Ulam Style Cut and Choose Games

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Figure: A piece of cake

Cut and Choose

Assume there's a piece of cake, and we want Ann and Bob to fairly share it. An easy way to do so is to let Ann *cut* the cake in two, then let Bob *choose* his piece, and let Ann have the remaining piece.

This principle is already mentioned in the bible, some hundred years BC. Modern investigation of *fair division* was initiated by Steinhaus, Banach and Knaster in the 1940ies. Initally, they extended fair division to a larger number of people, say now you have Ann, Bob and Chris...

In set theory, we tend to be very interested in *infinite games*. Mycielski and Ulam, in the 1960ies, proposed various *infinite cut and choose games*.

The Ulam game (sometimes called Mycielski game)

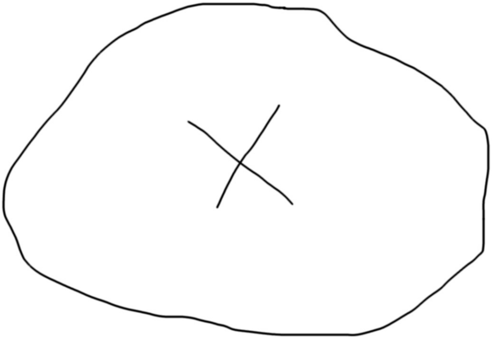
We have two players, going by the names of *Cut* and *Choose*. Instead of a piece of cake, we have a set X over which our game takes place. For a start, you may perhaps think of X being either $\mathbb{N}$ or $\mathbb{R}$.

- In their first move, *Cut* partitions X into two (disjoint) pieces, and *Choose* picks one of them, call it X_0.
- Now *Cut* partitions X_0 into two pieces, and *Choose* picks one of them, call it X_1.
- Now *Cut* partitions X_1 ...

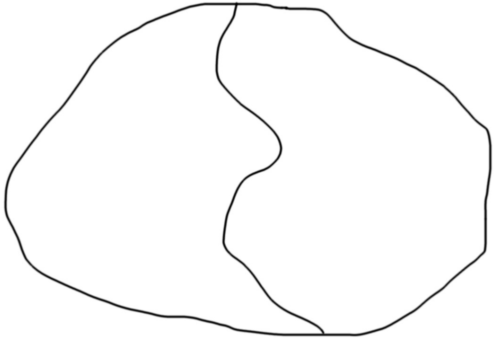
In the end, *Choose* wins a run of the game in case $\bigcap X_i$ has (at least) two distinct elements.

Let us denote the above game as $\mathcal{U}(X)$.

The Ulam Game



The Ulam Game



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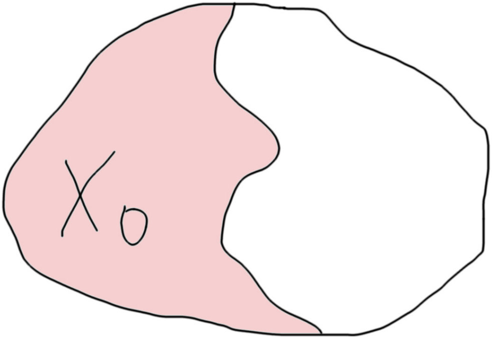

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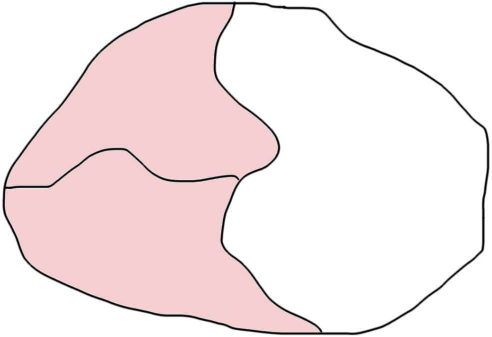

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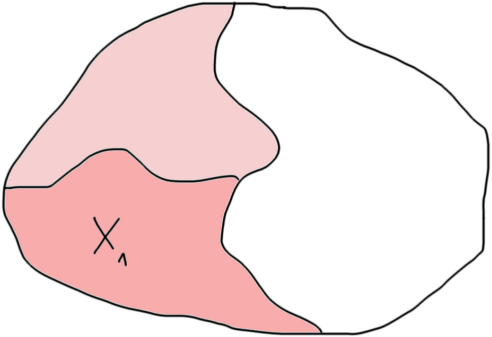
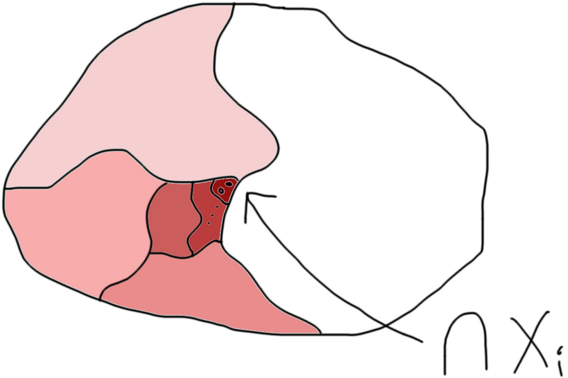


The Ulam Game



Why *two* elements?

Assume we declare *Choose* to win if only $\bigcap X_i$ were nonempty. Let X be any set, and let a be an element of X. Now in a run of $\mathcal{U}(X)$ with this modified winning condition, whatever partition *Cut* presents, let *Choose* simply pick the part X_i that contains a as an element. In the end, clearly $a \in \bigcap X_i$, and thus *Choose* would easily win all the time.

This is what we call a *winning strategy*, in this case for *Choose*.

A *strategy* (for one particular player) is a function which receives as input a play of the game so far, and its output is the next move to be made by that player. A strategy is *winning* if following it guarantees a win, regardless of the other player's moves.

In set theory, we like to say that some player *wins* a certain game to express that they have a winning strategy. We say that they win *a particular run* of a game otherwise.

Cut wins over $\mathbb{N}$

It is very easy to see that *Cut* wins $\mathcal{U}(\mathbb{N})$ – remember that in a run of the game, players cut and choose infinitely often, and *Choose* wins if the intersection of their choices has **two** distinct elements.

Cut wins by *removing* one natural number in each step:

- In their first move, they partition $\mathbb{N}$ into $\{0\}$ and $[1, \infty)$.
- *Choose* will have to pick $[1, \infty)$.
- In their second move, they present the choices $\{1\}$ and $[2, \infty)$.
- *Choose* will have to pick $[2, \infty)$.
- Next options will be $\{2\}$ and $[3, \infty)$, ...

Obviously, the intersection of all choices will be empty, hence *Cut* wins.

Note that this had nothing to do with any properties of the natural numbers other than their countability!

Cut wins over $\mathbb{R}$

It is still fairly easy for *Cut* to win $\mathcal{U}(\mathbb{R})$. For notational simplicity, let's play over $[0,1)$. *Cut* wins by simply cutting in half in each step:

- In their first move, they partition $[0,1)$ into $[0,\frac{1}{2})$ and $[\frac{1}{2},1)$.
- *Choose* picks one of these intervals $[a,b)$.
- Given $[a,b)$, let *Cut* partition it into $[a,\frac{a+b}{2})$ and $[\frac{a+b}{2},b)$.
- *Choose* again picks one of these intervals.

The lengths of these intervals *converge* to 0, so their intersection will contain at most one element. But this means that *Cut* wins.

Note again that this had nothing to do with any particular properties of $[0,1)$ other than its size. If any set stands in bijection with $[0,1)$, like $\mathbb{R}$ for example, then we can still make use of the above argument modulo this bijection. So the only thing that matters here is the *size*, or *cardinality*, of the underlying set. For any set X, let $|X|$ denote its cardinality.

Can *Choose* ever win?

If $|X| \leq |\mathbb{R}|$, that is there is an injection from X into $\mathbb{R}$, and thus into $[0,1)$, then by the above argument, *Cut* has a winning strategy in $\mathcal{U}(X)$.
(This means of course that *Choose* cannot have a winning strategy!)

If *Choose* wants to have a chance of winning, they need to play over a reasonably large set, at least larger than $|\mathbb{R}|$.

The question whether *Choose* can ever win is in fact strongly related to that of the possible existence of very large infinities, namely a certain type of *large cardinal*: the existence of *measurable cardinals* (introduced by Ulam in 1930).

Large Cardinals

Inaccessible cardinals (Hausdorff 1908, Sierpinski-Tarski-Zermelo 1930) can be seen as the *smallest* of large cardinals. They are sets X of uncountable size $|X|$ with the properties that

- If $|Y| < |X|$, then also $|\mathcal{P}(Y)| < |X|$, and
- X cannot be written as the union of less than $|X|$-many sets of size less than X.

Note that $|\mathbb{R}|$ is not inaccessible, nor is $|\mathcal{P}(\mathbb{R})|$, $|\mathcal{P}(\mathcal{P}(\mathbb{R}))|$, ... But also $|\bigcup \mathcal{P}^i(\mathbb{R})|$ is not inaccessible, for it clearly fails to obey the second condition above. In fact, inaccessible cardinals cannot be shown to actually exist. But their existence is often a useful extra assumption. For example, it is equivalent to the existence of *Grothendieck universes*, as used in algebraic geometry.

Measurable Cardinals

Measurable cardinals are uncountable sets κ for which there exists an object (called a *measurable ultrafilter*) deciding for each $A \subseteq \kappa$ whether A or its complement is considered *large* (and the other one *small*), in such a way that the intersection of less than $|\kappa|$-many large sets will still be large.

Choose wins $\mathcal{U}(\kappa)$ in case κ is a measurable cardinal!

Measurable cardinals are fairly large inaccessible cardinals. The smallest inaccessible cardinal is never measurable. In fact, below any measurable cardinal κ, there are $|\kappa|$-many smaller inaccessible cardinals.

Some classical results

Theorem (Silver-Solovay, 1970ies)

If there exists a set X such that Choose has a winning strategy in the game $\mathcal{U}(X)$, then there is a generically measurable cardinal, and hence a universe of mathematics in which there exists a measurable cardinal.

Corollary

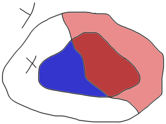
There are universes of mathematics in which *Choose* has no winning strategy for $\mathcal{U}(X)$ over any set X.

Theorem (Laver, 1970ies)

If a measurable cardinal exists, then there is a universe of mathematics in which Choose has a winning strategy in the game $\mathcal{U}(\mathcal{P}(\mathbb{R}))$, that is the Ulam game on the powerset of $\mathbb{R}$.

Leastness is interesting

If *Choose* has a winning strategy in the game $\mathcal{U}(X)$ and $|X| \leq |Y|$, then *Choose* has a winning strategy in $\mathcal{U}(Y)$ as well – simply use the strategy for the game $\mathcal{U}(X)$ modulo an injection from X to Y.



So what is interesting is the *smallest* size of a set X so that *Choose* can win $\mathcal{U}(X)$, if there is any at all.

A combination of known results

If there is a measurable cardinal, then there is a universe of mathematics with a measurable cardinal κ, such that *Choose* does not have a winning strategy in $\mathcal{U}(X)$ whenever $|X| < |\kappa|$.

Remember that *Choose* always has a winning strategy for $\mathcal{U}(\kappa)$ for measurable κ.

Ulam games at small inaccessibles

One of the main results of our paper is the following:

Theorem

If a measurable cardinal exists, then there is a universe of mathematics in which λ is an inaccessible cardinal that is not measurable (in fact, not even weakly compact), however λ is the least cardinal such that Choose has a winning strategy in the game $\mathcal{U}(\lambda)$.

This means that *Choose* can also *first win* the Ulam game over *fairly small* inaccessible cardinals.

Question

Can *Choose* also first win the Ulam game over the *least* inaccessible cardinal?

The weak Ulam game

Let's go back to the definition of the Ulam game. It seems somewhat annoying having to require two elements in the final intersection in order for *Choose* to win. This is taken care of by the *weak Ulam game* $\bar{\mathcal{U}}(X)$. It proceeds just like the Ulam game, however with two modifications:

- In each of their moves, *Choose* has to pick a set with at least two elements, however
- for *Choose* to win, the intersection of all choices only needs to be nonempty.

Observe that simply fixing some $a \in X$ in advance and always picking the part that contains a as an element is not a valid strategy for *Choose* anymore in the game $\bar{\mathcal{U}}(X)$, for it cannot be applied when presented a partition of the form $\langle \{a\}, Y \rangle$. So it doesn't seem that *Choose* can trivially win $\bar{\mathcal{U}}(X)$.

Cut does not win $\bar{\mathcal{U}}(\mathbb{R})$

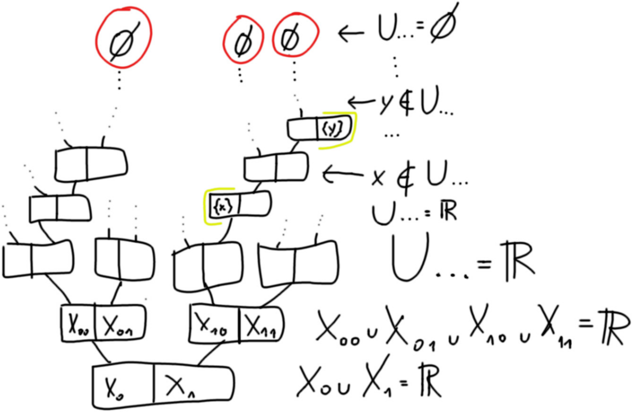
Remember that Cut had an easy strategy to win $\mathcal{U}([0,1))$, and thus also $\mathcal{U}(\mathbb{R})$ by always cutting given intervals in half. However:

Proposition

Cut does *not* have a winning strategy in $\bar{\mathcal{U}}(\mathbb{R})$.

Proof: By the picture on the next slide...

Cut does not win $\bar{\mathcal{U}}(\mathbb{R})$



However things aren't completely different...

Proposition

Choose does not have a winning strategy in the game $\bar{\mathcal{U}}(\mathbb{R})$.

Thus, *neither* player has a winning strategy in the game $\bar{\mathcal{U}}(\mathbb{R})$. We call such games *undetermined*.

In analogy to the classical theorems on $\mathcal{U}(X)$, we can show:

Observation

If there exists a set X such that *Choose* has a winning strategy in the game $\bar{\mathcal{U}}(X)$, then there is a *generically measurable cardinal*, and hence a universe of mathematics in which there exists a measurable cardinal.

Choose does not have a winning strategy in $\bar{\mathcal{U}}(\mathbb{R})$

Proof-Sketch: Assume that *Choose* has a winning strategy σ in the game $\bar{\mathcal{U}}(X)$ and $X \subseteq \mathbb{R}$ with $|X| = |\mathbb{R}|$.

Key technical lemma

σ is *dynamic*, that is, whenever *Choose* picks some set Y in a play of $\bar{\mathcal{U}}(\mathbb{R})$ according to σ, then there are two distinct (finite) continuations of that play (with *Cut* playing differently) in which *Choose* plays according to σ, such that in the respective last move, *Choose* picks a set Y_0 in one run, and a set Y_1 in another, so that Y_0 and Y_1 are disjoint.

$\Rightarrow$ X contains a continuous image of the Cantor set.

- Pick $X \subseteq \mathbb{R}$ of size $|\mathbb{R}|$ which doesn't contain a continuous image of the Cantor set (exists by an easy diagonalization argument).
- This means that *Choose* cannot have a winning strategy for $\bar{\mathcal{U}}(X)$.
- But having a winning strategy or not only depends on the cardinality of the starting set.
- Therefore, *Choose* does not have a winning strategy for $\bar{\mathcal{U}}(\mathbb{R})$. $\square$