Infinite Cut and Choose Games

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Assume there's a piece of cake, and we want Ann and Bob to fairly share it. An easy way to do so is to let Ann *cut* the cake in two, then let Bob *choose* his piece, and let Ann have the remaining piece.

This principle is already mentioned in the bible, some hundred years BC. Modern investigation of *fair division* was initiated by Steinhaus, Banach and Knaster in the 1940ies. Initially, they extended fair division to a larger number of people, say now you have Ann, Bob and Chris...

In set theory, we tend to be interested in *infinite games*. Galvin, Mycielski, Ulam, and possibly others, in the 1960ies, proposed various *infinite cut and choose games*.

Our basic Cut and Choose game

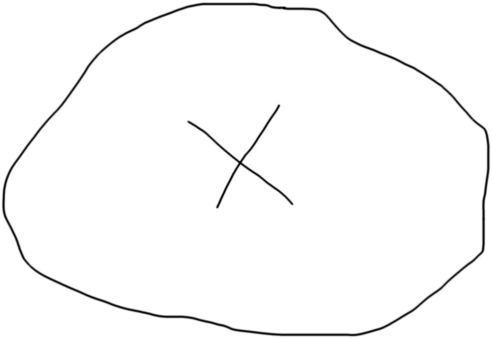
We have two players, going by the names of *Cut* and *Choose*. Instead of a piece of cake, we have a set X over which our game takes place. For a start, you may perhaps think of X being either $\mathbb{N}$ or $\mathbb{R}$.

- In their first move, *Cut* partitions X into two (disjoint) pieces, and *Choose* picks one of them, call it X_0.
- Now *Cut* partitions X_0 into two pieces, and *Choose* picks one of them, call it X_1.
- Now *Cut* partitions X_1 ...

In the end, *Choose* wins a run of the game in case $\bigcap X_i$ has (at least) two distinct elements.

Let us denote the above game as $\mathcal{U}(X)$.

Our basic Cut and Choose game



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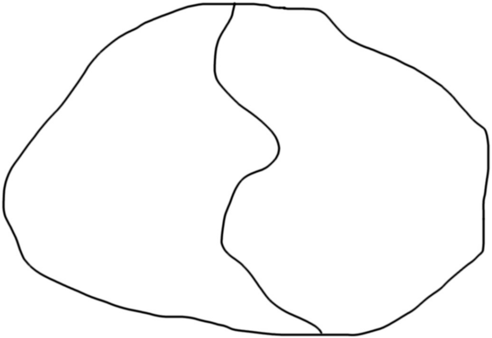

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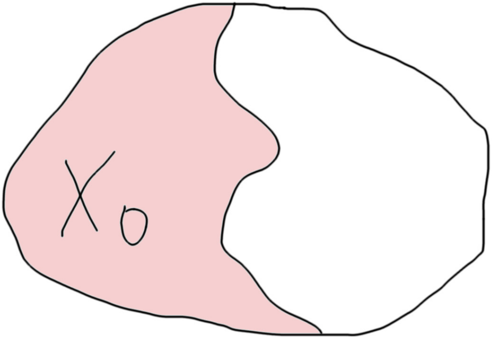

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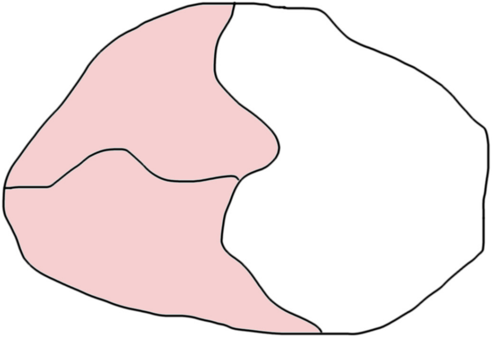

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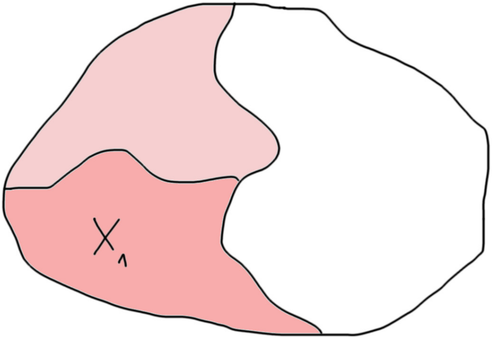

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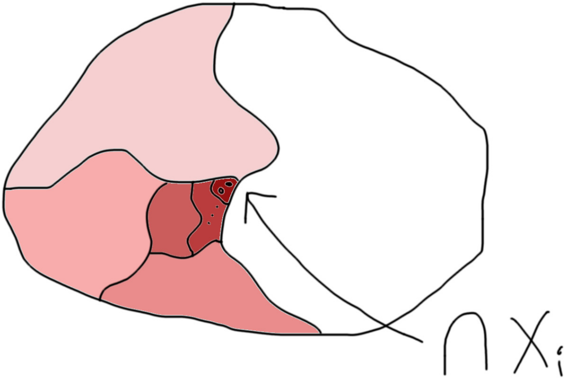


Why *two* elements?

Assume we declare *Choose* to win if only $\bigcap X_i$ were nonempty. Let X be any set, and let a be an element of X. Now in a run of $\mathcal{U}(X)$ with this modified winning condition, whatever partition *Cut* presents, let *Choose* simply pick the part X_i that contains a as an element. In the end, clearly $a \in \bigcap X_i$, and thus *Choose* would easily win all the time.

This is what we call a *winning strategy*, in this case for *Choose*.

A *strategy* (for one particular player) is a function which receives as input a play of the game so far, and its output is the next move to be made by that player. A strategy is *winning* if following it guarantees a win, regardless of the other player's moves.

In set theory, we like to say that some player *wins* a certain game to express that they have a winning strategy. We say that they win *a particular run* of a game otherwise.

Cut wins over $\mathbb{N}$

It is very easy to see that *Cut* wins $\mathcal{U}(\mathbb{N})$ – remember that in a run of the game, players cut and choose infinitely often, and *Choose* wins if the intersection of their choices has two distinct elements.

Cut wins by *removing* one natural number in each step:

- In their first move, they partition $\mathbb{N}$ into $\{0\}$ and $[1, \infty)$.
- *Choose* will have to pick $[1, \infty)$.
- In their second move, they present the choices $\{1\}$ and $[2, \infty)$.
- *Choose* will have to pick $[2, \infty)$.
- Next options will be $\{2\}$ and $[3, \infty)$, ...

Obviously, the intersection of all choices will be empty, hence *Cut* wins.

Note that this had nothing to do with any properties of the natural numbers other than their countability!

Cut wins over $\mathbb{R}$

It is still fairly easy for *Cut* to win $\mathcal{U}(\mathbb{R})$. For notational simplicity, let's play over $[0,1)$. *Cut* wins by simply cutting in half in each step:

- In their first move, they partition $[0,1)$ into $[0,\frac{1}{2})$ and $[\frac{1}{2},1)$.
- *Choose* picks one of these intervals $[a,b)$.
- Given $[a,b)$, let *Cut* partition it into $[a,\frac{a+b}{2})$ and $[\frac{a+b}{2},b)$.
- *Choose* again picks one of these intervals.

The lengths of these intervals *converge* to 0, so their intersection will contain at most one element. But this means that *Cut* wins.

Note again that this had nothing to do with any particular properties of $[0,1)$ other than its size. If any set stands in bijection with $[0,1)$, like $\mathbb{R}$ for example, then we can still make use of the above argument modulo this bijection. So the only thing that matters here is the *size*, or *cardinality*, of the underlying set. For any set X, let $|X|$ denote its cardinality.

Can *Choose* ever win?

If $|X| \leq |\mathbb{R}|$, that is there is an injection from X into $\mathbb{R}$, and thus into $[0, 1)$, then by the above argument, *Cut* has a winning strategy in $\mathcal{U}(X)$.
(This means of course that *Choose* cannot have a winning strategy!)

If *Choose* wants to have a chance of winning, they need to play over a reasonably large set, at least larger than $|\mathbb{R}|$.

The question whether *Choose* can ever win is in fact strongly related to that of the possible existence of very large infinities, namely a certain type of *large cardinal*: the existence of *measurable cardinals* (introduced by Ulam in 1930).

Large Cardinals

Inaccessible cardinals (Hausdorff 1908, Sierpinski-Tarski-Zermelo 1930) can be seen as the *smallest* of large cardinals. They are sets X of uncountable size $|X|$ with the properties that

- If $|Y| < |X|$, then also $|\mathcal{P}(Y)| < |X|$, and
- X cannot be written as the union of less than $|X|$-many sets of size less than X.

Note that $|\mathbb{R}|$ is not inaccessible, nor is $|\mathcal{P}(\mathbb{R})|$, $|\mathcal{P}(\mathcal{P}(\mathbb{R}))|$, ... But also $|\bigcup \mathcal{P}^i(\mathbb{R})|$ is not inaccessible, for it clearly fails to obey the second condition above. In fact, inaccessible cardinals cannot be shown to actually exist (by Gödel's incompleteness theorem, because they yield the consistency of the axioms of ZFC). But their existence is often a useful extra assumption. For example, it is equivalent to the existence of *Grothendieck universes*, as used in algebraic geometry.

Measurable Cardinals

Measurable cardinals are uncountable sets κ for which there exists a *measurable ultrafilter* – $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ deciding for each $A \subseteq \kappa$ whether A or its complement is considered *large* (and the other one *small*), in such a way that $\kappa \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, $\{a\} \notin \mathcal{F}$ for any $a \in \kappa$,

* the intersection of less than $|\kappa|$-many large sets will still be large, and
* if A is considered large, and $B \supseteq A$, then also B is considered large.

Measurable cardinals are fairly large inaccessible cardinals. The smallest inaccessible cardinal is never measurable. In fact, below any measurable cardinal κ, there are $|\kappa|$-many smaller inaccessible cardinals.

Choose wins $\mathcal{U}(\kappa)$ in case κ is a measurable cardinal!

Proof: Make choices according to a measurable ultrafilter $\mathcal{F}$. □

Some classical results

Theorem (Silver-Solovay, 1970ies)

If there exists a set X such that Choose has a winning strategy in the game $\mathcal{U}(X)$, then there is a generically measurable cardinal, and hence a universe of mathematics in which there exists a measurable cardinal.

Corollary

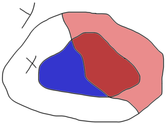
There are universes of mathematics in which *Choose* has no winning strategy for $\mathcal{U}(X)$ over any set X.

Theorem (Laver, 1970ies)

If a measurable cardinal exists, then there is a universe of mathematics in which Choose has a winning strategy in the game $\mathcal{U}(\mathcal{P}(\mathbb{R}))$, that is the Ulam game on the powerset of $\mathbb{R}$.

Leastness is interesting

If *Choose* has a winning strategy in the game $\mathcal{U}(X)$ and $|X| \leq |Y|$, then *Choose* has a winning strategy in $\mathcal{U}(Y)$ as well – simply use the strategy for the game $\mathcal{U}(X)$ modulo an injection from X to Y.



So what is interesting is the *smallest* size of a set X so that *Choose* can win $\mathcal{U}(X)$, if there is any at all.

A combination of known results

If there is a measurable cardinal, then there is a universe of mathematics with a measurable cardinal κ, such that *Choose* does not have a winning strategy in $\mathcal{U}(X)$ whenever $|X| < |\kappa|$.

Remember that *Choose* always has a winning strategy for $\mathcal{U}(\kappa)$ for measurable κ.

Small inaccessibles

Theorem

If a measurable cardinal exists, then there is a universe of mathematics in which λ is an inaccessible cardinal that is not measurable (in fact, not even weakly compact), however λ is the least cardinal such that Choose has a winning strategy in the game $\mathcal{U}(\lambda)$.

Proof: We add a countably closed, homogeneous κ-Suslin tree, and, working in that model, we argue using that the measurability of κ can be resurrected by adding a branch to that tree. □

This means that *Choose* can also *first win* the Ulam game over *fairly small* inaccessible cardinals.

Question

Can *Choose* also first win the Ulam game over the *least* inaccessible cardinal?

Longer Games (and a slightly modified winning condition)

We have a limit ordinal γ, which denotes the length of our game.

- In their first move, *Cut* partitions X into two (disjoint) pieces, and *Choose* picks one of them, call it X_0.
- Now *Cut* partitions X_0 into two pieces, and *Choose* picks one of them, call it X_1.
- Now *Cut* partitions X_1 ...
- At limit stages, intersections are taken, and then partitioned...
- If *Choose* ever picks a singleton or $\emptyset$, they immediately lose.
- Otherwise, this goes on for γ-many steps.

In the end, *Choose* wins if the intersection of all of their choices is nonempty. Otherwise, *Cut* wins.

Let us denote the above game as $\mathcal{U}(X, \gamma)$.

A simplification

For many practical purposes, rather than just cutting the intersection of all choices so far at any stage of our games, it is easier to think of *Cut* repeatedly cutting the starting set X into pieces, and *Choose* picking one of them. This is easily seen to be essentially the same game, for if we are only ever interested in intersections of choices in order to evaluate who wins, the cutting and choosing that happens outside of these intersections is clearly irrelevant. Let us thus *redefine* $\mathcal{U}(X, \gamma)$ as follows:

- In each of their moves, *Cut* partitions X into two (disjoint) pieces, and *Choose* picks one of them.
- If *Choose* ever picks a singleton or $\emptyset$, they immediately lose.
- This goes on for γ-many steps.

In the end, *Choose* wins if the intersection of all of their choices is nonempty. Otherwise, *Cut* wins.

Some fairly easy observations

It is not very interesting to investigate winning strategies for *Cut* in this game. Fairly easy arguments show the following:

Observation

Cut has a winning strategy for the game $\mathcal{U}(\kappa, \gamma)$ if and only if $\kappa \leq 2^{<\gamma}$.

Another fairly easy argument shows the following:

Observation

If γ is regular and $\kappa \leq 2^{\gamma}$, then *Choose* does not have a winning strategy for the game $\mathcal{U}(\kappa, \gamma)$.

Hence, if $2^{<\gamma} < \kappa \leq 2^{\gamma}$, then $\mathcal{U}(\kappa, \gamma)$ is undetermined.
In particular, $\mathcal{U}(\mathbb{R}, \omega)$ is undetermined.

Remember that for the very similar game $\mathcal{U}(\mathbb{R})$, *Cut* had a winning strategy.

Generically measurable cardinals

Definition

A cardinal κ is *generically measurable as witnessed by the notion of forcing* P if in every P-generic extension, there is a uniform V-normal V-ultrafilter on κ that induces a well-founded (generic) ultrapower of V. Equivalently, in every P-generic extension $V[G]$, there is an elementary embedding $j\colon V \to M$ with critical point κ for some transitive $M \subseteq V[G]$.

Observation

Choose wins $\mathcal{U}(\kappa, \gamma)$ whenever κ is generically measurable as witnessed by ${<}\gamma^+$-closed forcing.

Proof: Let $\dot{U}$ be a P-name for a uniform V-normal V-ultrafilter on κ. In each step, *Choose* picks conditions p_i forcing their choices X_i to be in $\dot{U}$, so that the p_i's are decreasing in P. By our closure assumption, the p_i's have a lower bound in P, which forces the intersection of their choices to be in $\dot{U}$, and thus in particular to be nonempty. □

Ideal Partitions

A natural generalization of cut and choose games is to allow *Cut* to cut into a larger number of pieces in each step, or even more generally, to fix an ideal I on κ and to let *Cut* play an *I-partition* in each move. It will turn out that we can use such generalized games to characterize central set theoretic notions. So let us fix a regular uncountable cardinal κ and a $<\kappa$-complete ideal I that contains all bounded subsets of κ.
Let I^+ denote $\mathcal{P}(\kappa) \setminus I$. We call elements of I^+ *I-positive*.

Definition

An I-partition P of $X \in I^+$ is a maximal collection of I-positive subsets of X such that $a \cap b \in I$ whenever $a \neq b$ are both elements of P.

Generalized cut and choose games

For ν a cardinal, or $\nu = \infty$,

- let $\mathcal{G}_\nu(X, I, {<}\gamma)$ denote the cut and choose game of length γ where in each move, *Cut* presents an I-partition of size at most ν, or of arbitrary size if $\nu = \infty$, and *Choose* picks one of its elements. *Choose* wins in case at any stage $\delta < \gamma$, the intersection of their choices up to stage δ is in I^+;
- let $\mathcal{G}_\nu(X, I, {\leq}\gamma)$ denote the variant where for *Choose* to win, we also require that the intersection of all of their choices is nonempty;
- let $\mathcal{G}_\nu(X, I, \gamma)$ denote the variant where for *Choose* to win, we require that the intersection of all of their choices is in I^+.

For these generalized games, unlike for our basic cut and choose games, it is very interesting to consider the existence of winning strategies for *Cut*.

Weak compactness

bd_κ denotes the bounded ideal on κ.

Observation

A cardinal κ is weakly compact if and only if *Cut* does not win $\mathcal{G}_2(\kappa, \mathrm{bd}_\kappa, {<}\kappa)$.

The subscript 2 means that *Cut* plays *I*-partitions of size 2 in each of their moves, which is really just equivalent to cutting into 2 pieces, as we did in our earlier games.

Distributivity and Precipitousness

Observation

An ideal I on κ is (γ,ν)-distributive if and only if the Boolean algebra $P(\kappa)/I$ is (γ,ν)-distributive if and only if for any $X \in I^+$, *Cut* does not have a winning strategy in the game $\mathcal{G}_\nu(X, I, \gamma)$.

Theorem (essentially Jech)

An ideal I on κ is precipitous if and only if for any $X \in I^+$, *Cut* does not have a winning strategy in the game $\mathcal{G}_\infty(X, I, \leq\omega)$.

Precipitous games

Let $\mathbb{P}(I, \gamma)$ denote the game of length γ in which players *Empty* and *Nonempty* take turns to play I-positive sets that form a $\subseteq$-decreasing sequence. *Empty* starts, and *Nonempty* goes first at all limit stages. *Nonempty* wins if the intersection of all of their choices is nonempty, and *Empty* wins otherwise.

It is well-known that I is precipitous if and only if *Empty* does not win $\mathbb{P}(I, \omega)$. The following thus generalizes our earlier characterization of precipitousness via cut and choose games.

Theorem (Jech and Velickovic for $\gamma = \omega$)

The games $\mathbb{P}(I, \gamma)$ and $\mathcal{G}_\infty(X, I, {\leq}\gamma)$ are essentially equivalent, that is:

Empty wins $\mathbb{P}(I, \gamma)$ iff $\forall X \in I^+$ *Cut* wins $\mathcal{G}_\infty(X, I, {\leq}\gamma)$, and

Nonempty wins $\mathbb{P}(I, \gamma)$ iff $\forall X \in I^+$ *Choose* wins $\mathcal{G}_\infty(X, I, {\leq}\gamma)$.

Strategic Closure

We can generalize precipitous games and cut and choose games to partial orders. Let Q be a partial order and $q \in Q$. In the precipitous game $\mathbb{P}(Q, \gamma)$ players *Empty* and *Nonempty* take turns playing increasingly stronger conditions in Q, and *Nonempty* wins in case they have a lower bound in Q. In the game $\mathcal{G}_\infty(q, Q, \gamma)$, *Cut* plays maximal antichains of Q, and *Choose* picks one of their elements. *Choose* wins if the set of their choices has a lower bound in Q. We have the same equivalence as before:

Theorem (Jech and Velickovic for $\gamma = \omega$)

The games $\mathbb{P}(Q, \gamma)$ and $\mathcal{G}_\infty(q, Q, \gamma)$ are essentially equivalent, that is:

Empty wins $\mathbb{P}(Q, \gamma)$ iff $\forall q \in Q$ *Cut* wins $\mathcal{G}_\infty(q, Q, \gamma)$, and

Nonempty wins $\mathbb{P}(Q, \gamma)$ iff $\forall q \in Q$ *Choose* wins $\mathcal{G}_\infty(q, Q, \gamma)$.

Note: *Nonempty* wins $\mathbb{P}(Q, \gamma)$ if and only if Q is $<\gamma^+$-strategically closed. By the above, we can thus characterize strategic closure in terms of cut and choose games on partial orders.

One out of four directions of proof

(Velickovic)

If *Choose* wins $\mathcal{G}_\infty(q, Q, \omega)$ for all $q \in Q$, then *Nonempty* wins $\mathbb{P}(Q, \omega)$.

Proof: Suppose that *Empty* starts a run of the game $\mathbb{P}(Q, \omega)$ by playing some $q_0 \in Q$. Let σ be a winning strategy for *Choose* in the game $\mathcal{G}_\infty(q_0, Q, \omega)$. We can identify σ with a function F which on input $\langle W_i \mid i \leq n \rangle$ for some $n < \omega$ considers the partial run in which the moves of *Cut* are given by the W_i, the moves of *Choose* at stages below n are given by the strategy σ, and $F(\langle W_i \mid i \leq n \rangle)$ produces a response $w_n \in W_n$ for *Choose* to this partial run. We describe a winning strategy for *Nonempty* in the game $\mathbb{P}(Q, \omega)$, making use of an auxiliary run of $\mathcal{G}_\infty(q_0, Q, \omega)$ according to σ. Let $Q(\leq q) = \{r \in Q \mid r \leq q\}$.

In order to define the first move of *Nonempty*, consider the set

$$\Sigma_\emptyset = \{F(\langle W\rangle) \mid W \text{ is a maximal antichain of } Q(\leq q_0)\}.$$

There is $r_0 \leq q_0$ such that $Q(\leq r_0) \subseteq \Sigma_\emptyset$, for otherwise the complement of $\Sigma_\emptyset$ is dense below q_0 and hence there is a maximal antichain W of $Q(\leq q_0)$ that is disjoint from $\Sigma_\emptyset$, however $F(\langle W\rangle) \in W \cap \Sigma_\emptyset$, which is a contradiction. Let *Nonempty* pick such r_0 as their first move.
In the next round, suppose that *Empty* plays $q_1 \leq r_0$. Let *Cut* play a maximal antichain W_0 of $Q(\leq q_0)$ such that $F(\langle W_0\rangle) = q_1$ as their first move in the game $\mathcal{G}_\infty(q_0, I, \gamma)$. Consider the set

$$\Sigma_{\langle W_0\rangle} = \{F(\langle W_0, W\rangle) \mid W \text{ is a maximal antichain of } Q(\leq q_0)\}.$$

As before, there is $r_1 \leq q_1$ such that $Q(\leq r_1) \subseteq \Sigma_{\langle W_0\rangle}$, and we let *Nonempty* respond with such r_1.

Proceeding in this way, the choices of *Choose* are exactly the choices of *Empty*, and hence they have a lower bound in Q, for *Choose* was following their winning strategy σ. So *Nonempty* wins $\mathbb{P}(Q, \gamma)$, as desired. □