Small embedding characterizations of large cardinals, and internal large cardinals

Peter Holy

University of Bonn

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Small Embedding Characterizations for Large Cardinals

Definition

Given cardinals $\kappa < \theta$, a non-trivial elementary embedding $j\colon M \to H(\theta)$ is a *small embedding for* κ if $M \in H(\theta)$ is transitive and $j(\mathrm{crit}\, j) = \kappa$ holds.

The properties of large cardinals κ that we study state that for sufficiently large cardinals θ, there is a small embedding $j\colon M \to H(\theta)$ for κ, with certain elements of $H(\theta)$ in its range, and such that the pair (M, V) satisfies certain correctness properties.

A Classic Example due to Magidor

A cardinal κ is supercompact if and only if for every $\eta > \kappa$ there is $\alpha < \kappa$ and an elementary embedding $j\colon H(\alpha) \to H(\eta)$ with $j(\mathrm{crit}\, j) = \kappa$.

$\exists\alpha\ M = H(\alpha)$ can indeed be seen as a correctness property between M and V, namely it can be reformulated as $\forall x \in M\ H(|x|^+) \subseteq M$.

Why do we want correctness properties?

If we didn't, consider the following small embedding characterization of measurability:

Lemma?

κ is measurable if for all sufficiently large cardinals θ there is a small embedding $j\colon M \to H(\theta)$ for κ such that

$$M \models \text{crit}\, j \text{ is measurable.}$$

This is a *trivial small embedding characterization*, and is excluded by our requiring small embedding characterizations to be correctness properties. $\text{crit}\, j$ is not necessarily measurable in V in the above situation, for example if κ was the least measurable cardinal.

A basic lemma

Lemma

For a parameter-free $\mathcal{L}_\in$-formula φ, the following statements are equivalent for every cardinal κ:

1. *κ is a regular uncountable cardinal, and the set of all ordinals $\alpha < \kappa$ such that $\varphi(\alpha)$ holds is stationary in κ.*
2. *For all sufficiently large cardinals θ, there is a small embedding $j\colon M \to H(\theta)$ for κ with $\varphi(\mathrm{crit}\, j)$.*

(1)$\to$(2): Let $\langle M_\alpha \mid \alpha < \kappa \rangle$ be a continuous and increasing sequence of elementary substructures of $H(\theta)$ of cardinality less than κ such that $\alpha \subseteq M_\alpha$ for all $\alpha < \kappa$.
(2)$\to$(1): similar.

Mahlo-like cardinals

Corollary

Let κ be a cardinal.

1. κ *is regular and uncountable iff for all sufficiently large cardinals* θ*, there is a small embedding* $j\colon M \to H(\theta)$ *for* κ*.*
2. κ *is weakly inaccessible iff for all sufficiently large cardinals* θ*, there is a small embedding* $j\colon M \to H(\theta)$ *for* κ *with the property that* $\operatorname{crit} j$ *is a cardinal.*
3. κ *is inaccessible iff …* $\operatorname{crit} j$ *is a strong limit cardinal.*
4. κ *is weakly Mahlo iff …* $\operatorname{crit} j$ *is regular.*
5. κ *is Mahlo iff …* $\operatorname{crit} j$ *is inaccessible.*

Note that while the basic lemma does not always produce characterizations of large cardinals that are correctness properties, all of the above are correctness properties, for example, because Mahloness implies inaccessibility, if κ is Mahlo and $j\colon M \to H(\theta)$ is a small embedding for κ, then $\operatorname{crit} j$ is inaccessible in M.

Mahlo-like cardinals are not weakly compact

Observation

Given an $\mathcal{L}_\in$-formula φ, the least cardinal κ with the property that for all sufficiently large cardinals θ, there is a small embedding $j\colon M \to H(\theta)$ for κ such that $\varphi(\operatorname{crit} j)$ holds is not weakly compact.

Proof: Assume for a contradiction that it is and use that weak compactness implies stationary reflection in order to contradict leastness. □

So, Mahlo-like cardinals are not weakly compact.

Indescribables

Based on an elementary embedding characterization of indescribables by Kai Hauser, we obtain the following small embedding characterization:

Lemma

Given $0 < n < \omega$, the following are equivalent for every cardinal κ:

1. κ *is* Π^1_n*-indescribable.*
2. *For all sufficiently large cardinals* θ*, there is a small embedding* $j\colon M \to H(\theta)$ *for* κ *such that* $H(\mathrm{crit}\, j^+)^M \prec_{\Sigma_n} H(\mathrm{crit}\, j^+)$.

Higher Indescribables

Similarly, we obtain a small embedding characterization for higher levels of indescribability:

Lemma

Given $0 < m, n < \omega$, the following are equivalent for every cardinal κ:

1. κ *is* Π^m_n*-indescribable.*
2. *For all sufficiently large cardinals θ, there is a small embedding $j\colon M \to H(\theta)$ for κ such that*

$$(V_{\mathrm{crit}\, j} \models \varphi)^M \leftrightarrow V_{\mathrm{crit}\, j} \models \varphi$$

for every Π^m_n*-formula φ with parameters in $M \cap V_{\mathrm{crit}\, j+1}$.*

Ineffables

Definition

A regular uncountable cardinal κ is *ineffable* if for every sequence $\vec{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ with $A_\alpha \subseteq \alpha$ (we call such sequences κ-*lists*), there exists $A \subseteq \kappa$ such that $S = \{\alpha < \kappa \mid A \cap \alpha = A_\alpha\}$ is stationary in κ.

Ineffability has another characterization in terms of the existence of homogeneous sets for certain colourings, however no embedding characterization was known so far.

Lemma

κ is ineffable iff for every κ-list $\vec{A}$ and for all sufficiently large cardinals θ, there is a small embedding $j\colon M \to H(\theta)$ for κ such that $\vec{A} \in \operatorname{ran} j$ and $A_{\operatorname{crit} j} \in M$.

Remark: It is enough to work with $\theta = \kappa^+$ in the above. *Note:* We also have a similar characterization for subtle cardinals.

Measurables

Lemma

κ is measurable iff for all sufficiently large cardinals θ, there is a small embedding $j\colon M \to H(\theta)$ for κ with

$$\bar{U} := \{A \in \mathcal{P}(\mathrm{crit}\, j)^M \mid \mathrm{crit}\, j \in j(A)\} \in M.$$

Proof: Assume that κ is measurable, fix a normal ultrafilter U on κ witnessing this, let $j_U\colon V \to Ult(V, U)$ be the induced ultrapower embedding. By elementarity, we find a desired small embedding for κ in V. □

Some additional structure

The small embeddings characterizing large cardinals cohere in a nice way that parallels the implication structure between the large cardinals.

Almost a theorem

Whenever there is a direct implication between large cardinal notions $A \to B$, then small embeddings witnessing A for κ also witness B for κ.

A sample lemma

If $j\colon M \to H(\theta)$ witnesses that κ is measurable (so that $\bar{U} \in M$), then $A_{\mathrm{crit}\, j} \in M$ whenever $\vec{A} \in \operatorname{ran} j$ is a κ-list.

Proof: $\vec{A}$ is mapped to a list of length greater than κ in the (usual) measurable ultrapower of V by $U = j(\bar{U})$, let $X \subseteq \kappa$ be its κ^{th} element, and note that $X \in \operatorname{ran} j$ by definability.

An auxiliary lemma

This sample lemma shows that small embeddings witnessing measurability also witness ineffability, modulo the following:

Almost a lemma

If we have a small embedding characterization of the form

$$\text{For all sufficiently large cardinals } \theta,$$

$$\text{there is a small embedding } j\colon M \to H(\theta) \text{ such that} \ldots,$$

we can equivalently replace it by

$$\text{For all sufficiently large cardinals } \theta \text{ and all } x \in H(\theta), \text{ there is}$$

$$\text{a small embedding } j\colon M \to H(\theta) \text{ with } x \in \operatorname{ran} j \text{ and such that} \ldots.$$

It is only *almost a lemma* as it only works under certain assumptions on the properties of the characterization, but it works for all characterizations that we have considered so far and will consider in the following.

More filter-based large cardinals

We can do small embedding characterizations similar to those of measurable cardinals for λ-supercompact cardinals whenever $\lambda = \lambda^{<\kappa}$, and thus for supercompact cardinals.

Also, the small embedding characterization of ineffability generalizes to λ-ineffability,

Theorem

κ is supercompact iff for every $\lambda \geq \kappa$, every $\vec{A} = \langle A_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ with $A_x \subseteq x$, and for all sufficiently large cardinals θ, there is $M \prec H(\theta)$ of size less than κ such that $M \cap \kappa \in Ord$, and such that both $\vec{A}$ and $A_{M\cap\lambda}$ are elements of M.

Internally Π^1_n-indescribables

Definition

A regular uncountable cardinal κ is internally Φ-Π^1_n-indescribable if

- a small embedding $j\colon M \to H(\theta)$ for κ with $x \in \operatorname{ran} j$,
- $N \models ZFC^-$ transitive with $N \cap Ord = \theta$, $N \subseteq H(\theta)$ and $\Phi(N, H(\theta))$,
- $M \in N$ (we call N with the above properties a Φ-*connector*) and
- $H(\operatorname{crit} j^+)^M \prec_{\Sigma_n} H(\operatorname{crit} j^+)^N$.

Examples for Φ:

- $H(\theta)$ is a forcing extension of N by a forcing that is $<\lambda$-closed / ...
- $(N, H(\theta))$ satisfies the ω_1-approximation property,

Properties of internally Π^1_n-indescribables

- Suitable Forcing turns Π^1_n-indescribables into internally Φ-Π^1_n-indescribables
- If κ becomes internally Φ-Π^1_n-indescribable after suitable forcing, then it was Π^1_n-indescribable in the ground model
- Internally Φ-Π^1_n-indescribable + inaccessible = Π^1_n-indescribable, for suitable Φ (for example closed or AP).
- The consistency strength of an internally Φ-Π^1_1-indescribable is exactly a Π^1_1-indescribable cardinal, as the former implies that there are no $\square(\kappa)$-sequences, for Φ = closed or AP.

It is not necessarily straightforward to come up with reasonable definitions for internal large cardinals. So far, we have internal versions for

- Mahlo cardinals
- Π^m_n-indescribable cardinals
- λ-ineffable cardinals
- λ-subcompact cardinals
- supercompact cardinals
- n-huge cardinals for $0 < n < \omega$

Some internal large cardinals may be new properties, some others correlate with properties that are already well-known.

ISP...

Definition

ISP(κ, λ) is the statement that every slender $\mathcal{P}_\kappa(\lambda)$-list has an ineffable branch.

These list properties were used by Matteo Viale and Christoph Weiß to show that in order to obtain *PFA* by reasonable forcing, one has to start with a supercompact cardinal.

...and internally supercompact cardinals

Definition

κ is *internally AP supercompact* if for all sufficiently large cardinals θ and all $x \in H(\theta)$, there is a small embedding $j\colon M \to H(\theta)$ for κ and an AP-connector N such that $x \in \operatorname{ran} j$ and $M = H(\nu)^N$ for some cardinal $\nu < \kappa$ in N.

Lemma

κ is internally AP supercompact iff $\forall\lambda$ ISP(κ, λ).A regular cardinal $\kappa > \omega_1$ is internally AP λ-ineffable iff ISP(κ, λ).

Theorem (Viale-Weiß)

PFA implies that ω_2 is internally AP supercompact.