On the distribution of distances between specified nodes in increasing trees☆

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Abstract

We study the quantity *distance between node j and node n in a random tree of size n* chosen from a family of increasing trees. For those subclass of increasing tree families, which can be constructed via a tree evolution process, we give closed formulæ for the probability distribution, the expectation and the variance. Furthermore we derive a distributional decomposition of the random variable considered and we show a central limit theorem of this quantity, for arbitrary labels $1 \leq j < n$ and $n \to \infty$.

Such tree models are of particular interest in applications, e.g., the widely used models of *recursive trees*, *plane-oriented recursive trees* and *binary increasing trees* are special instances and are thus covered by our results.

Key words: Increasing trees, Node distances, Limiting distribution
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1. Introduction

Various recent studies are devoted to a distributional analysis of distances between random nodes in a lot of tree families of interest. We mention here Mahmoud and Neininger [14] for binary search trees, Christophi and Mahmoud [5] for the digital data structure called Tries, and Panholzer [19] for simply generated trees (= conditioned Galton Watson trees). Considerably fewer studies are made to reveal the distribution of distances between *specified* nodes in *labelled* tree structures. Exceptions are the work of Dobrow [8] and Dobrow and Smythe [9], who have shown a central limit theorem for the distance between the nodes labelled by j and n (= the largest node), respectively, in a random recursive tree of size n for all sequences $(n, j(n))_{n\in\mathbb{N}}$, with $1 \leq j = j(n) < n$, and the work of Devroye and Neininger [7], who have shown a central limit theorem for the distance between the nodes labelled

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by j_1 and j_2 in a random binary search tree of size n, for all sequences $(n, j_1(n), j_2(n))_{n\in\mathbb{N}}$ with $1 \le j_1 = j_1(n) < j_2 = j_2(n) \le n$, provided that $j_2 - j_1 \to \infty$.

In the present paper we extend the results of [8, 9] from recursive trees to a larger class of tree families: we give a distributional analysis (by showing a central limit theorem) of the random variable (r.v. for short) $\Delta_{n,j}$, which counts the distance, measured by the number of edges lying on the connecting path, between node j and node n in a random tree of size n, for various members of so called *increasing tree families*.

Several important tree models, e.g., *recursive trees*, *plane-oriented recursive trees* (also known as non-uniform recursive trees or heap ordered trees) and *binary increasing trees* (and more generally *d-ary increasing trees*) are members of the family of increasing trees. These tree models turned out to be appropriate in order to describe the behaviour of a lot of quantities in various applications (see [16] for a survey). They are used, e.g., to describe the spread of epidemics, for pyramid schemes, and as a simplified growth model of the world wide web: plane-oriented recursive trees are a special instance of the so called Barabási-Albert model for scale-free networks (see, e.g., [2, 4]). Since for all tree families considered the distribution of the distance between nodes j_1 and j_2 will be independent from the size n ($\ge \max(j_1, j_2)$) of the tree, we get also as a corollary a central limit theorem for the random variable $\Delta_{n;j_1,j_2}$, which counts the distance between the nodes j_1 and j_2 in a random tree of size n.

Increasing trees can be considered as labelled trees, where the nodes of a tree of size n are labelled by distinct integers of the set $\{1, \ldots, n\}$ in such a way that each sequence of labels along any path starting at the root is increasing. To give examples: plane-oriented recursive trees are increasingly labelled ordered trees (= planted plane trees) and d-ary increasing trees are obtained from (unlabelled) d-ary trees via increasing labellings. It seems that this point of view appears first in [21]. A fundamental study of those increasing tree families that are generated from simply generated tree families (see [17]) by equipping the trees with increasing labellings was given in [3]. Such *simple families of increasing trees* are also the combinatorial objects, which are studied in the present paper. When analyzing parameters in trees chosen from a family of increasing trees we will throughout this paper always assume the so called *random tree model* as the model of randomness. Since increasing trees can be considered as weighted trees (see the exact definitions given in Section 2), we will always assume that, for the increasing tree family studied, every increasing tree of size n is chosen with a probability proportional to its weight. We might thus speak about *random increasing trees*.

But instead of using this combinatorial description it is probably more common to describe certain members of increasing tree families via a tree evolution process, i.e., for every tree T' of size n with vertices $v_1, \ldots, v_n$ one is giving probabilities $p_{T'}(v_1), \ldots, p_{T'}(v_n)$, such that when starting with a *random tree* T' of size n of the tree family considered, choosing a vertex v_i in T' according to the probabilities $p_{T'}(v_i)$ and attaching node $n+1$ to it, we obtain again a *random tree* T of size $n+1$ of the tree family considered. For the tree families mentioned above (i.e., recursive trees, plane-oriented recursive trees and d-ary increasing trees) the "insertion probabilities" $p_{T'}(v_i)$ are quite simple to describe, since they depend only on the size $|T'|$ of the tree T' and on the out-degree $d(v_i)$ of node v_i and are thus independent from the actual choice of the tree T'.

Most of the parameters previously studied are analyzed by using this description via the tree evolution process, e.g., by using Pólya urn models (see, e.g., [15]) or by translating results from continuous time branching processes (see, e.g., [10]). However, by means of these methods there are obtained quite few results that give insight into the behaviour of the node j (= the j-th individual) during the growth process in a tree of size n, in particular if the label $j = j(n)$ is growing with n. For showing the central limit theorem of $\Delta_{n,j}$ we will use the combinatorial description of increasing tree families, using an approach that turned out to be suitable for a distributional analysis of several label-dependent parameters (see [11, 20]).

Before continuing, we have to say a few words on the description of increasing tree families via tree evolution processes: it has been shown in [20] that it is not possible to describe every simple family of increasing trees by such a tree evolution process, even more, it has been given there a full characterization of such increasing tree families via the so called degree-weight generating function. Throughout this paper we will choose the term *evolving* simple families of increasing trees for the subclass of increasing tree families, which can be generated by a tree evolution process (in [11] the term *grown* simple families of increasing trees has been used, but evolving seems to be more appropriate). For these evolving simple families of increasing trees holds then that the distribution of the distance between the nodes with labels j_1 and j_2 is independent from the size n of the tree. Our recursive approach in combination with a treatment by suitable generating functions will in principle work for all simple families of increasing trees, even if there does not exist an evolution process, leading to a closed formula for the introduced generating functions of the probabilities $\mathbb{P}\{\Delta_{n,j} = m\}$ given as Proposition 1. But in the succeeding computations we will restrict ourselves to the evolving tree families, since then label-dependent parameters have a direct meaning in the tree evolution process; if there does not exist such a process the behaviour of node j does not seem to be of great importance. For all evolving simple families of increasing trees we obtain then closed formulæ for the probabilities $\mathbb{P}\{\Delta_{n,j} = m\}$, the expectation $\mathbb{E}(\Delta_{n,j})$ and the variance $\mathbb{V}(\Delta_{n,j})$. These explicit results are given in Theorem 1 and Theorem 2.

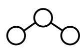
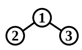
The advantage of having these explicit results for evolving simple families of increasing trees, in particular of a closed formula for the probability generating function (and thus also for the moment generating function) of $\Delta_{n,j}$, is, that they yield relatively easy the central limit theorem of $\Delta_{n,j}$, for arbitrary sequences $(n, j(n))_{n\in\mathbb{N}}$, with $1 \le j < n$, which is given as Theorem 4. As already mentioned above, for evolving simple families of increasing trees we can indeed obtain a central limit theorem for the random variable $\Delta_{n;j_1,j_2}$, for arbitrary sequences $(n, j_1(n), j_2(n))_{n\in\mathbb{N}}$, with $1 \le j_1, j_2 < n$, provided that $\max(j_1, j_2) \to \infty$. This result is given as Corollary 2. Of course, the exact and asymptotic formulæ presented here extend earlier results (in particular results for the instance of recursive trees and results for randomly selected nodes) for the distance between nodes in increasing trees. Moreover, the closed formula for the probability generating function enables us to obtain decomposition results of $\Delta_{n,j}$, stated in Theorem 3, which are improving the earlier results of [8, 9].

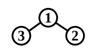
Note that our findings for $\Delta_{n,j}$ imply results concerning the *depth* (the depth of node v, also called the *height* of node v, is defined as the distance of v from the root) of nodes in families of increasing trees considered here, which can be obtained from the special case $j = 1$, $\Delta_{n,1}$. See Panholzer and Prodinger [20] and the references therein for results concerning depths in increasing trees.

Throughout this paper we use the abbreviations $x^{\underline{l}} := x(x-1)\cdots(x-l+1)$ and $x^{\overline{l}} := x(x+1)\cdots(x+l-1)$ for the falling and rising factorials, respectively. Moreover, we use E_x for the evaluation operator at $x = 1$, i.e., $E_x f(x) = f(1)$, and we denote by $\left[{n \atop k}\right]$ the signless Stirling numbers of the first kind. Furthermore $X \stackrel{(d)}{=} Y$ denotes the equality in distribution of the random variables X and Y, whereas $X_n \xrightarrow{(d)} X$ denotes the weak convergence, i.e., the convergence in distribution, of the sequence of random variables X_n to a random variable X. For independent random variables X and Y we write $X \oplus Y$ for the sum of X and Y. For not necessarily mutually independent random variables X and Y we write $X + Y$. We denote with $H_n := \sum_{k=1}^{n} \frac{1}{k}$ the n-th harmonic number and with $H_n^{(a)} := \sum_{k=1}^{n} \frac{1}{k^a}$ the n-th harmonic number of order a. For non-integer arguments we define the analytic continuations $H_z := \sum_{k\ge1} \left(\frac{1}{k} - \frac{1}{z+k}\right) = \gamma + \Psi(z+1)$, where $\Psi(z) = \Gamma'(z)/\Gamma(z)$ denotes the Psi function (= Digamma function) and $H_z^{(a)} := \sum_{k\ge1} \left(\frac{1}{k^a} - \frac{1}{(z+k)^a}\right) = \zeta(a) - \zeta(z, a)$, where $\zeta(z, a)$ denotes the Hurwitz Zeta function (= Generalized Zeta function). Furthermore $\delta_{m,n}$ denotes the Kronecker delta function.

2. Preliminaries

2.1. Combinatorial description of increasing tree families

Here we give a general formal definition of a class $\mathcal{T}$ of a simple family of increasing trees. Note that our basic objects are *ordered* trees, i.e., rooted trees, where, at each vertex, there is an order on the children. To give an example, this implies that the tree  has two different increasing labellings (with distinct labels $\{1, 2, 3\}$), namely 

and 

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A sequence of non-negative numbers $(\varphi_k)_{k\ge0}$ with $\varphi_0 > 0$ (we further assume that there exists a $k \ge 2$ with $\varphi_k > 0$) is used to define the weight $w(T)$ of any ordered tree T by $w(T) := \prod_v \varphi_{d(v)}$, where v ranges over all vertices of T and $d(v)$ is the out-degree of v. Furthermore, $\mathcal{L}(T)$ denotes the set of different increasing labellings of the tree T with distinct integers $\{1, 2, \ldots, |T|\}$, where $|T|$ denotes the size of the tree T, and $L(T) := |\mathcal{L}(T)|$ its cardinality. Then the family $\mathcal{T}$ consists of all trees T together with their weights $w(T)$ and the set of increasing labellings $\mathcal{L}(T)$.

For a given degree-weight sequence $(\varphi_k)_{k\ge0}$ with a degree-weight generating function $\varphi(t) := \sum_{k\ge0} \varphi_k t^k$, we define now the total weights by $T_n := \sum_{|T|=n} w(T) \cdot L(T)$. We might say that T_n

is given by the sum of the weights $w(T)$ over all increasingly labelled ordered trees. It follows then that the exponential generating function $T(z) := \sum_{n\ge1} T_n \frac{z^n}{n!}$ satisfies the *autonomous* first order differential equation

$$T'(z) = \varphi\big(T(z)\big), \quad T(0) = 0. \tag{1}$$

We remark that $T(z)$ can always be considered as a formal power series. However, when using analytic properties of $T(z)$ suitable growth conditions on the degree-weight sequence $(\varphi_k)_{k\ge0}$ have to be assumed in order to ensure that $T(z)$ converges in a neighbourhood of the origin.

Often it is advantageous to describe a simple family of increasing trees $\mathcal{T}$ by the formal recursive equation

$$\mathcal{T} = \textcircled{1} \times \Big(\varphi_0 \cdot \{\epsilon\} \mathbin{\dot\cup} \varphi_1 \cdot \mathcal{T} \mathbin{\dot\cup} \varphi_2 \cdot \mathcal{T} * \mathcal{T} \mathbin{\dot\cup} \varphi_3 \cdot \mathcal{T} * \mathcal{T} * \mathcal{T} \mathbin{\dot\cup} \cdots\Big) = \textcircled{1} \times \varphi(\mathcal{T}), \tag{2}$$

where $\textcircled{1}$ denotes the node labelled by 1, $\times$ the cartesian product, $\dot\cup$ the disjoint union, $*$ the partition product for labelled objects, and $\varphi(\mathcal{T})$ the substituted structure (see, e.g., [22]). We also use the notation $\mathcal{T}_n$ for all members of $\mathcal{T}$ of size n.

By specializing the degree-weight generating function $\varphi(t)$ in (1) we get the basic enumerative results for the three most interesting increasing tree families:

• *Plane-oriented recursive trees (PORTs)* are the family of plane increasing trees such that all node degrees are allowed. In other words every ordered increasing tree has equal weight one, so we set $\varphi_k = 1$, for $k \ge 0$, and obtain thus the degree-weight generating function $\varphi(t) = \frac{1}{1-t}$. Equation (1) leads here to

$$T(z) = 1 - \sqrt{1-2z}, \quad \text{and} \quad T_n = \tfrac{(n-1)!}{2^{n-1}}\tbinom{2n-2}{n-1} = 1\cdot3\cdot5\cdots(2n-3) = (2n-3)!!, \ \text{ for } n \ge 1.$$

• *Recursive trees* are the family of non-plane increasing trees such that all node degrees are allowed. Thus every unordered increasing tree should have equal weight one, so in order to compensate for the number of orderings we set $\varphi_k = 1/k!$, for $k \ge 0$, and obtain therefore the degree-weight generating function $\varphi(t) = \exp(t)$. Solving (1) gives

$$T(z) = \log\Big(\frac{1}{1-z}\Big), \quad \text{and} \quad T_n = (n-1)!, \ \text{ for } n \ge 1.$$

• *Binary increasing trees* have the degree-weight generating function $\varphi(t) = (1+t)^2$. Thus it follows

$$T(z) = \frac{z}{1-z}, \quad \text{and} \quad T_n = n!, \ \text{ for } n \ge 1.$$

2.2. Characterization of evolving simple families of increasing trees

We will describe now the characterization of evolving simple increasing tree families via the degree-weight generating function $\varphi(t)$ as obtained in [20]; see also [12]. Before doing that we remark that two degree-weight sequences $(\tilde\varphi_k)_{k\ge0}$ and $(\varphi_k)_{k\ge0}$, where it holds $\tilde\varphi_k = ab^k\varphi_k$ for all $k \ge 0$, with

$a, b > 0$ (or equivalently $\tilde{\varphi}(t) = a\varphi(bt)$ for the corresponding degree-weight generating functions), lead for both families to the same distribution of a random increasing tree. Therefore it fully suffices to consider degree-weight sequences after a suitable normalization. That the above statement is indeed true can be seen easily when considering the weights $\tilde{w}(T) = \prod_v \tilde{\varphi}_{d(v)}$ and $w(T) = \prod_v \varphi_{d(v)}$ of a tree T with respect to $(\tilde{\varphi}_k)_k$ and $(\varphi_k)_k$:

$$\tilde{w}(T) = \prod_{v \in T} \tilde{\varphi}_{d(v)} = \prod_{v \in T} a b^{d(v)} \varphi_{d(v)} = a^{|T|} b^{\sum_{v \in T} d(v)} \prod_{v \in T} \varphi_{d(v)} = a^{|T|} b^{|T|-1} w(T).$$

Thus when changing $(\varphi_k)_k$ to $(\tilde{\varphi}_k)_k$ the weight of any tree T of size n will be multiplied by the same factor $a^n b^{n-1}$, which will affect the weight of all trees of size n also by the same factor.

Lemma 1 ([20]). *A simple family of increasing trees $\mathcal{T}$ can be constructed via a tree evolution process and is thus an evolving simple family of increasing trees iff there exist positive constants $a, b > 0$, such that the degree-weight generating function $\tilde{\varphi}(t) = \sum_{k \geq 0} \tilde{\varphi}_k t^k$ satisfies $\tilde{\varphi}(t) = a\varphi(bt)$, where $\varphi(t)$ is given by one of the following three formulæ:*

$$\textbf{Case A (recursive trees)}: \quad \varphi(t) = e^t,$$

$$\textbf{Case B (d-ary trees)}: \quad \varphi(t) = (1+t)^d, \quad \text{for } d \in \{2, 3, 4, \dots\},$$

$$\textbf{Case C (generalized plane-oriented recursive trees)}: \quad \varphi(t) = \frac{1}{(1-t)^\alpha}, \quad \text{for } \alpha > 0.$$

Solving the differential equation (1) one obtains the following explicit formulæ for the exponential generating function $T(z)$:

$$T(z) = \begin{cases} \log\left(\frac{1}{1-z}\right), & \text{Case A}, \\ \frac{1}{\left(1-(d-1)z\right)^{\frac{1}{d-1}}} - 1, & \text{Case B}, \\ 1 - (1-(\alpha+1)z)^{\frac{1}{\alpha+1}}, & \text{Case C}. \end{cases} \tag{3}$$

Furthermore the coefficients T_n are given by the following formula, where we have to set $c = 0$ and $\kappa = 1$ for Case A, $c = \frac{1}{d-1}$ and $\kappa = d - 1$ for Case B, and $c = -\frac{1}{\alpha+1}$ and $\kappa = \alpha + 1$ for Case C:

$$T_n = \kappa^{n-1} (n-1)! \binom{n-1+c}{n-1}. \tag{4}$$

Finally we are going to describe in more detail the tree evolution process which generates random trees (of arbitrary size n) of evolving simple families of increasing trees. This description is a consequence of the considerations made in [20]:

- Step 1: The process starts with the root labelled by 1.
- Step $i+1$: At step $i+1$ the node with label $i+1$ is attached to any previous node v (with out-degree $d(v)$) of the already grown tree of size i with probabilities $x(v)$ proportional to

$$\frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}},$$

i.e.,

$$x(v) = \begin{cases} \frac{1}{i}, & \text{for Case A}, \\ \frac{d-d(v)}{(d-1)i+1}, & \text{for Case B}, \\ \frac{d(v)+\alpha}{(\alpha+1)i-1}, & \text{for Case C}. \end{cases} \tag{5}$$

3. Results for evolving simple families of increasing trees

3.1. Exact formulæ

Here we give the exact formulæ for the distribution, the expectation and the variance of the random variable $\Delta_{n,j}$.

In the following formula for the probabilities $\mathbb{P}\{\Delta_{n,j} = m\}$ we have to distinguish between the instance of plane-oriented recursive trees (Case C, with $\alpha = 1$) and the other instances of evolving simple families of increasing trees. We give here expressions for the exact probabilities to demonstrate that these quantities can be described explicitly by means of classical combinatorial sequences. However, we want to remark that the exact formulæ for the probability generating function $p_{n,j}(v) = \sum_{m\ge 0} \mathbb{P}\{\Delta_{n,j} = m\}v^m$ as given in equation (27) and (30), which also characterize the exact distribution of $\Delta_{n,j}$, are of more interest for considerations concerning the asymptotic behaviour.

Theorem 1. *The probabilities $\mathbb{P}\{\Delta_{n,j} = m\}$, which give the probability that the distance between the node with label j and the node with label n in a randomly chosen size-n tree of an evolving simple family of increasing trees as given by Lemma 1, is m, are, for $m \ge 1$ and $1 \le j < n$ given by the following formula.*

- *Case C for the instance $\alpha = 1$ (Plane-oriented recursive trees): it holds*

$$\mathbb{P}\{\Delta_{n,j} = m\} = \frac{2^{2n-3}}{(n-1)\binom{2n-2}{n-1}\binom{n-2}{j-1}} \sum_{k=0}^{m-1} \frac{1}{2^{m-1-k}} \Big(\sum_{l=0}^{j-1} \left[l \atop k \right] \frac{1}{l!} \binom{j-l-\frac{3}{2}}{j-l-1} \Big) \times \\ \times \Big(\sum_{l=0}^{n-j-1} \left[l \atop m-1-k \right] \frac{1}{l!} \binom{n-2-l}{j-1} \Big).$$

- *Case A, Case B, and Case C for instances $\alpha \neq 1$: setting $c = 0$ for Case A, $c = \frac{1}{d-1}$ for Case B, and $c = -\frac{1}{\alpha+1}$ for Case C then it holds*

$$\mathbb{P}\{\Delta_{n,j} = m\} = \frac{\binom{j-1+c}{j-1}(1+c)}{(n-1)\binom{n-2}{j-1}\binom{n-1-c}{n-1}} \Bigg(\sum_{l=0}^{n-j-1} \binom{n-l-2}{j-1} (1+c)^{m-1} \frac{1}{l!} \left[l \atop m-1 \right] \\ + \frac{1}{1+2c} \sum_{k=0}^{n-j-1} \binom{n-k-2}{j-1} \sum_{l=0}^{m-2} \frac{2^l(1+c)^l}{(1+2c)^l} (1+c)^{m-2-l} \frac{1}{k!} \left[k \atop m-2-l \right] \Bigg) \\ - \frac{1}{(n-1)\binom{n-2}{j-1}\binom{n-1-c}{n-1}} \sum_{l=0}^{m-2} \frac{2^{m-2-l}(1+c)^{m-1-l}}{(1+2c)^{m-1-l}} \sum_{k=0}^{l} \Bigg(\Big(\sum_{i=0}^{j-1} \binom{j-2-i-c}{j-1-i} 2^k (1+c)^k \frac{1}{k!} \left[i \atop k \right] \Big) \times$$

$$\times \Big(\sum_{i=0}^{n-j-1} \binom{n-2-i}{j-1} (1+c)^{l-k} \frac{1}{(l-k)!} \left[\begin{matrix} i \\ l-k \end{matrix} \right] \Big) \Big).$$

Theorem 2. *The expectation and the variance of the random variable $\Delta_{n,j}$, which counts the distance between the node with label j and the node with label n in a randomly chosen tree of size n, are for all evolving simple families of increasing trees as given by Lemma 1 (and $1 \le j < n$) given as follows, where we have to set $c = 0$ for Case A, $c = \frac{1}{d-1}$ for Case B, and $c = -\frac{1}{\alpha+1}$ for Case C. The $\mathcal{O}$-term in the asymptotic expansion holds uniformly for $1 \le j \le n$.*

$$\begin{aligned}
\mathbb{E}(\Delta_{n,j}) &= (1+c)\Big(H_{n+c-1} + H_{j+c} - 2H_{c+1} + \frac{1+2c}{j+c} \Big) - 2c \\
&= (1+c)\big(\log n + \log j\big) + \mathcal{O}(1), \\
\mathbb{V}(\Delta_{n,j}) &= (1+c)H_{n+c-1} + \Big((1+c) - 4\frac{(1+2c)(1+c)^2}{j+c} \Big) H_{j+c} \\
&\quad - 2\Big((1+c) - 2\frac{(1+2c)(1+c)^2}{j+c} \Big) H_{c+1} - (1+c)^2 \Big(H^{(2)}_{n+c-1} + 3H^{(2)}_{j+c} - 4H^{(2)}_{c+1} \Big) \\
&\quad + 2(1+c)(1+2c) - \frac{(1+2c)(1+c)}{j+c} - \frac{(1+2c)^2(1+c)^2}{(j+c)^2} \\
&= (1+c)\big(\log n + \log j\big) + \mathcal{O}(1).
\end{aligned}$$

For the reader's convenience we explicitly give the formulæ for the three most prominent members of evolving simple tree families. The result for recursive trees already appears in [18].

Corollary 1. *The expectation and the variance of the random variable $\Delta_{n,j}$ (for $1 \le j < n$) are for plane-oriented recursive trees (Case C, with $\alpha = 1$) given by*

$$\begin{aligned}
\mathbb{E}(\Delta_{n,j}) &= H_{2n-2} - \frac{1}{2}H_{n-1} + H_{2j} - \frac{1}{2}H_j - 1, \\
\mathbb{V}(\Delta_{n,j}) &= H_{2n-2} - \frac{1}{2}H_{n-1} + H_{2j} - \frac{1}{2}H_j - H^{(2)}_{2n-2} + \frac{1}{4}H^{(2)}_{n-1} - 3H^{(2)}_{2j} + \frac{3}{4}H^{(2)}_j + 2.
\end{aligned}$$

The expectation and the variance of the random variable $\Delta_{n,j}$ (for $1 \le j < n$) are for recursive trees (Case A) given by

$$\begin{aligned}
\mathbb{E}(\Delta_{n,j}) &= H_{n-1} + H_j + \frac{1}{j} - 2, \\
\mathbb{V}(\Delta_{n,j}) &= H_{n-1} + H_j - H^{(2)}_{n-1} - 3H^{(2)}_j - \frac{4}{j}H_j + 4 + \frac{3}{j} - \frac{1}{j^2}.
\end{aligned}$$

The expectation and the variance of the random variable $\Delta_{n,j}$ (for $1 \le j < n$) are for binary increasing trees (Case B, with $d = 2$) given by

$$\begin{aligned}
\mathbb{E}(\Delta_{n,j}) &= 2H_n + 2H_{j+1} + \frac{6}{j+1} - 8, \\
\mathbb{V}(\Delta_{n,j}) &= 2H_n + 2H_{j+1} - 4H^{(2)}_n - 12H^{(2)}_{j+1} - \frac{48}{j+1}H_{j+1} + 26 + \frac{66}{j+1} - \frac{36}{(j+1)^2}.
\end{aligned}$$

3.2. Distributional results

Dobrow and Smythe [9] already provided the distribution law of the distance $\Delta_{n,j}$ in terms of $\Delta_{j+1,j}$ and a sum of independent Bernoulli variables. Moreover, Dobrow [8] obtained a rather complicated decomposition of $\Delta_{j+1,j}$ for recursive trees . We can improve the distributional results of [9] and [8] by completely identifying the Bernoulli variables in the decomposition of $\Delta_{n,j}$ and by decomposing $\Delta_{j+1,j}$ in a simple manner. Recall that we use $X \oplus Y$ for the sum of independent random variables.

Theorem 3. *The random variable $\Delta_{n,j}$, which counts the distance between node j and node n in a randomly chosen tree of size n, admits for all evolving simple families of increasing trees as given by Lemma 1 the following distribution law:*

$$\Delta_{n,j} \stackrel{(d)}{=} \Delta_{j+1,j} \oplus \bigoplus_{k=j+1}^{n-1} \mathbb{1}(A_k), \tag{6}$$

where A_k denotes the event that node k is lying on the path from node n to node j. It holds that $\mathbb{1}(A_k) \stackrel{(d)}{=} \mathrm{Be}(p_k)$, with $p_k = \frac{1+c}{k+c}$, for $j+1 \le k \le n-1$, where we have to set $c = 0$ for Case A, $c = \frac{1}{d-1}$ for Case B, and $c = -\frac{1}{\alpha+1}$ for Case C.

The distribution law of the distance $\Delta_{j+1,j}$ between node $j+1$ and node j is characterized as follows.

- *For plane oriented recursive trees (Case C, with $\alpha = 1$) the following simple decomposition of $\Delta_{j+1,j}$ holds:*

$$\Delta_{j+1,j} = \bigoplus_{k=1}^{j} \mathbb{1}(\tilde{A}_k), \tag{7}$$

  where $\tilde{A}_k$ denotes the event that node k is lying on the unique path from node j to node $j+1$. It further holds that $\mathbb{1}(\tilde{A}_j) \stackrel{(d)}{=} 1$ and $\mathbb{1}(\tilde{A}_k) \stackrel{(d)}{=} \mathrm{Be}(\frac{2}{2k+1})$, for $1 \le k \le j-1$.

- *For Case A and for Case C, with $\alpha \neq 1$, the following distributional decomposition of $\Delta_{j+1,j}$ holds:*

$$\Delta_{j+1,j} \stackrel{(d)}{=} \bigoplus_{k=1}^{\eta_j} \tilde{B}_k, \tag{8}$$

  where $\tilde{B}_k \stackrel{(d)}{=} \mathrm{Be}(\tilde{p}_k)$, $\tilde{p}_0 = \tilde{p}_1 = 1$ and $\tilde{p}_k = \frac{2(1+c)}{k-1+c}$ for $3 \le k \le j$. Furthermore it holds $\mathbb{P}\{\eta_j = 1\} = \frac{1+c}{j+c}$ and $\mathbb{P}\{\eta_j = m\} = \frac{1}{j+c}$, for $2 \le m \le j$, where we always have to set $c = 0$ for Case A, and $c = -\frac{1}{\alpha+1}$ for Case C.

- *For Case B we do not obtain a simple distributional decomposition of $\Delta_{j+1,j}$. We are only able to give a simplified form of the probability generating function $p_{j+1,j}(v) = \sum_{m\ge0} \mathbb{P}\{\Delta_{j+1,j} = m\} v^m$ of $\Delta_{j+1,j}$, where we have to set $c = \frac{1}{d-1}$:*

$$p_{j+1,j}(v) = \frac{v(1+c)+v^2}{j+c} + \frac{v^2}{j+c}\Big(\frac{-c+2v(1+c)}{2+c}\Big) \times \sum_{k=2}^{j-2}\prod_{l=2}^{k}\Big(\frac{l-(1+c)+2v(1+c)}{l+1+c}\Big).$$

Remark 1. It should be pointed out that the distributional decompositions in Theorem 3 can be used for an alternative approach towards the Gaussian limit law of $\Delta_{n,j}$, stated in the theorem below. However, we refrain from using a Poisson approximation approach since it is very computational (compare with Dobrow and Smythe [9], containing the derivation for recursive trees), and more important, it fails to show the limit law in the whole range for d-ary increasing trees, where no suitable distributional decomposition is available. Hence, we rely on the continuity theorem of Lévy (as done in Dobrow [8]) in order to prove the limit law in a unified manner.

Remark 2. One can analyse weighted generalizations of depths and distances, as considered for binary search trees by Aguech, Lasmar and Mahmoud in [1]. Theorem 3 can be used to derive limiting distributions, whereas more combinatorial approaches seem to fail.

3.3. Limiting distribution results

In the following we give the main theorem of the paper, i.e., the central limit theorems for the r.v. $\Delta_{n,j}$ and $\Delta_{n;j_1,j_2}$, respectively. In the expressions appearing we always have to set $c = 0$ for Case A, $c = \frac{1}{d-1}$ for Case B, and $c = -\frac{1}{\alpha+1}$ for Case C.

Theorem 4. *Let $\Delta_{n,j}$ count the distance between the nodes with label j and label n in a randomly chosen size-n tree of an evolving simple family of increasing trees as given by Lemma 1. Then it holds that the random variable*

$$\Delta^*_{n,j} := \frac{\Delta_{n,j} - \mu_{n,j}}{\sigma_{n,j}},$$

with $\mu_{n,j} := \big(1+c\big)(\log n + \log j)$ and $\sigma^2_{n,j} := \big(1+c\big)(\log n + \log j)$, is, for arbitrary sequences $(n, j(n))_{n\in\mathbb{N}}$, with $1 \le j = j(n) < n$, asymptotically for $n \to \infty$ Gaussian distributed:

$$\Delta^*_{n,j} = \frac{\Delta_{n,j} - \mu_{n,j}}{\sigma_{n,j}} \xrightarrow{(d)} \mathcal{N}(0,1).$$

Corollary 2. *Let $\Delta_{n;j_1,j_2}$ count the distance between the nodes with label j_1 and label j_2 in a randomly chosen size-n tree of an evolving simple family of increasing trees as given by Lemma 1. Then it holds that the random variable*

$$\Delta^*_{n;j_1,j_2} := \frac{\Delta_{n,;j_1,j_2} - \mu_{n;j_1,j_2}}{\sigma_{n;j_1,j_2}},$$

with $\mu_{n;j_1,j_2} := \big(1+c\big)(\log j_1 + \log j_2)$ and $\sigma^2_{n;j_1,j_2} := \big(1+c\big)(\log j_1 + \log j_2)$, is, for arbitrary sequences $(n, j_1(n), j_2(n))_{n\in\mathbb{N}}$, with $1 \le j_1 = j_1(n), j_2 = j_2(n) \le n$ and $j_1 \ne j_2$, provided that $\max(j_1, j_2) \to \infty$, asymptotically for $n \to \infty$ Gaussian distributed:

$$\Delta^*_{n;j_1,j_2} = \frac{\Delta_{n,;j_1,j_2} - \mu_{n;j_1,j_2}}{\sigma_{n;j_1,j_2}} \xrightarrow{(d)} \mathcal{N}(0,1).$$

We have formulated here our theorems for random variables obtained after centering and normalizing by the *asymptotic* mean and standard deviation of $\Delta_{n,j}$ and $\Delta_{n;j_1,j_2}$, respectively. However, we want to remark that, as a direct consequence of the uniform asymptotic expansion given in Theorem 2, analogous results for the r.v. obtained after centering and normalizing by the *exact* mean and standard deviation also hold. Furthermore, one could give an alternative formulation of Corollary 2 saying that if the variance of $\Delta_{n,;j_1,j_2}$ tends to infinity then the centered and normalized distance is asymptotically standard normally distributed.

This paper is organized as follows. In Section 4 we treat a recurrence for the probabilities $\mathbb{P}\{\Delta_{n,j} = m\}$ by means of suitable generating functions. This leads for all simple families of increasing trees, i.e., not only for those characterized in Lemma 1, to a closed formula for the generating function under consideration, which will be given in Proposition 1. In Section 5 and Section 6 we prove the explicit results for evolving simple families of increasing trees that are given by Theorem 1, and the corresponding limiting distribution result of Theorem 4 is shown in Section 8. The results given in Theorem 3 concerning the decomposition of $\Delta_{n,j}$ are proven in Section 7.

4. A recurrence for the probabilities

By using the combinatorial description as given in Subsection 2.1 we will obtain a recursive description of $\Delta_{n,j}$ and thus of the probabilities $\mathbb{P}\{\Delta_{n,j} = m\}$ for simple families of increasing trees. For these considerations we also have to introduce the r.v. $D_{n,j}$, which counts the depth (= the number of edges lying on the path connecting the root, i.e., the node with label 1, with the node considered) of node j in a random size-n tree of a simple family of increasing trees. One may thus also define $D_{n,j} := \Delta_{n;j,1}$.

For increasing trees of size n with root-degree r and subtrees with sizes $k_1, \dots, k_r$, enumerated from left to right, we will distinguish between two cases that cover by symmetry arguments all possible cases. For the first case we assume that node j and node n are both lying in the leftmost subtree of the root, where the node labelled by j is the i-th smallest node in this subtree. We can then reduce the computation of the probabilities $\mathbb{P}\{\Delta_{n,j} = m\}$ to the probabilities $\mathbb{P}\{\Delta_{k_1,i} = m\}$. For the second case we assume that node j is lying in the leftmost subtree and is the i-th smallest node in this subtree, whereas node n is lying in the second subtree (from left to right). We can thus reduce the computation of the probabilities $\mathbb{P}\{\Delta_{n,j} = m\}$ to the probabilities of the depths $\mathbb{P}\{D_{k_1,i} = t\}$ and $\mathbb{P}\{D_{k_2,k_2} = m - 2 - t\}$.

In the first case we get as factor the total weight of the r subtrees and the root node $\varphi_r T_{k_1} \cdots T_{k_r}$, divided by the total weight T_n of trees of size n and multiplied by the number of order preserving relabellings of the r subtrees, which are given here by

$$\binom{j-2}{i-1}\binom{n-1-j}{k_1-1-i}\binom{n-1-k_1}{k_2, k_3, \dots, k_r}:$$

the $i-1$ labels smaller than j are chosen from $2, 3, \dots, j-1$, the $k_1 - 1 - i$ labels larger than j but different from n are chosen from $j+1, \dots, n-1$, and the remaining $n - 1 - k_1$ labels are distributed to the second, third, …, r-th subtree. Due to symmetry arguments we obtain a factor r, if the node j is the i-th smallest node in the second, third, …, r-th subtree.

Analogously, in the second case we get the factor $\varphi_r T_{k_1} \cdots T_{k_r}$ divided by the total weight T_n of trees of size n and multiplied by the number of order preserving relabellings of the r subtrees, which are given here by

$$\binom{j-2}{i-1}\binom{n-1-j}{k_1-i}\binom{n-2-k_1}{k_2-1, k_3, \dots, k_r}:$$

the $i-1$ labels smaller than j are chosen from $2, 3, \dots, j-1$, the $k_1 - i$ labels larger than j are chosen from $j+1, \dots, n-1$ (since node n must be in the second subtree), and the remaining $n-2-k_1$ labels are distributed to the second, third, ..., r-th subtree. Again due to symmetry arguments we obtain a factor $r(r-1)$.

Summing up over all choices for the rank i of label j in its subtree, the subtree sizes $k_1, \dots, k_r$, and the degree r of the root node gives the following recurrence (9).

$$\begin{aligned}\mathbb{P}\{\Delta_{n,j} = m\} = &\sum_{r\ge1} r\varphi_r \sum_{\substack{k_1+\dots+k_r=n-1,\\ k_1,\dots,k_r\ge1}} \frac{T_{k_1}\cdots T_{k_r}}{T_n} \times \\ &\times \sum_{i=1}^{\min\{k_1,j-1\}} \mathbb{P}\{\Delta_{k_1,i} = m\} \binom{j-2}{i-1}\binom{n-1-j}{k_1-1-i}\binom{n-1-k_1}{k_2, k_3, \dots, k_r} \\ &\quad + \sum_{r\ge1} r(r-1)\varphi_r \sum_{\substack{k_1+\dots+k_r=n-1,\\ k_1,\dots,k_r\ge1}} \frac{T_{k_1}\cdots T_{k_r}}{T_n} \times \\ \times \sum_{i=1}^{\min\{k_1,j-1\}} \sum_{t=0}^{m-2} &\mathbb{P}\{D_{k_1,i} = t\}\mathbb{P}\{D_{k_2,k_2} = m-2-t\} \binom{j-2}{i-1}\binom{n-1-j}{k_1-i}\binom{n-2-k_1}{k_2-1, k_3, \dots, k_r}, \end{aligned} \tag{9}$$

for $2 \le j \le n-1$, with $\mathbb{P}\{\Delta_{n,1} = m\} = \mathbb{P}\{D_{n,n} = m\}$ and $\mathbb{P}\{\Delta_{n,n} = m\} = \delta_{m,0}$.

To treat this recurrence (9) we set $n := k + j$ with $k \ge 0$ and define the trivariate generating functions

$$\begin{aligned} M(z,u,v) &:= \sum_{k\ge1}\sum_{j\ge1}\sum_{m\ge0} \mathbb{P}\{\Delta_{k+j,j} = m\} T_{k+j} \frac{z^{j-1}}{(j-1)!}\frac{u^{k-1}}{(k-1)!} v^m, \\ N(z,u,v) &:= \sum_{k\ge0}\sum_{j\ge1}\sum_{m\ge0} \mathbb{P}\{D_{k+j,j} = m\} T_{k+j} \frac{z^{j-1}}{(j-1)!}\frac{u^k}{k!} v^m. \end{aligned} \tag{10}$$

Multiplying (9) with $T_{k+j}\frac{z^{j-2}}{(j-2)!}\frac{u^{k-1}}{(k-1)!}v^m$ and summing up over $k \ge 1$, $j \ge 2$ and $m \ge 0$ gives then $\frac{\partial}{\partial z}M(z,u,v)$ for the left hand side and $\varphi'\big(T(z+u)\big)M(z,u,v)$ plus $v^2 N(z,u,v)N(z+u,0,v)\varphi''\big(T(z+u)\big)$ for the right hand side of (9). Since these are essentially straightforward, but quite lengthy computations, they are omitted here; similar considerations are done in [20] for a study of the r.v. $D_{n,j}$, where the (somewhat simpler) recurrences appearing there are treated analogously. In any case we obtain the following differential equation:

$$\frac{\partial}{\partial z}M(z,u,v) = \varphi'\big(T(z+u)\big)M(z,u,v) + v^2 N(z,u,v)N(z+u,0,v)\varphi''\big(T(z+u)\big), \tag{11}$$

together with the initial condition

$$M(0,u,v) = \sum_{k\ge 1}\sum_{m\ge 0} \mathbb{P}\{\Delta_{k+1,1} = m\} T_{k+1} \frac{u^{k-1}}{(k-1)!} v^m = \sum_{k\ge 1}\sum_{m\ge 0} \mathbb{P}\{D_{k+1,k+1} = m\} T_{k+1} \frac{u^{k-1}}{(k-1)!} v^m$$
$$= \frac{\partial}{\partial u} N(u,0,v). \tag{12}$$

Remark 3. The differential equation (11) can also be obtained by a combinatorial reasoning involving the counting of 4-colored increasing trees. Since the arguments used are very similar to [20] we just sketch the derivation and refer the interested reader to [20]. The combinatorial objects considered are all possible 4-colored increasing trees of size ≥ 2 with a coloring as specified next. In each increasing tree T the node with largest label is colored green. From the remaining nodes exactly one node is colored red, all nodes with a smaller label than the red node are colored black, and all remaining nodes (with a label larger than the red node) are colored white.

We are interested in the distance between the red node and the green node. Let us consider such a 4-colored increasing tree T, where we assume that the red node of T is not the root node. Then after a decomposition of T into the root and the subtrees of T there may occur the following two situations: (i) the red node and the green node are lying in the same subtree, or (ii) the red node and the green node are lying in different subtrees. We consider now these subtrees after an order-preserving relabelling with integers from 1 up to the size of the respective subtree. In (i) it holds then that exactly one subtree is again a suitably 4-colored increasing tree, whereas the remaining subtrees are only bicolored increasing trees colored black and white in a way that a white node never has a black child. In (ii) it holds that one subtree is a tricolored increasing tree colored black, red and white in a specific manner, one subtree is a tricolored increasing tree colored black, white and green in a specific manner, and the remaining subtrees are bicolored increasing trees colored black and white in a way that a white node never has a black child. By using exponential generating functions as defined in (10) (note that z counts the black nodes, u the white nodes, and v either the distance between the red and the green node or the depth of the red node, respectively) this decomposition according to the root node can be translated into equation (11). Note that case (i) gives $\varphi'\big(T(z+u)\big)M(z,u,v)$, whereas case ($ii$) leads to $v^2 N(z,u,v)N(z+u,0,v)\varphi''\big(T(z+u)\big)$; the left-hand side of (11) is given by $\frac{\partial}{\partial z}M(z,u,v)$, since in T the root node colored black has the fixed label 1. We further remark that considering 4-colored increasing trees T, where the root node of T is colored red, gives the initial condition $M(0,u,v) = \frac{\partial}{\partial u}N(u,0,v)$.

The random variable $D_{n,j}$ was already analyzed in [20], where the following result was obtained:

$$N(z,u,v) = \varphi\big(T(u)\big)\Big(\frac{\varphi\big(T(z+u)\big)}{\varphi\big(T(u)\big)}\Big)^v = T'(u)\Big(\frac{T'(z+u)}{T'(u)}\Big)^v.$$

Consequently we get

$$N(z,0,v) = \varphi_0\Big(\frac{\varphi\big(T(z)\big)}{\varphi_0}\Big)^v = \varphi_0\Big(\frac{T'(z)}{\varphi_0}\Big)^v,$$

and further

$$M(0,u,v) = \frac{\partial}{\partial u}N(u,0,v) = \frac{\partial}{\partial u}\Big(\varphi_0\big(\frac{T'(u)}{\varphi_0}\big)^v\Big) = \varphi_0 v\Big(\frac{T'(u)}{\varphi_0}\Big)^{v-1}\frac{T''(u)}{\varphi_0} = vT''(u)\Big(\frac{T'(u)}{\varphi_0}\Big)^{v-1}.$$

Thus the differential equation (11) can be rewritten into

$$\frac{\partial}{\partial z}M(z,u,v) = \varphi'\big(T(z+u)\big)M(z,u,v) + \frac{v^2\varphi''\big(T(z+u)\big)\big(T'(z+u)\big)^{2v}}{\big(T'(u)\big)^{v-1}\varphi_0^{v-1}},$$

with initial condition $M(0,u,v) = vT''(u)\Big(\frac{T'(u)}{\varphi_0}\Big)^{v-1}$. The corresponding homogeneous differential equation has the solution

$$M^{[h]}(z,u,v) = C(u,v)\exp\Big(\int_0^z \varphi'\big(T(t+u)\big)dt\Big) = C(u,v)\frac{\varphi\big(T(z+u)\big)}{\varphi\big(T(u)\big)} = C(u,v)\frac{T'(z+u)}{T'(u)},$$

with some function $C(u,v)$. Variation of the constants method leads to the particular solution

$$M^{[p]}(z,u,v) = \frac{v^2 T'(z+u)}{\varphi_0^{v-1}\big(T'(u)\big)^{v-1}} \int_0^z \varphi''\big(T(t+u)\big)\big(T'(t+u)\big)^{2v-1}dt.$$

Adapting to the initial condition leads to $C(u,v) = M(0,u,v) = vT''(u)\Big(\frac{T'(u)}{\varphi_0}\Big)^{v-1}$, and therefore to the following proposition.

Proposition 1. *The function $M(z,u,v)$ as defined in equation (10), which is the trivariate generating function of the probabilities $\mathbb{P}\{\Delta_{n,j} = m\}$, which give the probability that the distance (measured by the number of edges on the connecting path) between the node with label j and the node with label n in a randomly chosen size-n tree of a simple family of increasing trees with degree-weight generating function $\varphi(t)$, is m, is given by the following formula:*

$$M(z,u,v) = vT''(u)\Big(\frac{T'(u)}{\varphi_0}\Big)^{v-1}\frac{T'(z+u)}{T'(u)} + \frac{v^2T'(z+u)}{\varphi_0^{v-1}\big(T'(u)\big)^{v-1}}\int_0^z \varphi''\big(T(t+u)\big)\big(T'(t+u)\big)^{2v-1}dt.$$

This immediately has the following consequence.

Corollary 3. *The trivariate generating function $M(z,v,u)$ is for all evolving simple families of increasing trees as given by Lemma 1 given by the following formula:*

$$M(z,u,v) = \frac{\kappa(1+c)v\big(1-\frac{v}{(1+c)(2v-1)-c}\big)}{(1-\kappa u)^{(c+1)(v-1)+1}(1-\kappa(z+u))^{c+1}} + \frac{\kappa(1+c)v^2(1-\kappa u)^{(c+1)(v-1)}}{\big((1+c)(2v-1)-c\big)(1-\kappa(z+u))^{(c+1)(2v-1)+1}},$$

where we have to set $c=0$ and $\kappa=1$ for Case A, $c=\frac{1}{d-1}$ and $\kappa = d-1$ for Case B, and $c = -\frac{1}{\alpha+1}$ and $\kappa = \alpha+1$ for Case C.

For the special instance of plane-oriented recursive trees (Case C, with $\alpha=1$, i.e., $c=-1/2$ and $\kappa=1$) it holds $\frac{v}{(1+c)(2v-1)-c} = \frac{v}{\frac{1}{2}(2v-1)+\frac{1}{2}} = 1$, which leads to simplifications for the expression given in Corollary 3.

Corollary 4. *For plane-oriented recursive trees (Case C, with $\alpha=1$, as defined by Lemma 1) the generating function $M(z,u,v)$ simplifies to*

$$M(z,u,v) = \frac{v(1-2u)^{\frac{v-1}{2}}}{(1-2(z+u))^{v+\frac{1}{2}}}. \qquad (13)$$

5. Closed formulæ for the probabilities

For extracting coefficients from the trivariate generating function $M(z,u,v)$ as given by Corollary 3 it is convenient to split $M(z,u,v)$ into two parts $M(z,u,v) = M_1(z,u,v) + M_2(z,u,v)$, where the first part, i.e., $M_1(z,u,v)$, disappears for plane-oriented recursive trees (whose generating function is given by Corollary 4). Furthermore we use the well known relation for the Stirling numbers of the first kind

$$\sum_{n\ge 0}\sum_{m=0}^{m} \begin{bmatrix} n \\ m \end{bmatrix} \frac{z^n}{n!} v^m = \frac{1}{(1-z)^v}. \tag{14}$$

We only present the calculations for the instance of plane-oriented recursive trees trees and start with the generating function $M(z,u,v)$ given in Corollary 4. The general case can be treated by the same method; the calculations become a bit lengthier, but are only slightly more complicated. We extract coefficients according to (10):

$$\begin{aligned}
\mathbb{P}\{\Delta_{n,j} = m\} &= \frac{(j-1)!(n-j-1)!}{T_n}[z^{j-1}u^{n-j-1}v^m]M(z,u,v)\\
&= \frac{(j-1)!(n-j-1)!}{T_n}[z^{j-1}u^{n-j-1}v^m]\frac{v(1-2u)^{\frac{v-1}{2}}}{(1-2(z+u))^{v+\frac{1}{2}}}\\
&= \frac{(j-1)!(n-j-1)!2^{n-2}}{T_n}[z^{j-1}u^{n-j-1}v^{m-1}]\frac{(1-u)^{\frac{v-1}{2}}}{(1-z-u)^{v+\frac{1}{2}}}\\
&= \frac{(j-1)!(n-j-1)!}{2(n-1)!\binom{n-\frac{3}{2}}{n-1}}[z^{j-1}u^{n-j-1}v^{m-1}]\frac{1}{(1-u)^{\frac{v}{2}+1}(1-\frac{z}{1-u})^{v+\frac{1}{2}}},
\end{aligned}$$

where we have used $[z^n]f(qz) = q^n[z^n]f(z)$ and (4). We get further

$$\begin{aligned}
\mathbb{P}\{\Delta_{n,j} = m\} &= \frac{2^{2n-3}(j-1)!(n-j-1)!}{(n-1)!\binom{2n-2}{n-1}}[u^{n-j-1}v^{m-1}]\frac{\binom{v+j-\frac{3}{2}}{j-1}}{(1-u)^{\frac{v}{2}+j}}\\
&= \frac{2^{2n-3}}{(n-1)\binom{n-2}{j-1}\binom{2n-2}{n-1}}[v^{m-1}]\binom{v+j-\frac{3}{2}}{j-1}\binom{\frac{v}{2}+n-2}{n-j-1}.
\end{aligned} \tag{15}$$

The remaining part of the proof follows by using (14) and

$$\sum_{l\ge 0}\binom{v+K+l-1}{l}z^l = \frac{1}{(1-z)^{v+K}}, \quad [v^{m-1}]\binom{v+j-\frac{3}{2}}{j-1} = [z^{j-1}v^{m-1}]\frac{1}{(1-z)^{v+\frac{1}{2}}}.$$

6. Closed formulæ for expectation and variance

To avoid lengthy computations we again restrict our presentation to the instance of plane-oriented recursive trees (Case C, with $\alpha = 1$) and start with the generating function $M(z,u,v)$ given Corollary 4. We basically use (recall that E_v denotes the evaluation operator at $v = 1$)

$$\begin{aligned}
\mathbb{E}(\Delta_{n,j}) &= \frac{(j-1)!(n-j-1)!}{T_n}[z^{j-1}u^{n-j-1}]E_v\frac{\partial}{\partial v}M(z,u,v),\\
\mathbb{E}(\Delta_{n,j}^2) &= \frac{(j-1)!(n-j-1)!}{T_n}[z^{j-1}u^{n-j-1}]E_v\frac{\partial^2}{\partial v^2}M(z,u,v).
\end{aligned}$$

For the calculation of the expectation we begin with

$$E_v \frac{\partial}{\partial v} M(z,u,v) = \frac{1}{(1-c_1(z+u))^{\frac{3}{2}}}\Big(1 - \frac{1}{2}\log\big(\frac{1}{1-2u}\big) + \log\big(\frac{1}{1-2(z+u)}\big)\Big).$$

Now we use the relations

$$\frac{1}{(1-2(z+u))^{\beta}} = \frac{1}{(1-2u)^{\beta}(1-\frac{2z}{1-2u})^{\beta}}, \log^{\beta}\big(\frac{1}{1-2(z+u)}\big) = \Big(\log\big(\frac{1}{1-2u}\big) + \log\big(\frac{1}{1-\frac{2z}{1-2u}}\big)\Big)^{\beta} \tag{16}$$

and

$$[z^n]\frac{\log\big(\frac{1}{1-z}\big)}{(1-z)^{\beta+1}} = \binom{n+\beta}{n}(H_{n+\beta} - H_{\beta}), \tag{17}$$

to obtain

$$[z^{j-1}u^{n-j-1}]E_v \frac{\partial}{\partial v} M(z,u,v) = \frac{2^{n-1}}{2}\binom{n-\frac{3}{2}}{n-j-1}\binom{j-\frac{1}{2}}{j-1}\big(\frac{1}{2}H_{n-\frac{3}{2}} + \frac{1}{2}H_{j-\frac{1}{2}} - H_{\frac{1}{2}} + 1\big).$$

Together with

$$\frac{(j-1)!(n-j-1)!}{T_n} = \frac{(j-1)!(n-j-1)!}{2^{n-1}(n-1)!\binom{n-\frac{3}{2}}{n-1}} = \frac{2}{2^{n-1}\binom{n-\frac{3}{2}}{n-j-1}\binom{j-\frac{1}{2}}{j-1}},$$

and by converting into "integer" harmonic numbers we get the desired result for $\mathbb{E}(\Delta_{n,j})$.

For the second factorial moment we get

$$\begin{aligned} E_v \frac{\partial^2}{\partial v^2} M(z,u,v) = \frac{1}{4(1-2(z+u))^{\frac{3}{2}}}\Big(&-4\log\big(\frac{1}{1-2u}\big) + 8\log\big(\frac{1}{1-2(z+u)}\big) + \log^2\big(\frac{1}{1-2u}\big) \\ &- 4\log\big(\frac{1}{1-2u}\big)\log\big(\frac{1}{1-2(z+u)}\big) + 4\log^2\big(\frac{1}{1-2(z+u)}\big)\Big). \end{aligned}$$

For extracting coefficients we use again (16) and (17) together with

$$[z^n]\frac{\log^2\big(\frac{1}{1-z}\big)}{(1-z)^{\beta+1}} = \binom{n+\beta}{n}\big((H_{n+\beta} - H_{\beta})^2 - (H^{(2)}_{n+\beta} - H^{(2)}_{\beta})\big), \tag{18}$$

and obtain

$$\begin{aligned} \mathbb{E}(\Delta^2_{n,j}) = &\frac{1}{4}(H_{n-\frac{3}{2}} - H_{j-\frac{1}{2}})^2 + (H_{n-\frac{3}{2}} - H_{\frac{1}{2}})(H_{j-\frac{1}{2}} - H_{\frac{1}{2}} + 1) + 2(H_{j-\frac{1}{2}} - H_{\frac{1}{2}}) + (H_{j-\frac{1}{2}} - H_{\frac{1}{2}})^2 \\ &- \frac{1}{4}(H^{(2)}_{n-\frac{3}{2}} - H^{(2)}_{j-\frac{1}{2}}) - (H^{(2)}_{j-\frac{1}{2}} - H^{(2)}_{\frac{1}{2}}). \end{aligned} \tag{19}$$

The variance follows then via

$$\mathbb{V}(\Delta_{n,j}) = \mathbb{E}(\Delta^2_{n,j}) + \mathbb{E}(\Delta_{n,j}) - \big(\mathbb{E}(\Delta_{n,j})\big)^2.$$

For the general case of the variance the usage of a computer algebra system becomes handy for carrying out the very lengthy but routine calculations, which are omitted here.

To get the uniform asymptotic expansions of $\mathbb{E}(\Delta_{n,j})$ and $\mathbb{V}(\Delta_{n,j})$ stated in Theorem 2 we only have to apply the well-known asymptotic expansions of H_z and $H_z^{(2)}$ (for $z \to \infty$):

$$H_z = \log z + \gamma + \mathcal{O}\big(z^{-1}\big), \quad H_z^{(2)} = \frac{\pi^2}{6} + \mathcal{O}\big(z^{-1}\big).$$

We omit here the straightforward computations.

7. Providing the distributional decompositions

In this section we will give a proof of Theorem 3 and start to show the first part of it, i.e., equation (6). We will assume here that $n > j + 1$, otherwise nothing has to be shown.

Let $\{n <_c k\}$ denote the event that node n is a child of (i.e., it is attached to) node k in an increasing tree. In the following we use the simple fact that for any increasing tree of size n it holds $\{n <_c n-1\} = A_{n-1}$, with A_k as defined in Theorem 3, i.e., A_k denotes the event that node k is lying on the path from node n to node j. First this implies that for a random evolving increasing tree of size n we have

$$\mathbb{P}\{A_{n-1}\} = \mathbb{P}\{n <_c n-1\} = \frac{1+c}{n-1+c},$$

with $c = 0$ for Case A, $c = \frac{1}{d-1}$ for Case B, and $c = -\frac{1}{\alpha+1}$ for Case C, since $\{n <_c n-1\}$ occurs exactly if n is attached to $n-1$ during the tree evolution process and the probability is therefore given by (5). Thus $\mathbb{1}(A_{n-1}) \stackrel{(d)}{=} \text{Be}\big(\frac{1+c}{n-1+c}\big)$.

Now we describe a simple construction, which gives, for any increasing tree T of size n an increasing tree T' of size $n-1$:

Construction 1:

- (i) if T satisfies $\{n <_c n-1\}$, then T' is obtained from T by removing node n,
- (ii) if T satisfies $\{n \not<_c n-1\}$, then T' is obtained from T by first exchanging the labels $n-1$ and n, and afterwards removing node n.

Of course, it holds that for case (ii) the distance between node j and node n in T is equal to the distance between node j and node $n-1$ in T', whereas for case (i) the distance between j and n in T is the distance between j and $n-1$ in T' plus 1. We introduce now the random variable $\Delta'_{n,j}$, which counts the distance between node j and node $n-1$ in a tree obtained by selecting a random increasing tree of size n and applying Construction 1. Of course, it holds

$$\Delta_{n,j} = \Delta'_{n-1,j} + \mathbb{1}(A_{n-1}).$$

In order to simultaneously identify the distribution of $\Delta'_{n-1,j}$ and to show independence between $\Delta'_{n-1,j}$ and $\mathbb{1}(A_{n-1})$ we will consider the conditional r.v. $\Delta'_{n-1,j}|A_{n-1}$ and $\Delta'_{n-1,j}|A^c_{n-1}$, which count the distance between j and $n-1$ in a tree obtained after applying Construction 1 to a tree chosen at random from all increasing trees of size n satisfying A_{n-1} or not, respectively. Since we will show that

$$\Delta'_{n-1,j}|A_{n-1} \stackrel{(d)}{=} \Delta_{n-1,j}, \quad \text{and} \quad \Delta'_{n-1,j}|A^c_{n-1} \stackrel{(d)}{=} \Delta_{n-1,j}, \tag{20}$$

this will imply

$$\Delta_{n,j} = \Delta'_{n-1,j} \oplus \mathbb{1}(A_{n-1}), \quad \text{with} \quad \Delta'_{n-1,j} \stackrel{(d)}{=} \Delta_{n-1,j}. \tag{21}$$

We proceed now in proving (20), where we simply have to show that the distribution of the increasing trees obtained after applying Construction 1 to a tree chosen at random from all increasing trees of size n satisfying A_{n-1} or not, respectively, again follows the random tree model. Let us denote by $p'(T'|A_{n-1})$ and $p'(T'|A_{n-1}^c)$ the probabilities that a certain increasing tree T' of size $n-1$ is obtained after applying Construction 1 conditioned on the events A_{n-1} and A_{n-1}^c, respectively. Furthermore, for any increasing tree T, we define by $p(T)$ the probability that T is chosen when selecting a tree from $\mathcal{T}_{|T|}$ at random according to the random tree model (recall that $\mathcal{T}_n$ denotes all members of an increasing tree family $\mathcal{T}$ of size n). We also use $p(T|A_{n-1})$ and $p(T|A_{n-1}^c)$ for the probabilities that T is chosen under the random tree model conditioned on A_{n-1} and A_{n-1}^c, respectively.

We consider now an arbitrary tree $T' \in \mathcal{T}_{n-1}$ and compute the conditional probabilities $p'(T'|A_{n-1})$ and $p'(T'|A_{n-1}^c)$. We do this by considering all trees $T \in \mathcal{T}_n$ satisfying A_{n-1} or A_{n-1}^c, respectively, which give T' after applying Construction 1 (this corresponds to case (i) or case (ii), respectively).

- For case (i) it is obvious that T' can only be obtained by applying Construction 1 to the tree T, which itself is obtained from T' by attaching node n to $n-1$. This immediately gives:

$$p'(T'|A_{n-1}) = p(T|A_{n-1}) = \frac{p(T)}{\mathbb{P}\{A_{n-1}\}} = \frac{p(T')\mathbb{P}\{A_{n-1}\}}{\mathbb{P}\{A_{n-1}\}} = p(T').$$

- For case (ii) we consider now an arbitrary tree T, which gives T' after applying Construction 1. For the tree $\tilde{T}$, obtained from T by exchanging the labels n and $n-1$, it holds that $p(\tilde{T}) = p(T)$; furthermore $\tilde{T}$ can be obtained from T' by attaching node n to a node $\neq n-1$. This gives:

$$\begin{aligned} p'(T'|A_{n-1}^c) &= \sum_{\substack{T \in \mathcal{T}_n \text{ satisfying } A_{n-1}^c, \\ T \text{ gives } T' \text{ by Construction 1}}} p(T|A_{n-1}^c) = \frac{1}{\mathbb{P}\{A_{n-1}^c\}} \sum_{\substack{T \in \mathcal{T}_n \text{ satisfying } A_{n-1}^c, \\ T \text{ gives } T' \text{ by Construction 1}}} p(T) \\ &= \frac{1}{\mathbb{P}\{A_{n-1}^c\}} \sum_{\substack{\tilde{T} \in \mathcal{T}_n \colon \tilde{T} \text{ obtained from } T' \\ \text{by attaching } n \text{ to a node } \neq n-1}} p(\tilde{T}) = \frac{p(T')\mathbb{P}\{A_{n-1}^c\}}{\mathbb{P}\{A_{n-1}^c\}} = p(T'). \end{aligned}$$

Hence (20) and thus (21) is shown. The first part of Theorem 3 follows now by iterating the arguments presented (note that, e.g., when considering the tree T' obtained from T by Construction 1 it holds that the event $\{n-1 <_c n-2\}$ is equal to A_{n-2} in the original tree T): we obtain after $n-j-1$ iterations the decomposition

$$\Delta_{n,j} = \Delta'_{n;j+1,j} \oplus \mathbb{1}(A_{j+1}) \oplus \mathbb{1}(A_{j+2}) \oplus \cdots \oplus \mathbb{1}(A_{n-1}),$$

where $\Delta'_{n;j+1,j}$ counts the distance between the nodes j and $j+1$ in a tree obtained by selecting a random increasing tree of size n and applying Construction 1 successively $(n-j-1)$-times (to the original tree of size n, to the resulting tree of size $n-1$, ..., to the resulting tree of size $j+2$). It further holds $\Delta'_{n;j+1,j} \stackrel{(d)}{=} \Delta_{j+1,j}$ and $\mathbb{1}(A_{n-1}) \stackrel{(d)}{=} \text{Be}\left(\frac{1+c}{n-1+c}\right)$, which shows the distributional decomposition (6).

Next we are going to show the second part of Theorem 3 concerning the distribution law of the random variable $\Delta_{j+1,j}$.

First we will consider the instance of plane-oriented recursive trees (Case C, with $\alpha = 1$). We will assume here that $j > 1$, since due to $\Delta_{2,1} = 1$ nothing has to be shown for $j = 1$.

We consider now the event $\tilde{A}_{j-1}$, with $\tilde{A}_k$ as defined in Theorem 3, i.e., $\tilde{A}_k$ denotes the event that node k is lying on the path from node $j+1$ to node j. To consider the probability $\mathbb{P}\{\tilde{A}_{j-1}\}$ for a random plane-oriented recursive tree of size n we will condition on the event $\{\Delta_{j,j-1} = 1\}$. We use that $\{\Delta_{j,j-1} = 1\} = \{j <_c j-1\}$; furthermore if $\{j <_c j-1\}$ then $\tilde{A}_{j-1}$ holds exactly if $\{j+1 \not<_c j\}$, whereas if $\{j \not<_c j-1\}$ then $\tilde{A}_{j-1}$ holds only if $\{j+1 <_c j-1\}$. Since these probabilities can be computed easily by considering the growth rule of plane-oriented recursive trees we get:

$$\begin{aligned}
\mathbb{P}\{\tilde{A}_{j-1}\} &= \mathbb{P}\{\tilde{A}_{j-1}|\Delta_{j,j-1} = 1\}\mathbb{P}\{\Delta_{j,j-1} = 1\} + \mathbb{P}\{\tilde{A}_{j-1}|\Delta_{j,j-1} > 1\}\mathbb{P}\{\Delta_{j,j-1} > 1\} \\
&= \frac{\mathbb{P}\{\tilde{A}_{j-1}|\Delta_{j,j-1} = 1\}}{2j-3} + \frac{\mathbb{P}\{\tilde{A}_{j-1}|\Delta_{j,j-1} > 1\}(2j-4)}{2j-3} \\
&= \frac{2j-2}{(2j-3)(2j-1)} + \frac{2j-4}{(2j-3)(2j-1)} = \frac{2}{2j-1}.
\end{aligned} \tag{22}$$

Thus $\mathbb{1}(\tilde{A}_{j-1}) \stackrel{(d)}{=} \mathrm{Be}\left(\frac{2}{2j-1}\right)$.

Next we describe a simple construction, which gives, for any increasing tree T of size $j+1$ an increasing tree T'' of size j (see Figure 1):

Construction 2:

(*i*) if T satisfies $\{j+1 <_c j-1\}$ and $\{j <_c j-1\}$, then T'' is obtained from T by removing node $j+1$,

(*ii*) if T satisfies $\{j+1 <_c j-1\}$ and $\{j \not<_c j-1\}$, then T'' is obtained from T by first exchanging the labels $j-1$ and j, and afterwards removing node $j+1$,

(*iii*) if T satisfies $\{j <_c j-1\}$ and $\{j+1 \not<_c j-1\}$ and $\{j+1 \not<_c j\}$, then T'' is obtained from T by first exchanging the labels j and $j+1$, and afterwards removing node $j+1$,

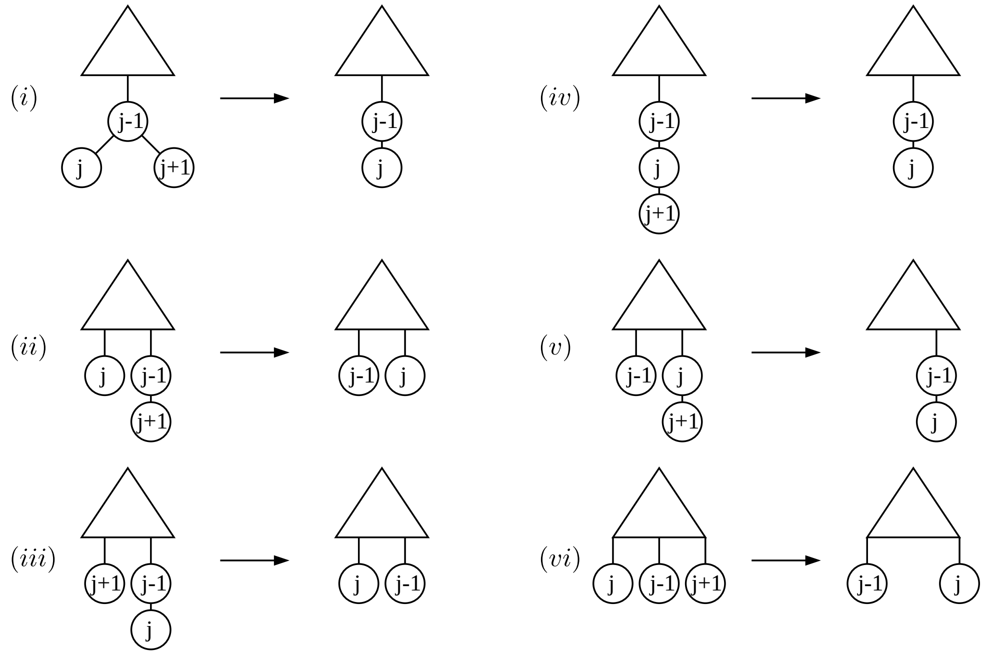
(*iv*) if T satisfies $\{j+1 <_c j <_c j-1\}$, then T'' is obtained from T by removing node $j+1$,

(*v*) if T satisfies $\{j+1 <_c j \not<_c j-1\}$, then T'' is obtained from T by first exchanging the labels $j-1$ and j, then exchanging the labels j and $j+1$, and afterwards removing node $j+1$,

(*vi*) if T satisfies $\{j+1 \not<_c j\}$ and $\{j+1 \not<_c j-1\}$ and $\{j \not<_c j-1\}$, then T'' is obtained from T by first exchanging the labels $j-1$ and j, then exchanging the labels j and $j+1$, and afterwards removing node $j+1$.

We observe that for a tree T the event $\tilde{A}_{j-1}$ holds exactly for the cases (*i*)–(*iii*), whereas $\tilde{A}_{j-1}^c$ holds for the cases (*iv*)–(*vi*). By construction we obtain that for cases (*iv*)–(*vi*) the distance between

Figure 1: Construction 2



node j and $j+1$ in T is equal to the distance between node $j-1$ and j in T'', whereas for cases (i)–(iii) the distance between j and $j+1$ in T is the distance between $j-1$ and j in T'' plus 1.

We introduce now the random variable $\Delta''_{j,j-1}$, which counts the distance between node $j-1$ and node j in a tree obtained by selecting a random increasing tree of size $j+1$ and applying Construction 2. Of course, it holds

$$\Delta_{j+1,j} = \Delta''_{j,j-1} + \mathbb{1}(\tilde{A}_{j-1}).$$

We will then show for the conditional r.v. $\Delta''_{j,j-1}|\tilde{A}_{j-1}$ and $\Delta''_{j,j-1}|\tilde{A}^c_{j-1}$:

$$\Delta''_{j,j-1}|\tilde{A}_{j-1} \stackrel{(d)}{=} \Delta_{j,j-1}, \quad \text{and} \quad \Delta''_{j,j-1}|\tilde{A}^c_{j-1} \stackrel{(d)}{=} \Delta_{j,j-1}, \tag{23}$$

which will imply

$$\Delta_{j+1,j} = \Delta''_{j,j-1} \oplus \mathbb{1}(\tilde{A}_{j-1}), \quad \text{with} \quad \Delta''_{j,j-1} \stackrel{(d)}{=} \Delta_{j,j-1}. \tag{24}$$

To prove this we only have to show that the distribution of the plane-oriented recursive trees obtained after applying Construction 2 to a tree chosen at random from all plane-oriented recursive trees of size $j+1$ satisfying $\tilde{A}_{j-1}$ or not, respectively, again follows the random tree model. Let us denote by $p''(T''|\tilde{A}_{j-1})$ and $p''(T''|\tilde{A}^c_{j-1})$ the probabilities that a certain plane-oriented recursive tree T'' of size j is obtained after applying Construction 2 conditioned on the events $\tilde{A}_{j-1}$ and $\tilde{A}^c_{j-1}$, respectively. The quantities $p(T)$, $p(T|\tilde{A}_{j-1})$ and $p(T|\tilde{A}^c_{j-1})$ are defined analogous to the previous computations.

We consider now an arbitrary plane-oriented recursive tree $T'' \in \mathcal{T}_j$ and compute the conditional probabilities $p''(T''|\tilde{A}_{j-1})$ and $p''(T''|\tilde{A}^c_{j-1})$. We do this by considering all plane-oriented recursive trees $T \in \mathcal{T}_{j+1}$ satisfying $\tilde{A}_{j-1}$ or $\tilde{A}^c_{j-1}$, respectively, which give T'' after applying Construction 2 (this corresponds to the cases (i)–(iii) or the cases (iv)–(vi), respectively).

- Computing $p''(T''|\tilde{A}_{j-1})$: we will distinguish, whether T'' satisfies $\{j <_c j-1\}$ or not.

  * If T'' satisfies $\{j <_c j-1\}$ then there are exactly two trees $T \in \mathcal{T}_{j+1}$ satisfying $\tilde{A}_{j-1}$, let us denote them by S_1 and S_2, which give T'' after applying Construction 2. S_1 and S_2 are obtained from T'' by attaching node $j+1$ to $j-1$ to the left of node j or to the right of node j, respectively. Considering the tree evolution process of plane-oriented recursive trees and using (22) this immediately gives:

$$\begin{aligned} p''(T''|\tilde{A}_{j-1}) &= p(S_1|\tilde{A}_{j-1}) + p(S_2|\tilde{A}_{j-1}) = \frac{p(S_1)}{\mathbb{P}\{\tilde{A}_{j-1}\}} + \frac{p(S_2)}{\mathbb{P}\{\tilde{A}_{j-1}\}} \\ &= \frac{p(T'')}{\mathbb{P}\{\tilde{A}_{j-1}\}} \frac{1}{2j-1} + \frac{p(T'')}{\mathbb{P}\{\tilde{A}_{j-1}\}} \frac{1}{2j-1} = p(T''). \end{aligned}$$

  * If T'' satisfies $\{j \not<_c j-1\}$ then there are exactly two trees $T \in \mathcal{T}_{j+1}$ satisfying $\tilde{A}_{j-1}$, let us denote them by S_1 and S_2, which give T'' after applying Construction 2. S_1 is obtained from T'' by first attaching node $j+1$ to $j-1$ and afterwards exchanging the labels $j+1$ and j. In order to describe S_2 we introduce the tree $\tilde{T}''$, which is obtained from T'' by exchanging the labels $j-1$ and j. S_2 is then obtained from $\tilde{T}''$ by attaching node $j+1$ to $j-1$. Using $p(\tilde{T}'') = p(T'')$ and $p(\tilde{S}_1) = p(S_1)$, where $\tilde{S}_1$ is the tree obtained from S_1 by exchanging the labels $j+1$ and j, we obtain:

$$\begin{aligned} p''(T''|\tilde{A}_{j-1}) &= p(S_1|\tilde{A}_{j-1}) + p(S_2|\tilde{A}_{j-1}) = \frac{p(S_1)}{\mathbb{P}\{\tilde{A}_{j-1}\}} + \frac{p(S_2)}{\mathbb{P}\{\tilde{A}_{j-1}\}} = \frac{p(\tilde{S}_1)}{\mathbb{P}\{\tilde{A}_{j-1}\}} + \frac{p(S_2)}{\mathbb{P}\{\tilde{A}_{j-1}\}} \\ &= \frac{p(T'')}{\mathbb{P}\{\tilde{A}_{j-1}\}} \frac{1}{2j-1} + \frac{p(\tilde{T}'')}{\mathbb{P}\{\tilde{A}_{j-1}\}} \frac{1}{2j-1} = \frac{p(T'')}{\mathbb{P}\{\tilde{A}_{j-1}\}} \frac{1}{2j-1} + \frac{p(T'')}{\mathbb{P}\{\tilde{A}_{j-1}\}} \frac{1}{2j-1} = p(T''). \end{aligned}$$

- Computing $p''(T''|\tilde{A}^c_{j-1})$: again we will distinguish, whether T'' satisfies $\{j <_c j-1\}$ or not.

  * If T'' satisfies $\{j <_c j-1\}$ then there are exactly $2j-3$ trees $T \in \mathcal{T}_{j+1}$ satisfying $\tilde{A}^c_{j-1}$, which give T'' after applying Construction 2. Either T is the tree S_1 obtained from T'' by attaching node $j+1$ to j, or it is a tree S obtained from T'' by first attaching node $j+1$ to a node $\neq j-1$ and $\neq j$, then exchanging the labels $j+1$ and j, and finally exchanging the labels j and $j-1$. In the latter case when considering such an arbitrary tree S we denote by $\tilde{S} \in \mathcal{T}_{j+1}$ the tree, which gives, after first exchanging the labels $j+1$ and j and then exchanging the labels j and $j-1$, the tree S; it holds then $p(\tilde{S}) = p(S)$. This gives:

$$p''(T''|\tilde{A}^c_{j-1}) = p(S_1|\tilde{A}^c_{j-1}) + \sum_{S \in \mathcal{T}_{j+1} \text{ as defined above}} p(S|A^c_{j-1})$$

$$= \frac{p(S_1)}{\mathbb{P}\{\tilde{A}_{j-1}^c\}} + \frac{1}{\mathbb{P}\{\tilde{A}_{j-1}^c\}} \sum_{S \in \mathcal{T}_{j+1} \text{ as defined above}} p(S)$$

$$= \frac{p(T'')}{\mathbb{P}\{\tilde{A}_{j-1}^c\}} \frac{1}{2j-1} + \frac{1}{\mathbb{P}\{\tilde{A}_{j-1}^c\}} \sum_{\substack{\tilde{S} \in \mathcal{T}_{j+1}:\ \tilde{S} \text{ obtained from } T'' \\ \text{by attaching } j+1 \text{ to a node } \neq j-1 \text{ and } \neq j}} p(\tilde{S})$$

$$= \frac{p(T'')}{\mathbb{P}\{\tilde{A}_{j-1}^c\}} \frac{1}{2j-1} + \frac{p(T'')}{\mathbb{P}\{\tilde{A}_{j-1}^c\}} \frac{2j-4}{2j-1} = p(T'').$$

* If T'' satisfies $\{j \not<_c j-1\}$ then there are exactly $2j-3$ trees $T \in \mathcal{T}_{j+1}$ satisfying $\tilde{A}_{j-1}^c$, which give T'' after applying Construction 2. Namely, T is a tree obtained from T'' by first attaching node $j+1$ to a node $\neq j-1$ and $\neq j$, then exchanging the labels $j+1$ and j, and finally exchanging the labels j and $j-1$. In the latter case when considering such an arbitrary tree T we denote by $\tilde{T} \in \mathcal{T}_{j+1}$ the tree, which gives, after first exchanging the labels $j+1$ and j and then exchanging the labels j and $j-1$, the tree T; it holds then $p(\tilde{T}) = p(T)$. This gives:

$$p''(T''|\tilde{A}_{j-1}^c) = \sum_{T \in \mathcal{T}_{j+1} \text{ as defined above}} p(T|A_{j-1}^c) = \frac{1}{\mathbb{P}\{\tilde{A}_{j-1}^c\}} \sum_{T \in \mathcal{T}_{j+1} \text{ as defined above}} p(T)$$

$$= \frac{1}{\mathbb{P}\{\tilde{A}_{j-1}^c\}} \sum_{\substack{\tilde{T} \in \mathcal{T}_{j+1}:\ \tilde{T} \text{ obtained from } T'' \\ \text{by attaching } j+1 \text{ to a node } \neq j-1 \text{ and } \neq j}} p(\tilde{T}) = \frac{p(T'')}{\mathbb{P}\{\tilde{A}_{j-1}^c\}} \frac{2j-3}{2j-1} = p(T'').$$

Hence (23) and thus (24) is shown. The second part of Theorem 3 for plane-oriented recursive trees follows now by iterating the arguments presented (note that, e.g., when considering the tree T'' obtained from T by Construction 2 it holds that the event $\{j-2 \text{ is lying on the path from } j-1 \text{ to } j\}$ is equal to $\tilde{A}_{j-2}$ in the original tree T): we obtain after $j-1$ iterations the decomposition

$$\Delta_{j+1,j} = \Delta''_{j+1;2,1} \oplus \mathbb{1}(\tilde{A}_1) \oplus \mathbb{1}(\tilde{A}_2) \oplus \cdots \oplus \mathbb{1}(\tilde{A}_{j-1}),$$

where $\Delta''_{j+1;2,1}$ counts the distance between the nodes 1 and 2 in a tree obtained by selecting a random increasing tree of size $j+1$ and applying Construction 2 successively $(j-1)$-times (to the original tree of size $j+1$, to the resulting tree of size j, ..., to the resulting tree of size 3). Since $\Delta''_{j+1;2,1}$ always measures the distance between the nodes 1 and 2 in the unique plane-oriented recursive tree of size 2 it further holds $\Delta''_{j+1;2,1} = 1 = \mathbb{1}(\tilde{A}_j)$ and we obtain (7).

Finally we show the second part of Theorem 3 for the remaining instances (Case A, Case B, and Case C, with $\alpha \neq 1$). We do this by studying the probability generating function $p_{n,j}(v)$ of $\Delta_{n,j}$, which will be given later as equation (30). It follows that $p_{j+1,j}(v)$ can be written as follows (with $c = 0$ for Case A, $c = \frac{1}{d-1}$ for Case B, and $c = -\frac{1}{\alpha+1}$ for Case C):

$$p_{j+1,j}(v) = \frac{v(1+c)}{(j+c)}\Big(1 - \frac{v}{(1+c)(2v-1)-c}\Big) + \frac{v^2(1+c)\big(j-1+(c+1)(2v-1)\big)^{\overline{j-1}}}{(j+c)\big((1+c)(2v-1)-c\big)(j-1+c)^{\underline{j-1}}}. \qquad (25)$$

It holds

$$\frac{\big(j-1+(c+1)(2v-1)\big)^{\underline{j-1}}}{\big((1+c)(2v-1)-c\big)(j-1+c)^{\underline{j-1}}} = \frac{(c+1)(2v-1)-c+j-1+c}{j-1+c}\cdot\frac{\big(j-2+(c+1)(2v-1)\big)^{\underline{j-2}}}{\big((1+c)(2v-1)-c\big)(j-2+c)^{\underline{j-2}}},$$

which can be written as follows:

$$a_{j-1} = \frac{\big((c+1)(2v-1)-c\big)}{j-1+c}a_{j-2} + a_{j-2}, \quad \text{with} \quad a_j := \frac{\big(j+(c+1)(2v-1)\big)^{\underline{j}}}{\big((1+c)(2v-1)-c\big)(j+c)^{\underline{j}}}.$$

Iterating this equation gives $a_{j-1} = \sum_{k=0}^{j-2}\frac{\big((c+1)(2v-1)-c\big)}{k+1+c}a_k + a_0$, and we get from (25) the following expression:

$$\begin{aligned} p_{j+1,j}(v) &= \frac{v(1+c)}{(j+c)} + \frac{v^2(1+c)}{(j+c)}\sum_{k=0}^{j-2}\frac{a_k}{k+1+c} \\ &= \frac{v(1+c)}{(j+c)} + \frac{v^2}{(j+c)}\sum_{k=0}^{j-2}\frac{\big(k+(c+1)(2v-1)\big)^{\underline{k}}}{(k+1+c)^{\underline{k}}} \\ &= \frac{v(1+c)}{(j+c)} + \frac{v^2}{(j+c)}\sum_{k=0}^{j-2}\prod_{l=1}^{k}\Big(\frac{l-(1+c)}{l+1+c} + \frac{2v(1+c)}{l+1+c}\Big). \end{aligned} \tag{26}$$

For Case A (recursive trees) and for Case C, with $\alpha \neq 1$, we can identify the right hand side of (26): it is the generating function of a mixture of sums of independent Bernoulli variables. This shows the second part of Theorem 3 for these tree families.

But for Case B (d-ary trees) the summand $l = 1$ in above expression gives the factor

$$\frac{l-(1+c)}{l+1+c} + \frac{2v(1+c)}{l+1+c} = \frac{-c}{2+c} + \frac{2v(1+c)}{2+c},$$

but $-c = -\frac{1}{d-1} < 0$. Thus we cannot identify the right hand side of (26) as before; we only give the following expression for $p_{j+1,j}(v)$:

$$\begin{aligned} p_{j+1,j}(v) &= \frac{v(1+c)}{(j+c)} + \frac{v^2}{(j+c)} \\ &\quad + \frac{v^2}{(j+c)}\Big(\frac{-c}{2+c} + \frac{2v(1+c)}{2+c}\Big)\cdot\sum_{k=2}^{j-2}\prod_{l=2}^{k}\Big(\frac{l-(1+c)}{l+1+c} + \frac{2v(1+c)}{l+1+c}\Big). \end{aligned}$$

Remark 4. Note that a decomposition of the form

$$\Delta_{j+1,j} = 1 + \sum_{k=1}^{j-1}\mathbb{1}(\tilde{A}_k)$$

is possible for arbitrary evolving simple families of increasing trees, but only for the instance of plane-oriented recursive trees the indicators are mutually independent. To give an example, for recursive trees we get

$$\begin{aligned} \mathbb{P}\{\tilde{A}_{j-1}\} &= \mathbb{P}\{\tilde{A}_{j-1}|\Delta_{j,j-1}=1\}\mathbb{P}\{\Delta_{j,j-1}=1\} + \mathbb{P}\{\tilde{A}_{j-1}|\Delta_{j,j-1}>1\}\mathbb{P}\{\Delta_{j,j-1}>1\} \\ &= \frac{j-1}{(j-1)j} + \frac{j-2}{(j-1)j} = \frac{2j-3}{j(j-1)}. \end{aligned}$$

Assuming that the $\tilde{A}_k$'s are mutually independent we would get further $\mathbb{P}\{\tilde{A}_k\} = \frac{2k-1}{k(k+1)}$. But it can be seen easily that $\mathbb{P}\{\tilde{A}_{j-1}\}\mathbb{P}\{\tilde{A}_{j-2}\} \neq \mathbb{P}\{\tilde{A}_{j-1}\tilde{A}_{j-2}\}$, which leads to a contradiction.

8. Proving the central limit theorem

As during the previous sections we consider mainly the instance of plane-oriented recursive trees (Case C, with $\alpha = 1$). At the end of this section we sketch the analogous calculations for the general case.

We start with an expression for the probability generating function $p_{n,j}(v) = \sum_{m\ge 0} \mathbb{P}\{\Delta_{n,j} = m\}v^m$ obtained by extracting coefficients from the generating function $M(z,u,v)$ as given in Corollary 4. We get (compare with the computations leading to (15)):

$$\begin{aligned} p_{n,j}(v) &= \frac{(j-1)!(n-j-1)!}{T_n}[z^{j-1}u^{n-j-1}]M(z,u,v) = \frac{2^{2n-3}}{(n-1)\binom{n-2}{j-1}\binom{2n-2}{n-1}}\binom{v+j-\frac{3}{2}}{j-1}\binom{\frac{v}{2}+n-2}{n-j-1} \\ &= \frac{2^{2n-3}(n-1)!\Gamma(v+j-\frac{1}{2})\Gamma(n-1+\frac{v}{2})}{(2n-2)!\Gamma(v+\frac{1}{2})\Gamma(\frac{v}{2}+j)}. \end{aligned} \tag{27}$$

The moment generating function $\mathcal{M}_{n,j}(t)$ of $\Delta^*_{n,j} := (\Delta_{n,j} - \mu_{n,j})/\sigma_{n,j}$ is then given by

$$\mathcal{M}_{n,j}(t) := \mathbb{E}(e^{t\Delta^*_{n,j}}) = e^{-\frac{\mu_{n,j}}{\sigma_{n,j}}t}\mathbb{E}(e^{\frac{\Delta_{n,j}}{\sigma_{n,j}}t}) = e^{-\frac{\mu_{n,j}}{\sigma_{n,j}}t}p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}).$$

For our further computations we split the region $1 \le j < n$ into two cases, namely j big, such that $j \ge \log n$, and j small, such that $j \le \log n$. In both cases we set $\mu_{n,j} := (\log n + \log j)/2$ and $\sigma^2_{n,j} := (\log n + \log j)/2$. In the former case $j \ge \log n$ we get by using Stirling's formula for the Gamma function

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \frac{\sqrt{2\pi}}{\sqrt{z}}\Big(1 + \mathcal{O}(\frac{1}{z})\Big), \tag{28}$$

the following expansion, which holds uniformly in a neighbourhood of $v = 1$:

$$p_{n,j}(v) = \frac{\sqrt{\pi}}{2\Gamma(v+\frac{1}{2})}n^{\frac{v-1}{2}}j^{\frac{v-1}{2}}\Big(1+\mathcal{O}(\frac{1}{n})+\mathcal{O}(\frac{1}{j})\Big) = \frac{\sqrt{\pi}}{2\Gamma(v+\frac{1}{2})}e^{(v-1)\mu_{n,j}}\Big(1+\mathcal{O}(\frac{1}{n})+\mathcal{O}(\frac{1}{j})\Big).$$

We get further, for an arbitrary but fixed real t,

$$\begin{aligned} p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}) &= \frac{\sqrt{\pi}}{2\Gamma(e^{\frac{t}{\sigma_{n,j}}}+\frac{1}{2})}e^{(\frac{t}{\sigma_{n,j}}+\frac{t^2}{2!\mu_{n,j}}+\mathcal{O}(\frac{1}{\sigma^3_{n,j}}))\mu_{n,j}}\Big(1+\mathcal{O}(\frac{1}{n})+\mathcal{O}(\frac{1}{j})\Big) \\ &= e^{t\sigma_{n,j}+\frac{t^2}{2!}}\Big(1+\mathcal{O}(\frac{1}{\sqrt{\log n}})+\mathcal{O}(\frac{1}{j})\Big), \end{aligned}$$

where we have used

$$\frac{\sqrt{\pi}}{2\Gamma(e^{\frac{t}{\sigma_{n,j}}}+\frac{1}{2})} = \frac{\sqrt{\pi}}{2\Gamma(\frac{3}{2})}\Big(1+\mathcal{O}(\frac{1}{\sqrt{\log n}})\Big) = 1 + \mathcal{O}(\frac{1}{\sqrt{\log n}}). \tag{29}$$

This leads, for t fixed, to

$$\mathcal{M}_{n,j}(t) = e^{-\sigma_{n,j}t}p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}) = e^{\frac{t^2}{2}}\Big(1+\mathcal{O}(\frac{1}{\sqrt{\log n}})+\mathcal{O}(\frac{1}{j})\Big).$$

Now we consider the case $j \le \log n$. We start with the following asymptotic expansion of $p_{n,j}(v)$, which holds uniformly in a neighbourhood of $v = 1$ (and uniformly for $1 \le j \le n$):

$$p_{n,j}(v) = \frac{\sqrt{\pi}}{2\Gamma(v+\frac{1}{2})} n^{\frac{v-1}{2}} \frac{\Gamma(v+j-\frac{1}{2})}{\Gamma(\frac{v}{2}+j)} \Big(1 + \mathcal{O}(\frac{1}{n})\Big).$$

For $j \le \log n$ this leads, for an arbitrary but fixed real t, to

$$\begin{aligned} p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}) &= \frac{\sqrt{\pi}\,\Gamma(e^{\frac{t}{\sigma_{n,j}}} + j - \frac{1}{2})}{2\Gamma(e^{\frac{t}{\sigma_{n,j}}} + \frac{1}{2})\Gamma(\frac{e^{\frac{t}{\sigma_{n,j}}}}{2} + j)} e^{(\frac{t}{\sigma_{n,j}} + \frac{t^2}{2!\mu_{n,j}} + \mathcal{O}(\frac{1}{\sigma_{n,j}^3}))(\mu_{n,j} - \frac{1}{2}\log j)} \Big(1 + \mathcal{O}(\frac{1}{n})\Big) \\ &= e^{t\sigma_{n,j} + \frac{t^2}{2!}} \Big(1 + \mathcal{O}(\frac{\log\log n}{\sqrt{\log n}})\Big), \end{aligned}$$

where we have used (29) and

$$\frac{\Gamma(e^{\frac{t}{\sigma_{n,j}}} + j - \frac{1}{2})}{\Gamma(\frac{e^{\frac{t}{\sigma_{n,j}}}}{2} + j)} = 1 + \mathcal{O}(\frac{\log\log n}{\sqrt{\log n}}).$$

The latter expansion can be obtained, e.g., by using Taylor's theorem applied to $\Gamma(j+1+x)$. Note that $\log\log n$ in the remainder term appears due to an estimate of $\Psi(j)$ for the considered region of j. This leads, for t fixed, to

$$\mathcal{M}_{n,j}(t) = e^{-\sigma_{n,j}t} p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}) = e^{\frac{t^2}{2}} \Big(1 + \mathcal{O}(\frac{\log\log n}{\sqrt{\log n}})\Big).$$

Thus for $1 \le j < n$ the moment generating function $\mathcal{M}_{n,j}(t)$ of $\Delta^*_{n,j}$ converges pointwise in a real neighbourhood of $t = 0$ to the moment generating function $e^{\frac{t^2}{2}}$ of the standard normal distribution. A result of Curtiss [6] (i.e., an analogue of the continuity theorem of Lévy, which holds for the characteristic function) for moment generating functions shows thus convergence in distribution of $\Delta^*_{n,j}$ to a Gaussian distributed random variable. As a referee remarks the asymptotic expression derived for $p_{n,j}(v)$ suggests that also a Poisson approximation could be given, which would lead to a better rate of convergence.

Now we will sketch the proof of the central limit theorem for the general case. We set, as stated in Theorem 2, $\mu_{n,j} := (1+c)(\log n + \log j)$ and $\sigma^2_{n,j} := (1+c)(\log n + \log j)$, where we have to set $c = 0$ for Case A, $c = \frac{1}{d-1}$ for Case B, and $c = -\frac{1}{\alpha+1}$ for Case C.

The probability generating function $p_{n,j}(v) = \sum_{m\ge 0} \mathbb{P}\{\Delta_{n,j} = m\} v^m$ of $\Delta_{n,j}$ can be obtained by extracting coefficients from $M(z,u,v)$ as given by Corollary 3 via $p_{n,j}(v) = \frac{(j-1)!(n-j-1)!}{T_n}[z^{j-1}u^{n-j-1}]M(z,u,v)$ and we get (again we have to set $c = 0$ for Case A, $c = \frac{1}{d-1}$ for Case B, and $c = -\frac{1}{\alpha+1}$ for Case C):

$$\begin{aligned} p_{n,j}(v) = \big(1+c\big) \frac{v\binom{n-2+v\left(1+c\right)}{n-j-1}}{(n-1)\binom{n-1+c}{n-1}\binom{n-2}{j-1}} \bigg(\Big(1 - \frac{v}{(1+c)(2v-1)-c}\Big)\binom{j-1+c}{j-1} \\ + \frac{v}{(1+c)(2v-1)-c}\binom{j-1+(c+1)(2v-1)}{j-1}\bigg). \end{aligned} \qquad (30)$$

The moment generating function $\mathcal{M}_{n,j}(t)$ of $\Delta^*_{n,j} := (\Delta_{n,j} - \mu_{n,j})/\sigma_{n,j}$ is given by

$$\mathcal{M}_{n,j}(t) := \mathbb{E}(e^{t\Delta^*_{n,j}}) = e^{-\frac{\mu_{n,j}}{\sigma_{n,j}}t}\mathbb{E}(e^{\frac{\Delta_{n,j}}{\sigma_{n,j}}t}) = e^{-\frac{\mu_{n,j}}{\sigma_{n,j}}t} p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}).$$

We split the region $1 \le j < n$ again into two cases: j big, such that $j \ge \log n$, and j small, such that $j \le \log n$. Since it holds, for an arbitrary real t fixed and uniformly for $1 \le j \le n$, that

$$\frac{e^{\frac{t}{\sigma_{n,j}}}}{(1+c)(2e^{\frac{t}{\sigma_{n,j}}} - 1) - c} = 1 + \mathcal{O}\Big(\frac{1}{\sqrt{\log n}}\Big), \tag{31}$$

it can be seen that only the second summand of $p_{n,j}(e^{\frac{t}{\sigma_{n,j}}})$, as given by (30), gives a main contribution to the asymptotic behaviour. By taking $v = e^{\frac{t}{\sigma_{n,j}}}$ we get from (30):

$$p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}) = e^{\frac{t}{\sigma_{n,j}}} \frac{\Gamma(n-1+e^{\frac{t}{\sigma_{n,j}}}(c+1))\Gamma(2+c)}{\Gamma(j+e^{\frac{t}{\sigma_{n,j}}}(c+1))\Gamma(n+c)} \Bigg(\Big(1 - \frac{e^{\frac{t}{\sigma_{n,j}}}}{(1+c)(2e^{\frac{t}{\sigma_{n,j}}} - 1) - c}\Big) \frac{\Gamma(j+c)}{\Gamma(1+c)} + \frac{e^{\frac{t}{\sigma_{n,j}}}}{(1+c)(2e^{\frac{t}{\sigma_{n,j}}} - 1) - c} \frac{\Gamma(j + (2e^{\frac{t}{\sigma_{n,j}}} - 1)(c+1))}{\Gamma(1 + (2e^{\frac{t}{\sigma_{n,j}}} - 1)(c+1))} \Bigg).$$

After using Stirling's formula for the Gamma function (28) and proceeding as in the proof of the special instance of plane-oriented recursive trees this leads to Theorem 4.

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