every $\gamma \leq \alpha$:

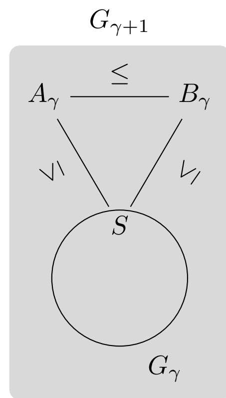
(i) $|G_\gamma| = \bigcup_{\delta < \gamma}(A_\delta \cup B_\delta)$.
(ii) If $\delta < \gamma$, G_δ is an induced subgraph of G_γ.
(iii) If $\delta < \gamma$, $(a, b) \in E(G_\gamma) \setminus E(G_\delta)$ and $a \in |G_\delta|$, then
$$\exists \eta < \delta\, \exists i \in [\delta, \gamma)\, \exists j \in S_i\ a = b_\eta^j.$$
(iv) If $i \geq \gamma$, $\delta < \gamma$ and $j \in S_i$, then $\operatorname{ot}\left(N_*^{G_\gamma}(b_\delta^j)\right) < \kappa$.
(v) G_γ is κ_*-splitting.
(vi) Assume $f\colon \mathrm{loi}(G_\gamma, H)$, $i < \kappa$ and $F := \{b_\delta^j \mid \delta < \gamma, j \in S_i\}$. Whenever $\bar{F}$ is a subset of F for which $F \setminus \bar{F}$ has size less than κ, then $\sup(f[\bar{F}]) \geq \kappa \cdot \gamma$.

Let G_0 be the empty graph. Let $G_1 = \langle B_0, \emptyset\rangle$. Note that G_1 is not connected. Actually, in our below construction, no G_γ is connected except for the final G_α.

Assume now we have constructed $\langle G_\delta \mid \delta \leq \gamma\rangle$ for some $\gamma \geq 1$, with the above properties, and we want to construct $G_{\gamma+1}$. Let $S = \{b_\delta^j \mid \delta < \gamma, j \in S_\gamma\}$, choose an enumeration so that $S = \{s_i \mid i < \kappa\}$. We let $G_{\gamma+1} = G_\gamma \cup \nabla(S, A_\gamma, B_\gamma)$, i.e. we let $|G_{\gamma+1}| = |G_\gamma| \cup A_\gamma \cup B_\gamma$ and let $E(G_{\gamma+1})$ be obtained from $E(G_\gamma)$ by adding the following edges:

- An edge between s_i and both a_γ^j and b_γ^j for $i \leq j < \kappa$.
- An edge between a_γ^i and b_γ^j for $i \leq j < \kappa$.

We picture this as follows:



We want to verify (i)-(vi) for $G_{\gamma+1}$.

(i) Immediate.

(ii) Immediate by the construction of $G_{\gamma+1}$.

(iii) Assume $\delta < \gamma + 1$ and $E = (a, b)$ is in $E(G_{\gamma+1}) \setminus E(G_\delta)$ with $a \in |G_\delta|$. If $\delta < \gamma$ and $E \in E(G_\gamma) \setminus E(G_\delta)$, then the result follows from (iii) for the pair (δ, γ). On the other hand if $E \in E(G_{\gamma+1}) \setminus E(G_\gamma)$, it follows from the construction of $G_{\gamma+1}$ that $a \in S$, i.e. $\exists \eta < \delta$ $a = b^j_\eta$ for some $j \in S_\gamma$.

(iv) Assume $i \geq \gamma + 1$, $\delta < \gamma + 1$ and $j \in S_i$. If $\delta < \gamma$, then by (iv) for γ, $\mathrm{ot}\left(N^{G_\gamma}_*(b^j_\delta)\right) < \kappa$. But by (iii) for the pair $(\gamma, \gamma+1)$, $N^{G_\gamma}_*(b^j_\delta) = N^{G_{\gamma+1}}_*(b^j_\delta)$. Otherwise, if $\delta = \gamma$, then $b^j_\delta \in B_\gamma$. But then it is easy to see from the construction of $G_{\gamma+1}$ that $\mathrm{ot}\left(N^{G_\gamma}_*(b^j_\delta)\right) < \kappa$.

(v) We have to show that $\mathrm{ot}\left(N^{G_{\gamma+1}}_*(a)\right) \leq \kappa$ for every $a \in |G_{\gamma+1}|$. If $a \in |G_\gamma| \setminus S$, this is immediate by (v) for γ and by the construction of $G_{\gamma+1}$. If $a \in S$, this is immediate from (v) for γ together with (iii). If $a \in |G_{\gamma+1}| \setminus |G_\gamma|$, then this is immediate from the construction of $G_{\gamma+1}$.

(vi) Fix any $i < \kappa$ and assume $f\colon \mathrm{loi}(G_{\gamma+1}, H)$. We will show that whenever $X \subseteq \kappa$ is of size κ, then $\sup\{f(b^j_\gamma) \mid j \in X\} \geq \kappa \cdot (\gamma + 1)$.

For any $x < \kappa$ and $y \geq x$, b^y_γ is adjancent to both a^x_γ and s_y. The above implies that $f(a^x_\gamma) > f(s_y)$ for every $y \geq x$. (vi) for γ implies that $\sup\{s_y \mid y \geq x\} \geq \kappa \cdot \gamma$. Thus $f(a^x_\gamma) \geq \kappa \cdot \gamma$ for every $x < \kappa$.

Now for any $x < \kappa$, s_0 is adjancent to both a^x_γ and b^x_γ, so $\kappa \cdot \gamma \leq f(a^x_\gamma) < f(b^x_\gamma)$. Since the b^x_γ are all distinct, it follows that $\sup\{f(b^x_\gamma) \mid x \in X\} \geq \kappa \cdot (\gamma + 1)$.

Now we assume that γ is a limit ordinal and that we have constructed $\langle G_\delta \mid \delta < \gamma \rangle$ with the above properties. We let $G_\gamma = \bigcup_{\delta<\gamma} G_\delta$ and want to verify (i)-(vi) hold for G_γ.

(i) Immediate.

(ii) Immediate.

(iii) Immediate.

(iv) Let $i \geq \gamma$, $\delta < \gamma$ and $j \in S_i$. Note that $b^j_\delta \in |G_{\delta+1}|$ and by (iv) for $\delta + 1$, $\mathrm{ot}\left(N^{G_{\delta+1}}_*(b^j_\delta)\right) < \kappa$. Now by (iii) for the pair $(\delta + 1, \gamma)$, $N^{G_\gamma}_*(b^j_\delta) = N^{G_{\delta+1}}_*(b^j_\delta)$, using that $j \in S_i$ and $i \geq \gamma$.

(v) We need to show that $\mathrm{ot}\left(N_*^{G_\gamma}(a)\right) \le \kappa$ for every $a \in G_\gamma$. If $a \in A_\delta$ for some $\delta < \gamma$, this is immediate by (v) for δ together with (iii) for the pair (δ, γ). Thus assume $a \in B_\delta$ for some $\delta < \gamma$ and pick j and i such that $a = b_\delta^j$ and $j \in S_i$.

If $i < \gamma$, let $k = \max(i, \delta+1)$. By (v) for k, $\mathrm{ot}\left(N_*^{G_k}(b_\delta^j)\right) \le \kappa$. By (iii) for the pair (k, γ), $N_*^{G_k}(b_\delta^j) = N_*^{G_\gamma}(b_\delta^j)$, using that $k > i$.

If $i \ge \gamma$, then by (v) for the pair $(\delta+1, \gamma)$, $\mathrm{ot}\left(N_*^{G_{\delta+1}}(b_\delta^j)\right) \le \kappa$ and by (iii) for $(\delta+1, \gamma)$, $N_*^{G_{\delta+1}}(b_\delta^j) = N_*^{G_\gamma}(b_\delta^j)$.

(vi) Immediate using (vi) inductively together with Lemma 2.1.

It only remains to verify the last statement of the theorem, that is, if $f\colon \mathrm{loi}(G, H)$, then H is κ^*-splitting. We will do this by showing that in fact $f\colon \mathrm{l_*oi}(G, H)$. By Observation 1.6, it suffices to show that whenever $(x, y) \in G$ and $x < y$, then $f(x) < f(y)$.

Thus assume that $(x, y) \in G$ and $x < y$. x, y will be contained in an induced subgraph of G that is of the form $\nabla(A, B, C)$, with $A = \{a_i \mid i < \kappa\}$, $B = \{b_i \mid i < \kappa\}$ and $C = \{c_i \mid i < \kappa\}$. Using that $f\colon \mathrm{loi}(G, H)$, it suffices to find z that is connected to both x and y in $\nabla(A, B, C)$.

- If $x = a_i$ and $y = b_j$, then $i \le j$ and we let $z = c_j$.
- If $x = a_i$ and $y = c_j$, then $i \le j$ and we let $z = b_j$.
- If $x = b_i$ and $y = c_j$, then $i \le j$ and we let $z = a_i$.

□

It seems natural to ask whether a similar construction can be performed with loti* in place of loi. We do not know the answer to this question, our best result in this direction is the following.

Theorem 3.5. *Assume κ is a regular and uncountable cardinal. There are κ_*-splitting graphs G and G_* such that $||G||^{\mathrm{loti}} = ||G_*||^{\mathrm{l_*oti}} = \kappa \cdot 2$.*

Proof. The proof will be based on the following easy standard claim, which we will provide the short proof of for the sake of completeness.

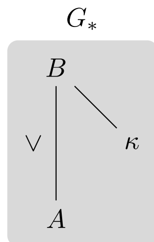
Claim 2. *Assume f is an injective function from κ to the ordinals and $\mathrm{ot}(f[\kappa]) = \kappa$. Then there is a club of $\alpha < \kappa$ such that $\mathrm{ot}(f[\alpha]) = \alpha$.*

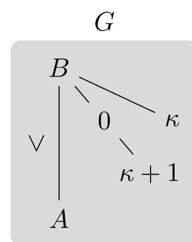
Proof. Let $C_0 = \{\alpha < \kappa \mid f[\kappa] \cap \sup(\{f(\beta)+1 \mid \beta < \alpha\}) = f[\alpha]\}$. We claim that C_0 is a club subset of κ. Obviously, C_0 is closed. To see that C is unbounded, let $\alpha_0 < \kappa$. Given α_i, let $\alpha_{i+1} < \kappa$ be such that $f[\kappa] \cap \sup(\{f(\beta)+1 \mid \beta < \alpha_i\}) \subseteq f[\alpha_{i+1}]$. Using the regularity of κ and the fact that $\mathrm{ot}(f[\kappa]) = \kappa$, it is easy to see that such α_{i+1} always exists. Let $\alpha = \bigcup_{i<\omega} \alpha_i$. Then $f[\kappa] \cap \sup(\{f(\beta) \mid \beta < \alpha\}) = f[\alpha]$, i.e. $\alpha \in C_0$.

Now let $C = \{\alpha \in C_0 \mid \mathrm{ot}(f[\alpha]) = \alpha\}$. We claim that C is club. Obviously, C is closed. To see that C is unbounded, let $\alpha_0 < \kappa$, $\alpha_0 \in C_0$. Given α_i and using the regularity of κ, let $\alpha_{i+1} \ge \alpha_i$ be such that $\alpha_{i+1} \in C_0$, $\alpha_{i+1} \ge \mathrm{ot}(f[\alpha_i])$ and $\mathrm{ot}(f[\alpha_{i+1}]) \ge \alpha_i$. Let $\alpha = \bigcup_{i<\omega} \alpha_i$. Then $\mathrm{ot}(f[\alpha]) = \bigcup_{i<\omega} \mathrm{ot}(f[\alpha_i]) = \alpha$, using that the α_i are in C_0. □

We will now define the graphs G and G_*. Let $A = [1, \kappa)$ and let $B = [\kappa+2, \kappa\cdot 2)$. Let $A = \{a_i \mid i < \kappa\}$ and $B = \{b_i \mid i < \kappa\}$ be increasing enumerations of A and

B respectively. G_* will have vertex set $A \cup B \cup \{\kappa\}$ where a_i is adjancent to b_j whenever $i < j$, and κ is adjancent to all nodes in B. G will have vertex set $\kappa \cdot 2$, where a_i is adjancent to b_j whenever $i < j$, κ is adjancent to all nodes in B and 0 is adjancent to all nodes in B and to $\kappa + 1$. We picture these graphs as follows:





We will provide the argument that G_* is as desired. The argument for G is similar.

- If $x \in A$, $\operatorname{ot}\left(N_*^{G_*}(x)\right) = \kappa$.
- If $x \in B$, $\operatorname{ot}\left(N_*^{G_*}(x)\right)$ is a double successor ordinal less than κ.
- $\operatorname{ot}\left(N_*^{G_*}(\kappa)\right) = \kappa$.

Assume now $f\colon \mathrm{l_*oti}(G_*, H_*)$. If $i < \kappa$, b_i is adjancent to a_j for $j < i$ and to κ. So $\operatorname{ot}\left(N_*^{G_*}(b_i)\right) = i + 2$, hence $\operatorname{ot}\left(N_*^{H_*}(f(b_i)\right) = i + 2$ and therefore $\operatorname{ot}(f[\{a_j \mid j < i\} \cup \{\kappa\}]) \geq i + 1$. Using the claim, it follows that $\operatorname{ot}(f[A \cup \{\kappa\}]) \geq \kappa + 1$. Let x be the κ^{th} element in the increasing enumeration of $f[A \cup \{\kappa\}]$. It follows that $x \geq \kappa$, and moreover $f[B] \setminus x$ must have cardinality κ, for otherwise $\operatorname{ot}\left(N_*^{H_*}(x)\right) > \kappa$. But this implies that $|H_*|$ has order-type at least $\kappa \cdot 2$. $\square$

Note: If G is the graph from Theorem 3.5 and $f\colon \mathrm{loti}(G, H)$, then H need not be κ_*-splitting.

4. Open Questions

The most obvious question left open seems to be whether we can improve the construction of Theorem 3.5 to get a positive answer to the following question:

Question 4.1. *Is there a κ-splitting graph G with $||G||^{\mathrm{loti}^*} > \kappa \cdot 2$? Is there, for every $\gamma < \kappa^+$, a κ-splitting graph G with $||G||^{\mathrm{loti}^*} > \gamma$?*

Our second question concerns the existence of small universal families for our classes of graphs: Assume $I \in \mathrm{OI}$. If we define the *(strong) I-universality number* for a given family $\mathcal{G}$ of graphs to be the least cardinality of some $\mathcal{X} \subseteq \mathcal{G}$ such that every element of $\mathcal{G}$ can be (strongly) I-embedded into some element of $\mathcal{X}$ (we call such $\mathcal{X}$ an *I-universal family*), then our results show that the I-universality number for α-splitting (or α_*-splitting) graphs is at least κ^+, for any $\kappa < \alpha < \kappa^+$, or also for $\alpha = \kappa$ in case $I \in \{\mathrm{loi}, \mathrm{l_*oi}, \mathrm{oi}\}$. Since the number of ordinal graphs of size κ modulo order-isomorphism is 2^κ, this number is largest possible if $2^\kappa = \kappa^+$. The following seems to be an obvious question.

Question 4.2. *Assume $2^\kappa > \kappa^+$, $\alpha \in [\kappa, \kappa^+)$ and $I \in \mathrm{OI}$. What is the (strong) I-universality number for α-splitting (or α_*-splitting) graphs?*

A question that was raised in the introduction of this paper is the following:

Question 4.3. *Assume κ is a singular cardinal. Is there (consistently) a universal object in the class of graphs of size κ that are locally of size less than κ?*

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