Small Models, Large Cardinals, and Large Cardinal Ideals (joint work with Philipp Lücke)

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Ramsey cardinals

Victoria Gitman isolated the following from work of William Mitchell from the late 70'ies.

Theorem

κ is a Ramsey cardinal if for every $A \subseteq \kappa$ there is a transitive weak κ-model M with $A \in M$ and with a (uniform) κ-amenable, countably complete and M-normal ultrafilter U on κ.

- A *weak κ-model* M is a model of ZFC^- such that $|M| = \kappa$ and $\kappa + 1 \subseteq M$.
- An M-ultrafilter U is *M-normal* if it closed under diagonal intersections in M.
- U is *countably complete* if any countable intersection (in $\mathbf{V}$) of filter elements is nonempty.
- U is *κ-amenable* if whenever X is a set of size κ in M, then $X \cap U \in M$.

Note: We will require all our filters to be uniform.

Varying the parameters

What happens if we vary the requirements on M and on U? For example:

- Instead of the countable completeness of U, only require the ultrapower of M by U to be well-founded.
- Do not require well-foundedness of the ultrapower.

Or require U to be ...

- *stationary-complete*: Every countable intersection from U (in $\mathbf{V}$) is stationary in κ.
- *genuine*: Every diagonal intersection of U is unbounded in κ.
- *normal*: Every diagonal intersection of U is stationary in κ.

We may also require that $M \prec H(\theta)$ for sufficiently large regular θ instead of transitivity of M in any of the above.

A table of results

U is κ-amenable and...	M is transitive	$M \prec H(\theta)$
$<\kappa$-complete for M	weakly compact	weakly compact
M-normal	$\mathbf{T}^\kappa_\omega$-Ramsey	completely ineffable
... and well-founded	weakly Ramsey	ω-Ramsey
... and countably complete	Ramsey	$\prec$-Ramsey
... and stationary-complete	ineffably Ramsey	Δ-Ramsey
genuine	∞^κ_ω-Ramsey	Δ-Ramsey
normal	Δ^κ_ω-Ramsey	Δ-Ramsey

Completely ineffable cardinals

Definition

$\mathcal{S} \subseteq \mathcal{P}(\kappa)$ is a *stationary class* if $\mathcal{S} \neq \emptyset$ is a collection of stationary subsets of κ.

Definition

A cardinal κ is *completely ineffable* if there is a stationary class $\mathcal{S} \subseteq \mathcal{P}(\kappa)$ such that whenever $A \in \mathcal{S}$ and $f\colon [A]^2 \to 2$, then there is $H \subseteq A$ in $\mathcal{S}$ that is homogeneous for f.

Theorem (Kleinberg, 1970ies)

κ is completely ineffable iff for every sufficiently large regular θ and every / some countable $M \prec H(\theta)$ with $\kappa \in M$, there is an M-normal, κ-amenable M-ultrafilter U on κ.

Another characterization of complete ineffability

Results from below papers essentially show the following theorem (using completely different proofs than the above result about countable models):

- Holy-Schlicht (2018): A hierarchy of Ramsey-like cardinals, characterized through the (non-)existence of winning strategies for certain infinite games, with ω-*Ramsey* cardinals at the bottom.
- Nielsen-Welch (2019): A characterization of complete ineffability as a weakening of ω-Ramseyness.

Theorem

κ is completely ineffable iff for every sufficiently large regular θ and every $x \in H(\theta)$ there is a weak κ-model $M \prec H(\theta)$ with $x \in M$ and with a κ-amenable, M-normal ultrafilter U on κ.

Uniform large cardinal ideals

These large cardinal characterizations also allow for highly uniform definitions of corresponding *large cardinal ideals*. Let φ denote a large cardinal property that is characterized (as are Ramseyness or complete ineffability above) through the existence of certain models M with M-ultrafilters U having a certain property φ^*. We define I_φ and $I_{\prec\varphi}$ as follows:

- $A \in I_\varphi$ if there is $x \subseteq \kappa$ such that for all transitive weak κ-models M with $x \in M$ and every M-ultrafilter U with Property φ^*, we have $A \notin U$.
- $A \in I_{\prec\varphi}$ if for all sufficiently large regular θ there is $x \in H(\theta)$ such that for all weak κ-models $M \prec H(\theta)$ with $x \in M$ and every M-ultrafilter U with Property φ^*, we have $A \notin U$.

Given that $\varphi(\kappa)$ holds, these ideals are easily seen to be proper ideals on κ. If φ^* implies the M-normality of U, then these ideals are normal ideals on κ.

Example: The completely ineffable ideal

In all cases of large cardinals for which corresponding large cardinal ideals have already been defined, these coincide with our definitions: weakly compact, Ramsey, ineffably Ramsey. In some other cases, our ideals correspond to well-known set theoretic objects, and sometimes they are *new*.

Let κ be completely ineffable, and let I denote the completely ineffable ideal on κ. An adaption of the proof of the previous theorem yields the following.

Theorem

I is the complement of the maximal (w.r.t. $\supseteq$) stationary class witnessing the complete ineffability of κ.

Hierarchy results

We can show in most cases that these ideals are strictly $\subseteq$-increasing, in a way which also implies that the related large cardinal notions are strictly increasing in terms of consistency strength. For example: Weakly compact ideal $\subsetneq$ Ineffable Ideal $\subsetneq$ Completely Ineffable ideal $\subsetneq$ weakly Ramsey ideal $\subsetneq$ Ramsey ideal $\subsetneq$ $\prec$-Ramsey ideal $\subsetneq$ *measurable ideal*.

The measurable ideal

The *measurable ideal* I^κ_{ms} on a measurable cardinal κ is the complement of the union of all normal ultrafilters on κ, and also fits into our framework of large cardinal ideals. This ideal is not very interesting in small inner models (for example in $L[U]$). Moreover:

Theorem

If any set of pairwise incomparable conditions in the Mitchell ordering at κ has size at most κ, then the partial order $\mathcal{P}(\kappa)/I^\kappa_{ms}$ is atomic.

However, it is consistently non-trivial – adapting classical arguments from Kunen and Paris yields the following:

Theorem

Every model with a measurable cardinal κ has a forcing extension in which $\mathcal{P}(\kappa)/I^\kappa_{ms}$ is atomless.

Normally Ramsey cardinals

Definition

An uncountable cardinal κ is S-Ramsey / ∞-Ramsey / Δ-Ramsey if for every regular $\theta > \kappa$, every $x \in H(\theta)$ is contained in a weak κ-model $M \prec H(\theta)$ with a κ-amenable, M-normal ultrafilter U on κ that is stationary-complete / genuine / normal.

Generalizing results from Holy and Schlicht shows the following.

Theorem

κ is S-Ramsey / ∞-Ramsey / Δ-Ramsey if for all regular $\theta > \kappa$, Player I does not have a winning strategy in the game of length ω in which Player I plays a $\subset$-increasing sequence of κ-models $M_i \prec H(\theta)$ with union M, and Player II responds with a $\subseteq$-increasing sequence of M_i-ultrafilters U_i with union U. Player I also has to ensure that M_i and U_i are both elements of M_{i+1} for every $i \in \omega$. Player II wins if U is an M-normal filter that is stationary-complete / genuine / normal.

... are equivalent to some seemingly weaker Ramsey-like cardinals

Lemma

S-Ramsey $\equiv$ ∞-Ramsey $\equiv$ Δ-Ramsey.

Proof: Assume that κ is S-Ramsey, that $\theta > \kappa$ is regular, and let $x \in H(\theta)$. Let $M_0 \prec H(\theta)$ with $x \in M_0$ be a weak κ-model. Consider a run of the game for S-Ramseyness, in which Player I starts by playing M_0, and which Player II wins – with resulting model $M = \bigcup_{i<\omega} M_i$ and M-ultrafilter $U = \bigcup_{i<\omega} U_i$. This means that $M \prec H(\theta)$ is a weak κ-model with $x \in M$, and U is κ-amenable, M-normal and stationary-complete. But $\Delta U \supseteq \bigcap_{i<\omega} \Delta U_i$ (modulo a non-stationary set). Since each $\Delta U_i \in U$, it follows that ΔU is stationary, for it is stationary-complete. But this means that U is normal, and hence κ is Δ-Ramsey. □