ON THE DEGREE DISTRIBUTION OF THE NODES IN INCREASING TREES

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Abstract. Simple families of increasing trees can be constructed from simply generated tree families, if one considers for every tree of size n all its increasing labellings, i. e. labellings of the nodes by distinct integers of the set $\{1,\dots,n\}$ in such a way that each sequence of labels along any branch starting at the root is increasing. Three such tree families are of particular interest: *recursive trees*, *plane-oriented recursive trees* and *binary increasing trees*. We study the quantity *degree of node j in a random tree of size n* and give closed formulæ for the probability distribution and all factorial moments for those subclass of tree families, which can be constructed via a tree evolution process. Furthermore limiting distribution results of this parameter are given, which completely characterize the phase change behaviour depending on the growth of j compared to n.

1. Introduction

There are several tree models, namely so called *recursive trees*, *plane-oriented recursive trees* (also known as non-uniform recursive trees or heap ordered trees) and *binary increasing trees* (and more generally *d-ary increasing trees*), which turned out to be appropriate in order to describe the behaviour of a lot of quantities in various applications (see [9] for a survey). E. g., these tree models are used to describe the spread of epidemics, for pyramid schemes, and they are used as a simplified growth model of the world wide web: plane-oriented recursive trees are a special instance of the so called Albert-Barabási model for scale-free networks (see e. g. [2]).

All the tree families mentioned above can be considered as so called increasing trees, i. e. labelled trees, where the nodes of a tree of size n are labelled by distinct integers of the set $\{1,\dots,n\}$ in such a way that each sequence of labels along any path starting at the root is increasing. E. g., plane-oriented recursive trees are increasingly labelled ordered trees (= planted plane trees) and d-ary increasing trees are obtained from (unlabelled) d-ary trees via increasing labellings. This point of view seems to be appearing first in [13]. A fundamental study of those increasing tree families that are generated from simply generated tree families (see [10]) by equipping the trees with increasing labellings was given in [1]. Such *simple families of increasing trees* are also the combinatorial objects, which are studied in the present paper.

But instead of using this combinatorial description it is more common to describe certain members of increasing tree families via a tree evolution process, i. e. for every tree T' of size n with vertices $v_1,\dots,v_n$ one is giving probabilities $p_{T'}(v_1),\dots,p_{T'}(v_n)$, such that when starting with a *random tree* T' of size n of the tree family considered, choosing a vertex v_i in T' according to the probabilities $p_{T'}(v_i)$ and attaching node $n+1$ to it, we obtain again a *random tree* T of size $n+1$ of the tree family considered. For the tree families mentioned above (i. e. recursive trees, plane-oriented recursive trees and d-ary increasing trees) the "insertion probabilities" $p_{T'}(v_i)$ are quite simple to describe, since they depend only on the size $|T'|$ of the tree T' and on the out-degree $d(v_i)$ of node v_i and are thus independent from the actual choice of the tree T'. For the model of randomness we always use the random tree model for weighted trees, i. e. since all tree families appearing can be considered as weighted trees, we assume that every tree of size n is chosen with probability proportional to its weight.

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Most of the parameters previously studied are analyzed by using this description via the tree evolution process, e. g. by using Pólya urn models (see e. g. [8]) or by translating results from continuous time branching processes (see e. g. [4]). However, up to now there are obtained very few results that give insight into the behaviour of the node j (= the j-th individual) during the growth process in a tree of size n, in particular if the label $j = j(n)$ is growing with n. This paper is now devoted to the presentation of a method suitable for a distributional study of such label-dependent parameters in increasing trees, where we are using the combinatorial description. The method is shown exemplarily on the random variable $X_{n,j}$, which counts the node degree (i. e. the out-degree) of a specified node j (with $1 \le j \le n$) in a random increasing tree of size n, but we want to remark that a distributional study of various parameters of interest is possible with the method presented (e. g. distances between specified nodes j_1 and j_2, subtree-size of node j, etc.).

Before continuing, we have to say a few words on the description of increasing tree families via tree evolution processes: it has been shown in [12] that it is not possible to describe every simple family of increasing trees by such a tree evolution process, even more, it has been given there a full characterization of such increasing tree families via the so called degree-weight generating function. Throughout this paper we will choose the term *grown* simple families of increasing trees for the subclass of increasing tree families, which can be generated by a tree evolution process. Our recursive approach in combination with a treatment by suitable generating functions will in principle work for all simple families of increasing trees, even if there does not exist an evolution process, leading to a closed formula for the introduced generating functions of the probabilities $\mathbb{P}\{X_{n,j} = m\}$ given as Proposition 1. But in the succeeding computations we will restrict ourselves to the grown tree families, since then label-dependent parameters have a direct meaning in the tree evolution process; if there does not exist such a process the behaviour of node j does not seem to be of great importance. For all grown simple increasing tree families we obtain then closed formulæ for the probabilities $\mathbb{P}\{X_{n,j} = m\}$ and the s-th factorial moments $\mathbb{E}(X_{n,j}^{\underline{s}}) = \mathbb{E}\big(X_{n,j}(X_{n,j}-1)\cdots(X_{n,j}-s+1)\big) = \sum_{m\ge0} m^{\underline{s}}\mathbb{P}\{X_{n,j} = m\}$. These explicit results are given in Theorem 1.

The advantage of having these explicit results is that they yield relatively easy a full characterization of the limiting distribution of $X_{n,j}$, for $n \to \infty$, depending on the growth of $j = j(n)$ compared to n, which is given as Theorem 2. That means we can describe in detail the phase change behaviour of the node-degree of node j in a random tree of size n for all increasing tree families that are of interest in this context. Of course, the exact and asymptotic formulæ presented here extend the (very few) known results on this subject.

Throughout this paper we use the abbreviations $x^{\underline{l}} := x(x-1)\cdots(x-l+1)$ and $x^{\overline{l}} := x(x+1)\cdots(x+l-1)$ for the falling and rising factorials, respectively. Moreover, we use the abbreviations D_x for the differential operator with respect to x, and E_x for the evaluation operator at $x = 1$. Further we denote with $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ the Stirling numbers of the first and second kind, respectively, with $X \overset{(d)}{=} Y$ the equality in distribution of the random variables X and Y, and with $X_n \xrightarrow{(d)} X$ the weak convergence, i. e. the convergence in distribution, of the sequence of random variables X_n to a random variable X. We denote with $H_n := \sum_{k=1}^{n} \frac{1}{k}$ the n-th harmonic number and with $H_n^{(a)} := \sum_{k=1}^{n} \frac{1}{k^a}$ the n-th harmonic number of order a.

2. Preliminaries

2.1. **Combinatorial description of increasing tree families.** Formally, a class $\mathcal{T}$ of a simple family of increasing trees can be defined in the following way. A sequence of non-negative numbers $(\varphi_k)_{k\ge0}$ with $\varphi_0 > 0$ (we further assume that there exists a $k \ge 2$ with $\varphi_k > 0$) is used to define the weight $w(T)$ of any ordered tree T by $w(T) := \prod_v \varphi_{d(v)}$, where v ranges over all vertices of T and $d(v)$ is the out-degree of v. Furthermore, $\mathcal{L}(T)$ denotes the set of different increasing labellings of the tree T with distinct integers $\{1, 2, \ldots, |T|\}$, where $|T|$ denotes the size of the tree T, and $L(T) := \big|\mathcal{L}(T)\big|$ its cardinality. Then the family $\mathcal{T}$ consists of all trees T together with their weights $w(T)$ and the set of increasing labellings $\mathcal{L}(T)$.

For a given degree-weight sequence $(\varphi_k)_{k\ge0}$ with a degree-weight generating function $\varphi(t) := \sum_{k\ge0} \varphi_k t^k$, we define now the total weights by $T_n := \sum_{|T|=n} w(T) \cdot L(T)$. It follows then that the exponential generating function $T(z) := \sum_{n\ge1} T_n \frac{z^n}{n!}$ satisfies the *autonomous* first order differential equation

$$T'(z) = \varphi\big(T(z)\big), \quad T(0) = 0. \tag{1}$$

Often it is advantageous to describe a simple family of increasing trees $\mathcal{T}$ by the formal recursive equation

$$\mathcal{T} = ① \times \Big(\varphi_0 \cdot \{\epsilon\} \,\dot\cup\, \varphi_1 \cdot \mathcal{T} \,\dot\cup\, \varphi_2 \cdot \mathcal{T} * \mathcal{T} \,\dot\cup\, \varphi_3 \cdot \mathcal{T} * \mathcal{T} * \mathcal{T} \,\dot\cup\, \cdots\Big) = ① \times \varphi(\mathcal{T}), \tag{2}$$

where ① denotes the node labelled by 1, $\times$ the cartesian product, $*$ the partition product for labelled objects, and $\varphi(\mathcal{T})$ the substituted structure (see, e. g., [14]).

By specializing the degree-weight generating function $\varphi(t)$ in (1) we get the basic enumerative results for the three most interesting increasing tree families:

• *Recursive trees* are the family of non-plane increasing trees such that all node degrees are allowed. The degree-weight generating function is $\varphi(t) = \exp(t)$. Solving (1) gives

$$T(z) = \log\Big(\frac{1}{1-z}\Big), \quad \text{and} \quad T_n = (n-1)!, \ \text{for } n \ge 1.$$

• *Plane-oriented recursive trees* are the family of plane increasing trees such that all node degrees are allowed. The degree-weight generating function is $\varphi(t) = \frac{1}{1-t}$. Equation (1) leads here to

$$T(z) = 1 - \sqrt{1-2z}, \quad \text{and} \quad T_n = \tfrac{(n-1)!}{2^{n-1}}\tbinom{2n-2}{n-1} = 1\cdot 3\cdot 5\cdots(2n-3) = (2n-3)!!, \ \text{for } n \ge 1.$$

• *Binary increasing trees* have the degree-weight generating function $\varphi(t) = (1+t)^2$. Thus it follows

$$T(z) = \frac{z}{1-z}, \quad \text{and} \quad T_n = n!, \ \text{for } n \ge 1.$$

2.2. **Characterization of grown simple families of increasing trees.** We will describe now the characterization of grown simple increasing tree families via the degree-weight generating function $\varphi(t)$ as obtained in [12].

Lemma 1 ([12])**.** *A simple family of increasing trees $\mathcal{T}$ can be constructed via a tree evolution process and is thus a grown simple family of increasing trees iff the degree-weight generating function $\varphi(t) = \sum_{k\ge0} \varphi_k t^k$ is given by one of the following three formulæ (with constants $c_1, c_2 \in \mathbb{R}$).*

Case A (recursive trees) :

$$\varphi(t) = \varphi_0 e^{\frac{c_1 t}{\varphi_0}}, \ \textit{for } \varphi_0 > 0,\ c_1 > 0, \quad (\textit{defining } c_2 := 0),$$

Case B (d-ary trees) :

$$\varphi(t) = \varphi_0\Big(1 + \frac{c_2 t}{\varphi_0}\Big)^d, \ \textit{for } \varphi_0 > 0,\ c_2 > 0,\ d := \frac{c_1}{c_2} + 1 \in \{2,3,4,\dots\},$$

Case C (generalized plane-oriented recursive trees) :

$$\varphi(t) = \frac{\varphi_0}{(1 + \frac{c_2 t}{\varphi_0})^{-\frac{c_1}{c_2}-1}}, \ \textit{for } \varphi_0 > 0,\ 0 < -c_2 < c_1.$$

The constants c_1, c_2 appearing in Lemma 1 are coming from an equivalent characterization of grown simple increasing tree families obtained in [5]: The total weights T_n of trees of size n of $\mathcal{T}$ satisfy for all $n \in \mathbb{N}$ the equation

$$\frac{T_{n+1}}{T_n} = c_1 n + c_2. \tag{3}$$

Solving either the differential equation (1) or using (3) one obtains the following explicit formulæ for the exponential generating function $T(z)$:

$$T(z) = \begin{cases} \frac{\varphi_0}{c_1}\log\big(\frac{1}{1-c_1 z}\big), & \text{Case A},\\ \frac{\varphi_0}{c_2}\Big(\frac{1}{(1-(d-1)c_2 z)^{\frac{1}{d-1}}} - 1\Big), & \text{Case B},\\ \frac{\varphi_0}{c_2}\Big(\frac{1}{(1-c_1 z)^{\frac{c_2}{c_1}}} - 1\Big), & \text{Case C}. \end{cases} \tag{4}$$

Furthermore the coefficients T_n are given by the following formula, which holds for all three cases of grown simple increasing tree families (setting $c_2 = 0$ in Case A and $d = \frac{c_1}{c_2} + 1$ in Case B):

$$T_n = \varphi_0 c_1^{n-1} (n-1)! \binom{n-1+\frac{c_2}{c_1}}{n-1}. \tag{5}$$

Next we are going to describe in more detail the tree evolution process which generates random trees (of arbitrary size n) of grown simple families of increasing trees. This description is a consequence of the considerations made in [12]:

- Step 1: The process starts with the root labelled by 1.
- Step $i+1$: At step $i+1$ the node with label $i+1$ is attached to any previous node v (with out-degree $d(v)$) of the already grown tree of size i with probabilities proportional to
$$\frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}},$$
i. e.
$$p(v) = \begin{cases} \frac{1}{i}, & \text{for Case A,} \\ \frac{d-d(v)}{(d-1)i+1}, & \text{for Case B,} \\ \frac{d(v)+\alpha}{(\alpha+1)i-1}, \quad \text{with } \alpha := -1 - \frac{c_1}{c_2} > 0, & \text{for Case C.} \end{cases}$$

Thus we see that from a probabilistic point of view one could completely reduce the considerations for Case A and Case B to recursive trees ($\varphi_0 = c_1 = 1$) and d-ary trees ($\varphi_0 = c_2 = 1$, $c_1 = d-1$). For Case C we see that plane-oriented recursive trees are contained due to $\varphi_0 = 1$, $c_1 = 2$, and $c_2 = -1$ (leading to $\alpha = 1$), but we have the possibility of choosing an arbitrary $\alpha > 0$, such that this case can indeed be seen as a generalization of plane-oriented recursive trees.

3. Results for grown simple families of increasing trees

3.1. **Exact formulæ.** In the following we give exact formulæ for the distribution and the s-th factorial moments of the random variable $X_{n,j}$.

Theorem 1. *The probabilities $\mathbb{P}\{X_{n,j} = m\}$, which give the probability that the node with label j in a randomly chosen size-n tree of a grown simple family of increasing trees as given by Lemma 1, has out-degree m, are, for $m \geq 1$ given by the following formulæ:*

$$\mathbb{P}\{X_{n,j} = m\} = \begin{cases} \frac{1}{\binom{n-1}{j-1}} \sum_{k=m}^{n-j} \binom{n-k-2}{j-2} \frac{\left[{k \atop m}\right]}{k!}, \quad \textit{Case A (recursive trees)}, \\ \binom{d}{m} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{\Gamma(n-1+\frac{k}{d-1})\Gamma(j+\frac{1}{d-1})}{\Gamma(j-1+\frac{k}{d-1})\Gamma(n+\frac{1}{d-1})}, \\ \textit{Case B (d-ary increasing trees)}, \\ \binom{m-2-\frac{c_1}{c_2}}{m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{\Gamma(n-1+k\frac{c_2}{c_1})\Gamma(j+\frac{c_2}{c_1})}{\Gamma(j-1+k\frac{c_2}{c_1})\Gamma(n+\frac{c_2}{c_1})}, \\ \textit{Case C (generalized plane-oriented recursive trees)}. \end{cases} \tag{6}$$

The s-th factorial moments $\mathbb{E}\big((X_{n,j})^{\underline{s}}\big) = \sum_{m\ge 0} m^{\underline{s}}\mathbb{P}\{X_{n,j}=m\}$ *are for* $s \ge 1$ *given by the following formulæ:*

$$\mathbb{E}(X_{n,j}^{\underline{s}}) = \begin{cases} \dfrac{s!}{\binom{n-1}{j-1}} \displaystyle\sum_{l=0}^{n-j} \binom{n-l-1}{j-1} \frac{\left[{l \atop s}\right]}{l!}, \quad \text{Case A (recursive trees)}, \\[2ex] \dfrac{d^{\underline{s}}\binom{\frac{1}{d-1}+j-1}{j-1}(n+\frac{s}{d-1}-2)^{\underline{n-j}}}{\binom{n-1}{j-1}\binom{n-1+\frac{1}{d-1}}{n-1}(n-j)!}, \quad \text{Case B (d-ary increasing trees)}, \\[2ex] \dfrac{\Gamma(s-1-\frac{c_1}{c_2})}{\Gamma(-1-\frac{c_1}{c_2})} \displaystyle\sum_{k=0}^{s}\binom{s}{k}(-1)^k \frac{\Gamma(n-\frac{c_2}{c_1}(s-1-k))\Gamma(j+\frac{c_2}{c_1})}{\Gamma(j-\frac{c_2}{c_1}(s-1-k))\Gamma(n+\frac{c_2}{c_1})}, \\ \quad \text{Case C (generalized plane-oriented recursive trees)}. \end{cases} \tag{7}$$

3.2. **Limiting distribution results.** The exact results obtained in Subsection 3.1 lead for all three classes of grown simple families of increasing trees to a full characterization of the distributional behaviour of $X_{n,j}$, for $n \to \infty$, depending on the growth of $j = j(n)$ compared to n.

Theorem 2. *The limiting distribution behaviour of the random variable* $X_{n,j}$*, which counts the out-degree of the node with label* j *in a randomly chosen size-*n *tree of a grown simple family of increasing trees of Case A (recursive trees) as given by Lemma 1, is, for* $n \to \infty$ *and depending on the growth of* j*, given as follows.*

- *The region for* j *small:* $j = o(n)$*. The centralized and normalized random variable* $X^*_{n,j}$ *is asymptotically Gaussian distributed,*

$$X^*_{n,j} = \frac{X_{n,j} - (\log n - \log j)}{\sqrt{\log n - \log j}} \xrightarrow{(d)} \mathcal{N}(0,1). \tag{8}$$

- *The central region for* j*:* $j \to \infty$ *such that* $j = \rho n$*, with* $0 < \rho < 1$*. The random variable* $X_{n,j}$ *is asymptotically Poisson distributed* Poisson(λ) *with parameter* $\lambda = -\log\rho$*.*

$$X_{n,j} \xrightarrow{(d)} X_\rho, \quad \text{with} \quad \mathbb{P}\{X_\rho = m\} = \frac{\rho(-\log\rho)^m}{m!}. \tag{9}$$

- *The region for* j *large:* $l := n - j = o(n)$*.* $\mathbb{P}\{X_{n,j} = 0\} \to 1$*.*

Theorem 3. *The limiting distribution behaviour of the random variable* $X_{n,j}$ *in a randomly chosen size-*n *tree of a grown simple family of increasing trees of Case B (*d*-ary increasing trees) as given by Lemma 1, is, for* $n \to \infty$ *and depending on the growth of* j*, given as follows.*

- *The region for* j *small:* $j = o(n)$*:* $\mathbb{P}\{X_{n,j} = d\} \to 1$*.*
- *The central region for* j*:* $j \to \infty$ *such that* $j = \rho n$*, with* $0 < \rho < 1$*. The random variable* $X_{n,j}$ *is asymptotically binomial distributed* Binomial(a, p) *with parameters* $a = d$ *and* $p = 1 - \rho^{\frac{1}{d-1}}$*.*

$$X_{n,j} \xrightarrow{(d)} X_\rho, \quad \text{with} \quad \mathbb{P}\{X_\rho = m\} = \binom{d}{m}\left(1-\rho^{\frac{1}{d-1}}\right)^m \left(\rho^{\frac{1}{d-1}}\right)^{d-m}. \tag{10}$$

- *The region for* j *large:* $l := n - j = o(n)$*:* $\mathbb{P}\{X_{n,j} = 0\} \to 1$*.*

Theorem 4. *The limiting distribution behaviour of the random variable* $X_{n,j}$ *in a randomly chosen size-*n *tree of a grown simple family of increasing trees of Case C (generalized plane-oriented recursive trees) as given by Lemma 1, is, for* $n \to \infty$ *and depending on the growth of* j*, given as follows.*

- *The region for* j *fixed. For general* c_1 *and* c_2 *the normalized random variable* $n^{\frac{c_2}{c_1}} X_{n,j}$ *has a limiting distribution, which can be characterized by its moments.*

$$n^{\frac{c_2}{c_1}} X_{n,j} \xrightarrow{(d)} X_j, \quad \text{with moments} \quad \mathbb{E}(X_j^s) = \frac{\Gamma(s-1-\frac{c_1}{c_2})\Gamma(j+\frac{c_2}{c_1})}{\Gamma(-1-\frac{c_1}{c_2})\Gamma(j-\frac{(s-1)c_2}{c1})}. \tag{11}$$

Alternatively, the limiting distribution of $X_{n,j}$ can be characterized by its density $f_j(x)$. It holds $f_j(x) = 0$ for $x < 0$, whereas for $x \geq 0$ one has

$$f_1(x) = \frac{-\Gamma(\frac{c_2}{c_1})x^{-2-\frac{c_1}{c_2}}}{\Gamma(-1-\frac{c_1}{c_2})\pi} \int_0^\infty e^{-(r^{-\frac{c_1}{c_2}})} r^{-\frac{c_1}{c_2}-1} e^{-xr\cos(\frac{c_2}{c_1}\pi)} \sin\big(xr\sin(-\frac{c_2}{c_1}\pi)\big)dr, \tag{12a}$$

$$f_j(x) = \frac{\Gamma(j+\frac{c_2}{c_1})}{\Gamma(1+\frac{c_2}{c_1})\Gamma(j-1)} \int_0^1 t^{2\frac{c_2}{c_1}}(1-t)^{j-2} f_1\big(xt^{\frac{c_2}{c_1}}\big)dt, \quad \text{for } j \geq 2. \tag{12b}$$

For plane-oriented recursive trees we get by specialization ($c_1 = 2$, $c_2 = -1$) the following results. The moments of the limiting distribution X_j are for $j \geq 1$ given by

$$\mathbb{E}(X_j^s) = \frac{(2j-2)!2^s s!\Gamma(j+\frac{s}{2})}{(j-1)!(s+2j-2)!}. \tag{13}$$

Thus the normalized r. v. $n^{-\frac{1}{2}}X_{n,1}$ (corresponding to the instance $j = 1$, i. e. the degree of the root) is asymptotically Rayleigh distributed with parameter $\sigma = \sqrt{2}$.

$$n^{-\frac{1}{2}}X_{n,1} \xrightarrow{(d)} X_1, \quad \text{with density} \quad f_{X_1}(x) = \frac{x}{2}e^{-\frac{x^2}{2}}. \tag{14a}$$

For $j > 1$ we get the following:

$$n^{-\frac{1}{2}}X_{n,j} \xrightarrow{(d)} X_j, \quad \text{with density} \quad f_{X_j}(x) = \frac{2j-3}{2^{2j-3}(j-2)!}\int_x^\infty (t-x)^{2j-4}e^{-\frac{t^2}{4}}dt. \tag{14b}$$

- *The region for j small, but not fixed: $j \to \infty$ such that $j = o(n)$. The normalized random variable $\big(\frac{n}{j}\big)^{\frac{c_2}{c_1}}X_{n,j}$ is asymptotically Gamma distributed $\gamma(a,\lambda)$ with parameters $a = -1-\frac{c_1}{c_2}$ and $\lambda = 1$.*

$$\big(\frac{n}{j}\big)^{\frac{c_2}{c_1}}X_{n,j} \xrightarrow{(d)} X, \quad \text{with moments} \quad \mathbb{E}(X^s) = \frac{a^{\overline{s}}}{\lambda^s} = \big(-\frac{c_1}{c_2}-1\big)^{\overline{s}} = \frac{\Gamma(s-1-\frac{c_1}{c_2})}{\Gamma(-1-\frac{c_1}{c_2})}. \tag{15}$$

- *The central region for j: $j \to \infty$ such that $j = \rho n$, with $0 < \rho < 1$. The random variable $X_{n,j}$ is asymptotically negative binomial distributed* $\mathrm{NegBin}(r,p)$ *with parameters $r = -1-\frac{c_1}{c_2}$ and $p = \rho^{-\frac{c_2}{c_1}}$.*

$$X_{n,j} \xrightarrow{(d)} X_\rho, \quad \text{with} \quad \mathbb{P}\{X_\rho = m\} = \binom{m-2-\frac{c_1}{c_2}}{m}\rho^{1+\frac{c_2}{c_1}}\big(1-\rho^{-\frac{c_2}{c_1}}\big)^m. \tag{16}$$

- *The region for j large: $l := n-j = o(n)$. $\mathbb{P}\{X_{n,j} = 0\} \to 1$.*

Let U_n be a random variable uniformly distributed on $\{1,2,\dots,n\}$. Then $X_n := X_{n,U_n}$ counts the out-degree of a random node in a randomly chosen increasing tree of size n. The following result is a consequence of the previous theorems.

Corollary 1. *The limiting distribution of the random variable X_n, which counts the out-degree of a random node in a randomly chosen size-n tree of a grown simple family of increasing trees as given by Lemma 1, is, for $n \to \infty$, given as follows.*

$$X_n \xrightarrow{(d)} X, \quad \text{with}$$

$$\mathbb{P}\{X = m\} = \begin{cases} \frac{1}{2^{m+1}}, \quad \text{for } m \geq 0, \quad \text{Case A (recursive trees)}, \\ \frac{d-1}{2d-1}\frac{\binom{d}{m}}{\binom{2d-2}{m}}, \quad \text{for } 0 \leq m \leq d, \quad \text{Case B (d-ary increasing trees)}, \\ \frac{-\frac{c_1}{c_2}}{-\frac{2c_1}{c_2}-1}\frac{\binom{m-2-\frac{c_1}{c_2}}{m}}{\binom{m-1-\frac{2c_1}{c_2}}{m}}, \quad \text{for } m \geq 0, \\ \quad \text{Case C (generalized plane-oriented recursive trees)}. \end{cases} \tag{17}$$

This paper is organized as follows. In Section 4 we treat a recurrence for the probabilities $\mathbb{P}\{X_{n,j} = m\}$ via generating functions. This leads for all simple families of increasing trees to a closed formula for the generating function, which is given in Proposition 1. In Section 5 we prove the explicit results for grown simple families of increasing trees that are given by Theorem 1, and the corresponding limiting distribution results of Theorem 2-4 are shown in Section 6.

4. A recurrence for the probabilities

We consider in this section the random variable $X_{n,j}$, which counts the out-degree of node j in a random increasing tree of size n, for general simple families of increasing trees with degree-weight generating function $\varphi(t)$. In the following we give a recurrence for the probabilities $\mathbb{P}\{X_{n,j} = m\}$, which is obtained from the formal recursive description (2). We begin with the case $j = 1$. At first we introduce a bivariate generating function $M(z, v)$ for the root-degree $j = 1$, which is defined as follows.

$$M(z,v) := \sum_{n\ge 1}\sum_{m\ge 0} \mathbb{P}\{X_{n,1} = m\} T_n \frac{z^n}{n!} v^m. \tag{18}$$

Either directly from the fact that the exponential generating function of trees with root degree m is given by

$$\varphi_m \int_0^z T^m(t)dt, \tag{19}$$

or by using

$$\mathbb{P}\{X_{n,1} = m\} = \varphi_m \sum_{\substack{k_1+\cdots+k_m=n-1,\\ k_1,\dots,k_m\ge 1}} \frac{T_{k_1}\cdots T_{k_m}}{T_n}\binom{n-1}{k_1,k_2,k_3,\dots,k_m}, \tag{20}$$

one gets the following lemma, which already appeared in [1].

Lemma 2 ([1])**.** *The bivariate generating function of the root-degree is given by*

$$M(z,v) = \int_0^z \varphi\big(vT(t)\big)dt. \tag{21}$$

Now we turn to the case $2 \le j \le n$. For increasing trees of size n with root-degree r and subtrees with sizes $k_1, \dots, k_r$, enumerated from left to right, where the node labelled by j lies in the leftmost subtree and is the i-th smallest node in this subtree, we can reduce the computation of the probabilities $\mathbb{P}\{X_{n,j} = m\}$ to the probabilities $\mathbb{P}\{X_{k_1,i} = m\}$. We get as factor the total weight of the r subtrees and the root node $\varphi_r T_{k_1}\cdots T_{k_r}$, divided by the total weight T_n of trees of size n and multiplied by the number of order preserving relabellings of the r subtrees, which are given here by

$$\binom{j-2}{i-1}\binom{n-j}{k_1-i}\binom{n-1-k_1}{k_2,k_3,\dots,k_r}:$$

the $i-1$ labels smaller than j are chosen from $2, 3, \dots, j-1$, the $k_1 - i$ labels larger than j are chosen from $j+1, \dots, n$, and the remaining $n-1-k_1$ labels are distributed to the second, third, …, r-th subtree. Due to symmetry arguments we obtain a factor r, if the node j is the i-th smallest node in the second, third, …, r-th subtree. Summing up over all choices for the rank i of label j in its subtree, the subtree-sizes $k_1, \dots, k_r$, and the degree r of the root node gives the following recurrence.

$$\begin{aligned}\mathbb{P}\{X_{n,j} = m\} = \sum_{r\ge 1} r\varphi_r \sum_{\substack{k_1+\cdots+k_r=n-1,\\ k_1,\dots,k_r\ge 1}} \frac{T_{k_1}\cdots T_{k_r}}{T_n}\times \\ \times \sum_{i=1}^{\min\{k_1,j-1\}} \mathbb{P}\{X_{k_1,i} = m\}\binom{j-2}{i-1}\binom{n-j}{k_1-i}\binom{n-1-k_1}{k_2,k_3,\dots,k_r},\end{aligned} \tag{22}$$

for $n \ge j \ge 2$.

To treat this recurrence (22) we set $n := k + j$ with $k \geq 0$ and define the trivariate generating function

$$N(z,u,v) := \sum_{k\geq 0}\sum_{j\geq 1}\sum_{m\geq 0} \mathbb{P}\{X_{k+j,j} = m\} T_{k+j} \frac{z^{j-1}}{(j-1)!}\frac{u^k}{k!} v^m. \tag{23}$$

Multiplying (22) with $T_{k+j}\frac{z^{j-2}}{(j-2)!}\frac{u^k}{k!}v^m$ and summing up over $k \geq 0$, $j \geq 2$ and $m \geq 0$ gives then $\frac{\partial}{\partial z}N(z,u,v)$ and $\varphi'\big(T(z+u)\big)N(z,u,v)$ for the left and right hand side of (22), respectively. Since these are essentially straightforward, but somehow lengthy computations if figured out in detail, they are omitted here. On the other hand it should be mentioned that it is maybe not that apparent, how recurrences like (22) can be treated in an appropriate way, i. e. that introducing the generating function (23) is successful. In any case we obtain the following differential equation

$$\frac{\partial}{\partial z}N(z,u,v) = \varphi'\big(T(z+u)\big)N(z,u,v), \tag{24}$$

satisfying the initial condition

$$N(0,u,v) = \sum_{k\geq 0}\sum_{m\geq 0}\mathbb{P}\{X_{k+1,1} = m\}T_{k+1}\frac{u^k}{k!}v^m = \frac{\partial}{\partial u}M(u,v) = \varphi\big(vT(u)\big). \tag{25}$$

The general solution of equation (24) is given by

$$N(z,u,v) = C(u,v)\exp\Big(\int_0^z \varphi'\big(T(t+u)\big)dt\Big), \tag{26}$$

with some function $C(u,v)$. Adapting to the initial condition (25) gives the required solution

$$N(z,u,v) = \varphi\big(vT(u)\big)\exp\Big(\int_0^z \varphi'\big(T(t+u)\big)dt\Big). \tag{27}$$

Due to the equation $T'(z) = \varphi(T(z))$ we further get the simplifications

$$\int_0^z \varphi'\big(T(t+u)\big)dt = \int_0^z \frac{\varphi'\big(T(t+u)\big)T'(t+u)}{\varphi\big(T(t+u)\big)}dt = \int_{T(u)}^{T(z+u)} \big(\log\varphi(w)\big)'dw = \log\Big(\frac{\varphi\big(T(z+u)\big)}{\varphi\big(T(u)\big)}\Big),$$

which leads from (27) to the following result.

Proposition 1. *The function $N(z,u,v)$ as defined in equation (23), which is the trivariate generating function of the probabilities $\mathbb{P}\{X_{n,j} = m\}$ that give the probability that the node with label j in a randomly chosen size-n tree of a simple family of increasing trees with degree-weight generating function $\varphi(t)$ has exactly m descendants, is given by the following formula:*

$$N(z,u,v) = \frac{\varphi\big(vT(u)\big)\varphi\big(T(z+u)\big)}{\varphi\big(T(u)\big)}. \tag{28}$$

5. Computing the probabilities and moments

5.1. **An exact formula for the probabilities.** ¿From Proposition 1 we can easily compute explicit formulæ for the probabilities $\mathbb{P}\{X_{n,j} = m\}$ for grown simple increasing tree families, i. e. increasing tree families, which can be constructed via a tree evolution process. We will figure out Case A (recursive trees) and Case B (d-ary increasing trees). The result for Case C (generalized plane-oriented recursive trees) follows from analogous computations. We observe that by specializing $\varphi(t)$, as given by Lemma 1, in Proposition 1 yields

$$N(z,u,v) = \begin{cases} \dfrac{\varphi_0}{(1-c_1u)^v(1-\frac{c_1z}{1-c_1u})}, & \text{Case A,} \\ \dfrac{\varphi_0\Big(1+v\big(\frac{1}{(1-(d-1)c_2u)^{\frac{1}{d-1}}}-1\big)\Big)^d}{\Big(1-\frac{(d-1)c_2z}{1-(d-1)c_2u}\Big)^{\frac{d}{d-1}}}, & \text{Case B,} \\ \dfrac{\varphi_0}{\Big(1+v\big(\frac{1}{(1-c_1u)^{\frac{c_2}{c_1}}}-1\big)\Big)^{-\frac{c_1}{c_2}-1}\Big(1-\frac{c_1z}{1-c_1u}\Big)^{\frac{c_2}{c_1}+1}}, & \text{Case C.} \end{cases} \tag{29}$$

By extracting coefficients we obtain for the probabilities for Case A:

$$\begin{aligned} \mathbb{P}\{X_{n,j}=m\} &= \frac{1}{c_1^{n-1}\binom{n-1}{j-1}}[u^{n-j}v^m]\frac{c_1^{j-1}}{(1-c_1u)^{v+j-1}} \\ &= \frac{1}{\binom{n-1}{j-1}}\sum_{k=0}^{n-j}[u^{n-j-k}]\frac{1}{(1-c_1u)^{j-1}}[u^kv^m]\frac{c_1^{j-1}}{(1-c_1u)^v}, \end{aligned} \tag{30}$$

which give together with the generating function identity for the Stirling numbers of first kind,

$$\sum_{n\geq 0}\sum_{m=0}^{m}\left[{n \atop m}\right]\frac{z^n}{n!}v^m = \frac{1}{(1-z)^v}, \tag{31}$$

to the desired result given in Theorem 1. For Case B we proceed as follows.

$$\begin{aligned} \mathbb{P}\{X_{n,j}=m\} &= \frac{\binom{\frac{1}{d-1}+j-1}{j-1}}{c_1^{n-j}\binom{n-1}{j-1}\binom{n-1+\frac{1}{d-1}}{n-1}}[u^{n-j}v^m]\frac{\Big(1+v\big(\frac{1}{(1-(d-1)c_2u)^{\frac{1}{d-1}}}-1\big)\Big)^d}{(1-(d-1)c_2u)^{j-1}} \\ &= \frac{\binom{\frac{1}{d-1}+j-1}{j-1}\binom{d}{m}}{c_1^{n-j}\binom{n-1}{j-1}\binom{n-1+\frac{1}{d-1}}{n-1}}[u^{n-j}]\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^{m-k}}{(1-(d-1)c_2u)^{j-1+\frac{k}{d-1}}} \\ &= \frac{\binom{\frac{1}{d-1}+j-1}{j-1}\binom{d}{m}}{\binom{n-1}{j-1}\binom{n-1+\frac{1}{d-1}}{n-1}}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\binom{n-2+\frac{k}{d-1}}{n-j} \\ &= \binom{d}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{\Gamma(n-1+\frac{k}{d-1})\Gamma(j+\frac{1}{d-1})}{\Gamma(j-1+\frac{k}{d-1})\Gamma(n+\frac{1}{d-1})}. \end{aligned} \tag{32}$$

A computation analogous to Case B leads also to the given result for Case C.

5.2. **An exact formula for the factorial moments.** We will only present the calculations for Case C, the other cases are treated completely analogous. To obtain the s-th factorial moments of $X_{n,j}$ we use again Proposition 1, but differentiate equation (28) s times w. r. t. v and evaluate then at $v=1$. For Case C this gives

$$E_vD_v^sN(z,u,v) = \frac{\varphi_0(-\frac{c_1}{c_2}-1)^{\overline{s}}(1-(1-c_1u)^{-\frac{c_2}{c_1}})^s}{(1-c_1u)^{1-\frac{c_2}{c_1}(s-1)}\Big(1-\frac{c_1z}{1-c_1u}\Big)^{\frac{c_2}{c_1}+1}}. \tag{33}$$

Extracting coefficients of (33) leads then by using (5) to

$$\begin{aligned} \mathbb{E}(X_{n,j}^{\underline{s}}) &= \frac{(j-1)!(n-j)!}{T_n}[z^{j-1}u^{n-j}]E_vD_v^sN(z,u,v) \\ &= \frac{(-\frac{c_1}{c_2}-1)^{\overline{s}}\varphi_0c_1^{j-1}\binom{j-1+\frac{c_2}{c_1}}{j-1}}{\varphi_0c_1^{n-1}\binom{n-1}{j-1}\binom{n-1+\frac{c_2}{c_1}}{n-1}}[u^{n-j}]\frac{(1-(1-c_1u)^{-\frac{c_2}{c_1}})^s}{(1-c_1u)^{j-\frac{c_2}{c_1}(s-1)}} \end{aligned}$$

$$= \frac{\Gamma(s-1-\frac{c_1}{c_2})\binom{j-1+\frac{c_2}{c_1}}{j-1}}{\Gamma(-1-\frac{c_1}{c_2})c_1^{n-j}\binom{n-1}{j-1}\binom{n-1+\frac{c_2}{c_1}}{n-1}} \sum_{k=0}^{s}\binom{s}{k}(-1)^k \frac{1}{(1-c_1u)^{j-\frac{c_2}{c_1}(s-k-1)}}$$
$$= \frac{\Gamma(s-1-\frac{c_1}{c_2})}{\Gamma(-1-\frac{c_1}{c_2})} \sum_{k=0}^{s}\binom{s}{k}(-1)^k \frac{\binom{j-1+\frac{c_2}{c_1}}{j-1}\binom{n-1-\frac{c_2}{c_1}(s-k-1)}{n-j}}{\binom{n-1}{j-1}\binom{n-1+\frac{c_2}{c_1}}{n-1}}$$
$$= \frac{\Gamma(s-1-\frac{c_1}{c_2})}{\Gamma(-1-\frac{c_1}{c_2})} \sum_{k=0}^{s}\binom{s}{k}(-1)^k \frac{\Gamma(n-\frac{c_2}{c_1}(s-1-k))\Gamma(j+\frac{c_2}{c_1})}{\Gamma(j-\frac{c_2}{c_1}(s-1-k))\Gamma(n+\frac{c_2}{c_1})},$$

which shows thus the corresponding part in Theorem 1.

6. Proofs of the limiting distribution results

6.1. **Case A (recursive trees).**

(i) First we turn our attention to the region $j = o(n)$. For the expectation and the second factorial moment we get by using

$$[z^n]\frac{\log\left(\frac{1}{1-z}\right)}{(1-z)^{j+1}} = \binom{n+j}{j}(H_n - H_j), \quad [z^n]\frac{\log^2\left(\frac{1}{1-z}\right)}{(1-z)^{j+1}} = \binom{n+j}{j}\left((H_n-H_j)^2 - (H_n^{(2)} - H_j^{(2)})\right), \quad (34)$$

and defining $M_1(u) := E_v D_v M(u,v)$ and $M_2(u) := E_v D_v^2 M(u,v)$, respectively, the following explicit result

$$\begin{aligned}
\mathbb{E}(X_{n,j}) &= \frac{1}{\varphi_0 c_1^{n-j}\binom{n-j}{j-1}}[u^{n-j}]\frac{M_1'(u)}{(1-c_1u)^{j-1}} = \frac{[u^{n-j}]}{c_1^{n-j}\binom{n-j}{j-1}}\frac{\log\left(\frac{1}{1-c_1u}\right)}{(1-c_1u)^j}\\
&= H_{n-1} - H_{j-1} = \log n - \log j + \mathcal{O}(1),\\
\mathbb{E}(X_{n,j}^2) &= \frac{1}{\varphi_0 c_1^{n-j}\binom{n-j}{j-1}}[u^{n-j}]\frac{M_2'(u)}{(1-c_1u)^{j-1}} = \frac{[u^{n-j}]}{c_1^{n-j}\binom{n-j}{j-1}}\frac{\log^2\left(\frac{1}{1-c_1u}\right)}{(1-c_1u)^j}\\
&= (H_{n-1}-H_{j-1})^2 - (H_{n-1}^{(2)} - H_{j-1}^{(2)}),
\end{aligned} \quad (35)$$

and thus for the variance

$$\mathbb{V}(X_{n,j}) = \mathbb{E}(X_{n,j}^2) + \mathbb{E}(X_{n,j}) - \mathbb{E}(X_{n,j})^2 = H_{n-1} - H_{j-1} - (H_{n-1}^{(2)} - H_{j-1}^{(2)}) = \log n - \log j + \mathcal{O}(1). \quad (36)$$

The $O(1)$ bound appearing in the previous equations (35) and (36) holds independently from j, i. e. for all $1 \le j \le n$. In the following we will use the abbreviation $\mu_{n,j} := \log n - \log j$ and $\sigma_{n,j}^2 = \log n - \log j$. Since we want to apply Lévys continuity theorem to the moment generating function of $X_{n,j}^* := (X_{n,j} - \mu_{n,j})/\sigma_{n,j}$ we calculate first the probability generating function $p_{n,j}(v)$ of $X_{n,j}$. We get

$$\begin{aligned}
p_{n,j}(v) &:= \mathbb{E}(v^{X_{n,j}}) = \sum_{m\ge 0}\mathbb{P}\{X_{n,j} = m\}v^m = \frac{1}{c_1^{n-j}\binom{n-j}{j-1}}[u^{n-j}]\frac{1}{(1-c_1u)^{v+j-1}} = \frac{\binom{n+v-2}{v+j-2}}{\binom{n-j}{j-1}} = \frac{\binom{n+v-2}{n-1}}{\binom{j+v-2}{j-1}}\\
&= \frac{\Gamma(n+v-1)\Gamma(j)}{\Gamma(n)\Gamma(j+v-1)}.
\end{aligned} \quad (37)$$

The moment generating function $\mathcal{M}_{n,j}(t)$ of $X_{n,j}^*$ is then given by

$$\mathcal{M}_{n,j}(t) := \mathbb{E}(e^{tX_{n,j}^*}) = e^{-\frac{\mu_{n,j}}{\sigma_{n,j}}t}\mathbb{E}\big(e^{\frac{X_{n,j}}{\sigma_{n,j}}t}\big) = e^{-\sigma_{n,j}t}p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}). \quad (38)$$

Using Stirling's formula for the Gamma function

$$\Gamma(z) = \left(\frac{z}{e}\right)^z\frac{\sqrt{2\pi}}{\sqrt{z}}\Big(1 + \frac{1}{12z} + \frac{1}{288z^2} + \mathcal{O}(\frac{1}{z^3})\Big), \quad (39)$$

we get for v fixed

$$\frac{\Gamma(n+v-1)}{\Gamma(n)} = n^{v-1}\Big(1+\mathcal{O}(\frac{1}{n})\Big) = e^{(v-1)\log n}\Big(1+\mathcal{O}(\frac{1}{n})\Big). \tag{40}$$

For our further computations we split the region $j = o(n)$ into two cases: $j \le \log n$ and $j > \log n$. We begin with considering the region $j = o(n)$, such that $j \le \log n$. It holds for t fixed:

$$\begin{aligned}
e^{-\sigma_{n,j} t} &= e^{-t\sqrt{\log n}\left(1+\mathcal{O}(\frac{\log j}{\log n})\right)} = e^{-t\sqrt{\log n}\left(1+\mathcal{O}(\frac{\log\log n}{\log n})\right)} = e^{-t\sqrt{\log n}}\Big(1+\mathcal{O}(\frac{\log\log n}{\sqrt{\log n}})\Big),\\
\frac{1}{\sigma_{n,j}} &= \frac{1}{\sqrt{\log n}\sqrt{1-\frac{\log j}{\log n}}} = \frac{1}{\sqrt{\log n}}\Big(1+\mathcal{O}(\frac{\log j}{\log n})\Big),\\
e^{\frac{t}{\sigma_{n,j}}} &= e^{\frac{t}{\sqrt{\log n}}}\Big(1+\mathcal{O}(\frac{\log j}{\log^{\frac{3}{2}} n})\Big) = e^{\frac{t}{\sqrt{\log n}}}\Big(1+\mathcal{O}(\frac{\log\log n}{\log^{\frac{3}{2}} n})\Big),
\end{aligned} \tag{41}$$

and consequently

$$e^{\frac{t}{\sigma_{n,j}}}-1 = \Big(1+\frac{t}{\sqrt{\log n}}+\frac{t^2}{2\log n}+\mathcal{O}(\frac{1}{\log^{\frac{3}{2}} n})\Big)\Big(1+\mathcal{O}(\frac{\log\log n}{\log^{\frac{3}{2}} n})\Big)-1 = \frac{t}{\sqrt{\log n}}+\frac{t^2}{2\log n}+\mathcal{O}(\frac{\log\log n}{\log^{\frac{3}{2}} n}). \tag{42}$$

Since $j \le \log n$ we also get the expansion

$$\frac{\Gamma(j+e^{\frac{t}{\sigma_{n,j}}}-1)}{\Gamma(j)} = 1+\mathcal{O}(\frac{\log\log n}{\sqrt{\log n}}), \tag{43}$$

by applying the trivial estimate

$$\frac{\Gamma^{(k)}(j)}{\Gamma(j)} \le k!(\log\log n)^k, \quad \text{for } k \ge 2 \tag{44}$$

to

$$\frac{\Gamma(j+e^{\frac{t}{\sigma_{n,j}}}-1)}{\Gamma(j)} = \sum_{k=0}^{\infty}\frac{\Gamma^{(k)}(j)}{k!\Gamma(j)}(e^{\frac{t}{\sigma_{n,j}}}-1)^k. \tag{45}$$

By combining the previous results (37), (40), (42) and (43) we obtain

$$\begin{aligned}
p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}) &= n^{e^{\frac{t}{\sigma_{n,j}}}-1}\Big(1+\mathcal{O}(\frac{\log\log n}{\sqrt{\log n}}\Big) = e^{\left(e^{\frac{t}{\sigma_{n,j}}}-1\right)\log n}\Big(1+\mathcal{O}(\frac{\log\log n}{\sqrt{\log n}})\Big)\\
&= e^{\left(t\sqrt{\log n}+\frac{t^2}{2}+\mathcal{O}(\frac{\log\log n}{\sqrt{\log n}})\right)}\Big(1+\mathcal{O}(\frac{\log\log(n)}{\sqrt{\log n}})\Big).
\end{aligned} \tag{46}$$

This leads from (38) to the required expansion

$$\begin{aligned}
\mathcal{M}_{n,j}(t) &= e^{-\sigma_{n,j}t}p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}) = e^{-t\sqrt{\log n}}e^{\left(t\sqrt{\log n}+\frac{t^2}{2}+\mathcal{O}(\frac{\log\log n}{\sqrt{\log n}}))\right)}\Big(1+\mathcal{O}(\frac{\log\log n}{\sqrt{\log n}})\Big)\\
&= e^{\frac{t^2}{2}}\Big(1+\mathcal{O}(\frac{\log\log n}{\sqrt{\log n}})\Big).
\end{aligned} \tag{47}$$

We proceed with considering the region $j = o(n)$, such that $j > \log n$. We simply observe

$$\frac{\Gamma(j+v-1)}{\Gamma(j)} = j^{v-1}\Big(1+\mathcal{O}(\frac{1}{j})\Big) = e^{(v-1)\log j}\Big(1+\mathcal{O}(\frac{1}{j})\Big), \tag{48}$$

which gives

$$p_{n,j}(v) = e^{(v-1)\mu_{n,j}}\Big(1+\mathcal{O}(\frac{1}{j})\Big). \tag{49}$$

This leads to

$$\begin{aligned}\mathcal{M}_{n,j}(t) &= e^{-\sigma_{n,j}t}p_{n,j}(e^{\frac{t}{\sigma_{n,j}}}) = e^{-\sigma_{n,j}t}e^{(e^{\frac{t}{\sigma_{n,j}}}-1)\mu_{n,j}}(1+\mathcal{O}(\frac{1}{j}))\\ &= e^{-\sigma_{n,j}t}e^{(\frac{t}{\sigma_{n,j}}+\frac{t^2}{2\mu_{n,j}}+\mathcal{O}(\frac{1}{\sigma_{n,j}^3}))\mu_{n,j}}(1+\mathcal{O}(\frac{1}{j})) = e^{\frac{t^2}{2}+\mathcal{O}(\frac{1}{\sigma_{n,j}})}(1+\mathcal{O}(\frac{1}{j}))\\ &= e^{\frac{t^2}{2}}(1+\mathcal{O}(\frac{1}{j}))(1+\mathcal{O}(\frac{1}{\sigma_{n,j}})) = e^{\frac{t^2}{2}}(1+\mathcal{O}(\frac{1}{j})+\mathcal{O}(\frac{1}{\sigma_{n,j}})).\end{aligned} \tag{50}$$

Thus for $j = o(n)$ the moment generating function $\mathcal{M}_{n,j}(t)$ of $X_{n,j}^*$ converges in a real neighbourhood of $t = 0$ to the moment generating function $e^{\frac{t^2}{2}}$ of the standard normal distribution. The continuity theorem of Lévy shows thus convergence in distribution of $X_{n,j}^*$ to a Gaussian distributed r. v.

(ii) For the region $j = \rho n$, with $0 < \rho < 1$, we use (48) to obtain that the probability generating function (and thus also the moment generating function) converges for $n \to \infty$ to the probability generating function (resp. the moment generating function) of a Poisson distributed r. v.

$$p_{n,j}(v) = e^{-(v-1)\log\rho}\Big(1+\mathcal{O}(\frac{1}{n})\Big) \xrightarrow{n\to\infty} p_j(v) = e^{-(v-1)\log\rho}. \tag{51}$$

(iii) For the region j: $l := n-j = o(n)$ we simply obtain:

$$\mathbb{P}\{X_{n,j} = 0\} = \frac{\binom{n-2}{j-2}}{\binom{n-1}{j-1}} = \frac{j-1}{n-1} = \frac{n-1-l}{n-1} = 1 - \frac{l}{n-1} = 1 - o(1). \tag{52}$$

6.2. **Case B (d-ary increasing trees).**

(i) First we consider the region $j = o(n)$ and obtain

$$\mathbb{P}\{X_{n,j} = d\} = 1 + \sum_{k=0}^{d-1}\binom{d}{k}(-1)^{d-k}\frac{\binom{j-1+\frac{1}{d-1}}{j-2+\frac{k}{d-1}}}{\binom{n-1+\frac{1}{d-1}}{n-2+\frac{k}{d-1}}}. \tag{53}$$

For $0 \le k \le d-1$ it holds:

$$\frac{\binom{j-1+\frac{1}{d-1}}{j-2+\frac{k}{d-1}}}{\binom{n-1+\frac{1}{d-1}}{n-2+\frac{k}{d-1}}} = \frac{(n-2+\frac{k}{d-1})^{\underline{n-j}}}{(n-1+\frac{1}{d-1})^{\underline{n-j}}} \le \frac{(n-1)^{\underline{n-j}}}{(n-1+\frac{1}{d-1})^{\underline{n-j}}} = \frac{(n-1)!(j-1+\frac{1}{d-1})!}{(n-1+\frac{1}{d-1})!(j-1)!}.$$

Now we split the region $j = o(n)$ into two cases: $j \le \log n$ and $j > \log n$. First we consider $j \le \log n$ and use that for $j \ge 2$:

$$\frac{(j-1+\frac{1}{d-1})!}{(j-1)!} = \frac{(j-1+\frac{1}{d-1})!j}{j!} \le j \le \log n = \mathcal{O}(\log n),$$

where the $O(\log n)$ bound holds of course also for $j = 1$. Together with

$$\frac{(n-1+\frac{1}{d-1})!}{(n-1)!} = n^{\frac{1}{d-1}}\Big(1+\mathcal{O}(\frac{1}{n})\Big),$$

we obtain thus

$$\frac{(n-1)!(j-1+\frac{1}{d-1})!}{(n-1+\frac{1}{d-1})!(j-1)!} = \mathcal{O}\Big(\frac{\log n}{n^{\frac{1}{d-1}}}\Big),$$

and further

$$\sum_{k=0}^{d-1}\binom{d}{k}(-1)^{d-k}\frac{\binom{j-1+\frac{1}{d-1}}{j-2+\frac{k}{d-1}}}{\binom{n-1+\frac{1}{d-1}}{n-2+\frac{k}{d-1}}} = \mathcal{O}\Big(\frac{\log n}{n^{\frac{1}{d-1}}}\Big). \tag{54}$$

For $j > \log n$ we simply use

$$\frac{(j-1+\frac{1}{d-1})!}{(j-1)!} = j^{\frac{1}{d-1}}\Big(1+\mathcal{O}(\frac{1}{j})\Big),$$

which gives

$$\frac{(n-1)!(j-1+\frac{1}{d-1})!}{(n-1+\frac{1}{d-1})!(j-1)!} = \big(\frac{j}{n}\big)^{\frac{1}{d-1}}\Big(1+\mathcal{O}\big(\frac{1}{j}\big)+\big(\frac{1}{n}\big)\Big),$$

and further

$$\sum_{k=0}^{d-1}\binom{d}{k}(-1)^{d-k}\frac{\binom{j-1+\frac{1}{d-1}}{j-2+\frac{k}{d-1}}}{\binom{n-1+\frac{1}{d-1}}{n-2+\frac{k}{d-1}}} = \mathcal{O}\Big(\big(\frac{j}{n}\big)^{\frac{1}{d-1}}\Big). \tag{55}$$

Combining these two cases equations (54) and (55) yields from (53)

$$\mathbb{P}\{X_{n,j}=d\} = 1+\mathcal{O}\Big(\big(\frac{j}{n}\big)^{\frac{1}{d-1}}\Big)+\mathcal{O}\Big(\frac{\log n}{n^{\frac{1}{d-1}}}\Big) \to 1. \tag{56}$$

(*ii*) For the region j: $j\to\infty$ such that $j=\rho n$, with $0<\rho<1$, we can apply Stirling's formula (39) and get

$$\begin{aligned}\mathbb{P}\{X_{n,j}=m\} &= \binom{d}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{\Gamma(n-1+\frac{k}{d-1})\Gamma(\rho n+\frac{1}{d-1})}{\Gamma(\rho n-1+\frac{k}{d-1})\Gamma(n+\frac{1}{d-1})}\\ &= \binom{d}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\rho^{1-\frac{k-1}{d-1}}\Big(1+\mathcal{O}\big(\frac{1}{n}\big)\Big)\\ &= \binom{d}{m}\rho^{1+\frac{1}{d-1}}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\rho^{-\frac{k}{d-1}}+\mathcal{O}\big(\frac{1}{n}\big) = \binom{d}{m}\rho^{1+\frac{1}{d-1}}\big(\rho^{-\frac{1}{d-1}}-1\big)^m+\mathcal{O}\big(\frac{1}{n}\big)\\ &= \binom{d}{m}\Big(1-\rho^{\frac{1}{d-1}}\Big)^m\Big(\rho^{\frac{1}{d-1}}\Big)^{d-m}+\mathcal{O}\big(\frac{1}{n}\big).\end{aligned} \tag{57}$$

(*iii*) For the region j: $l:=n-j=o(n)$ we finally compute

$$\mathbb{P}\{X_{n,j}=0\} = \frac{\Gamma(n-1)\Gamma(j+\frac{1}{d-1})}{\Gamma(j-1)\Gamma(n+\frac{1}{d-1})} = \frac{(n-2)^{\underline{l}}}{(n-1+\frac{1}{d-1})^{\underline{l}}} \geq \frac{(n-2)^{\underline{l}}}{(n-1)^{\underline{l}}} = \frac{j-1}{n-1} = 1-\frac{l}{n-1}\to 1. \tag{58}$$

6.3. **Case C (generalized plane-oriented recursive trees).**

(i') Since we already know the factorial moments of $X_{n,j}$ from Subsection 5.2, we can express the ordinary moments by using the Stirling numbers of the second kind $\left\{{n \atop k}\right\}$.

$$\mathbb{E}\big(X_{n,j}^s\big) = \mathbb{E}\big(X_{n,j}^{\underline{s}}\big)+\sum_{k=1}^{s-1}\left\{{s \atop k}\right\}\mathbb{E}\big(X_{n,j}^{\underline{k}}\big). \tag{59}$$

Using again Stirling's formula for the Gamma function (39) we get for the factorial moments given by Theorem 1 the asymptotic expansion

$$\mathbb{E}(X_{n,j}^{\underline{s}}) = \frac{\Gamma(s-1-\frac{c_1}{c_2})\Gamma(j+\frac{c_2}{c_1})}{\Gamma(-1-\frac{c_1}{c_2})\Gamma(j-\frac{(s-1)c_2}{c1})}n^{-\frac{c_2}{c_1}s}+\mathcal{O}(n^{-\frac{c_2}{c_1}(s-1)}), \tag{60}$$

which leads due to (59) also to an asymptotic expansion of the ordinary s-th moment. Thus the moments of $n^{\frac{c_2}{c_1}}X_{n,j}$ converge to a random variable X_j with s-th moments

$$\mathbb{E}(X_j^s) = \frac{\Gamma(s-1-\frac{c_1}{c_2})\Gamma(j+\frac{c_2}{c_1})}{\Gamma(-1-\frac{c_1}{c_2})\Gamma(j-\frac{(s-1)c_2}{c1})}. \tag{61}$$

Due to simple growth estimates for these moments it follows that they are fully characterizing the distribution and an application of the Theorem of Fréchet and Shohat (see e. g. [7]) shows then (11).

(i'') For the region $j \to \infty$ such that $j = o(n)$ we proceed exactly as before arriving at $\left(\frac{n}{j}\right)^{\frac{c_2}{c_1}} X_{n,j} \xrightarrow{(d)} X \stackrel{(d)}{=} \gamma(a, \lambda)$, where X is Gamma distributed with parameters $a = -\frac{c_1}{c_2} - 1$ and $\lambda = 1$, i. e.

$$\mathbb{E}(X) = \frac{\Gamma(s - 1 - \frac{c_1}{c_2})}{\Gamma(-1 - \frac{c_1}{c_2})}. \tag{62}$$

(ii) For the region $j \to \infty$ such that $j = \rho n$, with $0 < \rho < 1$, we get after applying (39) to the formula for the probabilities $\mathbb{P}\{X_{n,j} = m)$ given by Theorem 1 the asymptotic expansion

$$\mathbb{P}\{X_{n,j} = m\} = \binom{m - 2 - \frac{c_1}{c_2}}{m} \rho^{1+\frac{c_2}{c_1}} \big(1 - \rho^{-\frac{c_2}{c_1}}\big)^m + \mathcal{O}(\frac{1}{n}), \tag{63}$$

which proves the given result.

(iii) The result corresponding to the region $l := n - j = o(n)$ follows by a similar observation.

6.4. **The density appearing in Case C (generalized plane-oriented recursive trees) for j fixed.** In Subsection 6.3 we have shown that for j fixed the normalized r. v. $n^{\frac{c_2}{c_1}} X_{n,j}$ has a limiting distribution, which can be characterized by the moments given in equation (11). In the following we will figure out very briefly how one can characterize this limiting distribution also by its density. In principle one could show even more, namely local limit theorems (the normalized probability $n^{-\frac{c_2}{c_1}} \mathbb{P}\{X_{n,j} = xn^{-\frac{c_2}{c_1}}\}$ converge to the density $f_j(x)$), but we will restrict ourselves to show that the distribution characterized by the density given in equation (12) has the right moments.

The density $f_1(x)$ can be derived from the generating function (29) by using Cauchy's integration formula with a Hankel contour like integration path. We will not figure out this but refer to [3, 11] for applications of that technique to similar types of generating functions. This leads to the following formula for the density $f_1(x)$ of the limiting distribution of the out-degree of the root:

$$f_1(x) = \frac{-\Gamma(\frac{c_2}{c_1}) x^{-2-\frac{c_1}{c_2}}}{\Gamma(-1 - \frac{c_1}{c_2})\pi} \int_0^\infty e^{-(r^{-\frac{c_1}{c_2}})} r^{-\frac{c_1}{c_2}-1} e^{-xr\cos(\frac{c_2}{c_1}\pi)} \sin\big(xr \sin(-\frac{c_2}{c_1}\pi)\big) dr. \tag{64}$$

To obtain the density $f_j(x)$ we combine a previous result (obtained in [6]) about the random variable $D_{n,j}$, which counts the size of the subtree rooted at node j in a random increasing tree of size n, with the result obtained for the shape of $f_1(x)$. One knows (see [6] for details), that, for j fixed, it holds $\frac{D_{n,j}}{n} \xrightarrow{(d)} \beta(\frac{c_2}{c_1} + 1, j - 1)$, where $\beta(a, b)$ denotes a Beta distribution with parameter a and b. One cannot only show convergence in distribution but even local limit law, which gives

$$\lim_{n\to\infty} n\mathbb{P}\{\frac{D_{n,j}}{n} = t\} = \frac{t^{-\frac{c_2}{c_1}} (1 - t)^{j-2}}{\mathrm{B}(\frac{c_2}{c_1} + 1, j - 1)}, \tag{65}$$

where $\mathrm{B}(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ denotes the Beta function of p and q. By conditioning on the size of the subtree rooted at node j we get

$$\mathbb{P}\{X_{n,j} = m\} = \sum_{k\ge 0} \mathbb{P}\{X_{k,1} = m | D_{n,j} = k\} \mathbb{P}\{D_{n,j} = k\} = \sum_{k\ge 0} \mathbb{P}\{X_{k,1} = m\} \mathbb{P}\{D_{n,j} = k\}. \tag{66}$$

A standard argument leads then to the following formula for the density $f_j(x)$ of the limiting distribution of the out-degree of node j:

$$f_j(x) = \frac{\Gamma(j + \frac{c_2}{c_1})}{\Gamma(1 + \frac{c_2}{c_1})\Gamma(j - 1)} \int_0^1 t^{2\frac{c_2}{c_1}} (1 - t)^{j-2} f_1\big(xt^{\frac{c_2}{c_1}}\big) dt, \tag{67}$$

where $f_1(x)$ is given by (64).

To show that the distribution characterized by the density $f_1(x)$ has indeed the required moments we proceed as follows.

$$\begin{aligned}
\int_0^\infty x^s f(x)dx &= \frac{-\Gamma(\frac{c_2}{c_1})}{\Gamma(-1-\frac{c_1}{c_2})\pi}\int_0^\infty e^{-(r^{-\frac{c_1}{c_2}})} r^{-\frac{c_1}{c_2}-1}\int_0^\infty x^{s-2-\frac{c_1}{c_2}} e^{-xr\cos(\frac{c_2}{c_1}\pi)} \sin\big(xr\sin(-\frac{c_2}{c_1}\pi)\big)dx\,dr\\
&= \Im\bigg(\frac{-\Gamma(\frac{c_2}{c_1})}{\Gamma(-1-\frac{c_1}{c_2})\pi}\int_0^\infty e^{-(r^{-\frac{c_1}{c_2}})} r^{-\frac{c_1}{c_2}-1}\int_0^\infty x^{s-2-\frac{c_1}{c_2}} e^{-xr\cos(\frac{c_2}{c_1}\pi)} e^{ixr\sin(-\frac{c_2}{c_1}\pi)}dx\,dr\bigg)\\
&= \Im\bigg(\frac{-\Gamma(\frac{c_2}{c_1})}{\Gamma(-1-\frac{c_1}{c_2})\pi}\int_0^\infty e^{-(r^{-\frac{c_1}{c_2}})} r^{-\frac{c_1}{c_2}-1}\int_0^\infty x^{s-2-\frac{c_1}{c_2}} e^{-xr(e^{i\frac{c_2}{c_1}\pi})}dx\,dr\bigg),
\end{aligned} \tag{68}$$

Using the substitution $u = xre^{i\frac{c_2}{c_1}\pi}$, $\frac{du}{dx} = re^{i\frac{c_2}{c_1}\pi}$ we further get

$$\begin{aligned}
\int_0^\infty x^s f_1(x)dx &= \Im\bigg(\frac{-\Gamma(\frac{c_2}{c_1})}{\Gamma(-1-\frac{c_1}{c_2})\pi}\int_0^\infty e^{-(r^{-\frac{c_1}{c_2}})} r^{-\frac{c_1}{c_2}-1}\int_0^\infty \big(\frac{ue^{-i\frac{c_2}{c_1}\pi}}{r}\big)^{s-2-\frac{c_1}{c_2}} e^{-u}\frac{e^{-i\frac{c_2}{c_1}\pi}}{r}du\,dr\bigg)\\
&= \Im\bigg(\frac{-\Gamma(\frac{c_2}{c_1})e^{-\frac{c_2}{c_1}\pi i(s-1-\frac{c_1}{c_2})}}{\Gamma(-1-\frac{c_1}{c_2})\pi}\int_0^\infty e^{-(r^{-\frac{c_1}{c_2}})} r^{-s}\int_0^\infty u^{s-2-\frac{c_1}{c_2}} e^{-u}du\,dr\bigg)\\
&= \frac{-\Gamma(\frac{c_2}{c_1})\sin(1-\frac{c_2}{c_1}(s-1))\Gamma(s-1-\frac{c_1}{c_2})}{\Gamma(-1-\frac{c_1}{c_2})\pi}\int_0^\infty e^{-(r^{-\frac{c_1}{c_2}})} r^{-s}dr\\
&= \frac{-\Gamma(\frac{c_2}{c_1})\sin(1-\frac{c_2}{c_1}(s-1))\Gamma(s-1-\frac{c_1}{c_2})(-\frac{c_2}{c_1})\Gamma(\frac{c_2}{c_1}(s-1))}{\Gamma(-1-\frac{c_1}{c_2})\pi}\\
&= \frac{\Gamma(s-1-\frac{c_1}{c_2})\Gamma(1+\frac{c_2}{c_1})}{\Gamma(-1-\frac{c_1}{c_2})\Gamma(1-\frac{(s-1)c_2}{c1})},
\end{aligned} \tag{69}$$

where we have used the identity $\frac{\pi}{\sin(\pi x)} = \Gamma(x)\Gamma(1-x)$. Thus we have indeed obtained the moments given by (11).

Using the substitution $u = xt^{\frac{c_2}{c_1}}$ one can also show that the distribution characterized by the density $f_j(x)$ has the required moments (as given by (11)).

6.5. **Results for random nodes.** To show the limiting distribution results for X_n, which counts the out-degree of a random node in a randomly chosen tree of size n, as given by Corollary 1 we use the limiting distribution results for the central region of $X_{n,j}$, i. e. $j = \rho n$, with $0 < \rho < 1$, as given in Theorem 2-4. Since $X_{n,j} \xrightarrow{(d)} X_\rho$ we obtain $X_n \xrightarrow{(d)} X$, where the probabilities $\mathbb{P}\{X = m\}$ of the discrete r. v. X can be obtained via

$$\mathbb{P}\{X = m\} = \int_0^1 \mathbb{P}\{X_\rho = m\}\,d\rho. \tag{70}$$

We obtain in all three cases of grown simple families of increasing trees closed formulæ for these integrals.

Case A (recursive trees):

$$\begin{aligned}
\mathbb{P}\{X = m\} &= \int_0^1 \frac{\rho(-\log\rho)^m}{m!}\,d\rho = \frac{1}{m!}\int_0^\infty e^{-2u}u^m du = \frac{1}{2^{m+1}m!}\int_0^\infty e^{-t}t^m dt = \frac{\Gamma(m+1)}{2^{m+1}m!}\\
&= \frac{1}{2^{m+1}}, \quad \text{for } m \geq 0.
\end{aligned}$$

Case B (d-ary increasing trees):

$$\mathbb{P}\{X = m\} = \int_0^1 \binom{d}{m}(1-\rho^{\frac{1}{d-1}})^m(\rho^{\frac{1}{d-1}})^{d-m}d\rho = \binom{d}{m}(d-1)\int_0^1 t^{2d-m-2}(1-t)^m dt$$

$$= \binom{d}{m}(d-1)B(m+1,2d-m-1) = \binom{d}{m}(d-1)\frac{\Gamma(m+1)\Gamma(2d-m-1)}{\Gamma(2d)}$$
$$= \frac{d-1}{2d-1}\frac{\binom{d}{m}}{\binom{2d-2}{m}}, \quad \text{for } 0 \le m \le d.$$

Case C (generalized plane-oriented recursive trees):

$$\mathbb{P}\{X=m\} = \int_0^1 \binom{m-2-\frac{c_1}{c_2}}{m}\rho^{1+\frac{c_2}{c_1}}(1-\rho^{-\frac{c_2}{c_1}})^m d\rho = \binom{m-2-\frac{c_1}{c_2}}{m}\big(-\frac{c_1}{c_2}\big)\int_0^1 t^{-\frac{2c_1}{c_2}-2}(1-t)^m dt$$
$$= \binom{m-2-\frac{c_1}{c_2}}{m}\big(-\frac{c_1}{c_2}\big)B\big(-\frac{2c_1}{c_2}-1,m+1\big) = \binom{m-2-\frac{c_1}{c_2}}{m}\big(-\frac{c_1}{c_2}\big)\frac{\Gamma(-\frac{2c_1}{c_2}-1)\Gamma(m+1)}{\Gamma(-\frac{2c_1}{c_2}+m)}$$
$$= \frac{-\frac{c_1}{c_2}}{-\frac{2c_1}{c_2}-1}\frac{\binom{m-2-\frac{c_1}{c_2}}{m}}{\binom{m-1-\frac{2c_1}{c_2}}{m}} = \frac{(\alpha+1)\Gamma(\alpha+m)\Gamma(2\alpha+1)}{\Gamma(2\alpha+m+2)\Gamma(\alpha)}, \quad \text{for } m \ge 0,$$

with $\alpha := -\frac{c_1}{c_2} - 1$.

7. Conclusion

With our methods a full characterization of the limiting distribution of the parameter $X_{n,j}$, the out-degree of node j in a random tree of size n, was obtained for all grown simple families of increasing trees. Table 1 summarizes for sequences (n, j) the phase change behaviour of $X_{n,j}$ depending on the growth of $j = j(n)$ compared to n for all three possible cases.

	recursive trees (Case A)	d-ary increasing trees (Case B)	generalized plane-oriented trees (Case C)
j fixed	Gaussian	degenerate	characterized via moments
$j \to \infty$, $j = o(n)$			Gamma
$j \sim \rho n$, $0 < \rho < 1$	Poisson	binomial	negative binomial
$l := n - j = o(n)$	degenerate	degenerate	degenerate

Table 1. The limiting distribution behaviour of the r. v. $X_{n,j}$, which gives the out-degree of node j in a randomly chosen grown increasing tree of size n.

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