ON FUNCTIONS OF ARAKAWA AND KANEKO AND MULTIPLE ZETA FUNCTIONS

MARKUS KUBA

ABSTRACT. We study for $s \in \mathbb{N}$ the functions $\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t-1} \operatorname{Li}_k(1-e^{-t})dt$, and more generally $\xi_{k_1,\dots,k_r}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t-1} \operatorname{Li}_{k_1,\dots,k_r}(1-e^{-t})dt$, introduced by Arakawa and Kaneko [2] and relate them with (finite) multiple zeta functions, partially answering a question of [2]. In particular, we give an alternative proof of a result of Ohno [8].

1. INTRODUCTION

Let $\operatorname{Li}_{k_1,\dots,k_r}(z)$ denote the multiple polylogarithm function defined by

$$\operatorname{Li}_{k_1,\dots,k_r}(z) = \sum_{n_1>n_2>\dots>n_r\geq 1} \frac{z^{n_1}}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}},$$

with $k_1 \in \mathbb{N} \setminus \{1\}$ and $k_i \in \mathbb{N} = \{1, 2, \dots\}$, $2 \leq i \leq r$, and $|z| \leq 1$. For $z = 1$ the multiple polylogarithm function $\operatorname{Li}_{k_1,\dots,k_r}(1) = \zeta(k_1, \dots, k_r)$ simplifies to a multiple zeta function, also called multiple zeta value, where $\zeta(k_1, \dots, k_r)$ and $\zeta_N(k_1, \dots, k_r)$ denote the (finite) multiple zeta function defined by

$$\zeta(k_1, \dots, k_r) = \sum_{n_1>n_2>\dots>n_r\geq 1} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}},$$

$$\zeta_N(k_1, \dots, k_r) = \sum_{N\geq n_1>n_2>\dots>n_\ell\geq 1} \frac{1}{n_1^{ka_1} n_2^{k_2} \dots n_r^{k_r}},$$

with $k_1 \in \mathbb{N} \setminus \{1\}$, and $k_2, \dots, k_r \in \mathbb{N}$ for the infinite series and $N, k_1, \dots, k_r \in \mathbb{N}$ for the finite counterpart. Arakawa and Kaneko [2] introduced the function $\xi_k(s)$, and the more general function $\xi_{k_1,\dots,k_r}(s)$, defined by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \operatorname{Li}_k(1 - e^{-t})dt,$$

$$\xi_{k_1,\dots,k_r}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \operatorname{Li}_{k_1,\dots,k_r}(1 - e^{-t})dt,$$

respectively, being absolut convergent for $\Re(s) > 0$, and related them for special choices of s and $k_1, \dots k_r$ to multiple zeta functions. Ohno [8] obtained a result for $\xi_k(n)$, with $n \in \mathbb{N}$, using his generalization of the duality and sum formulas for multiple zeta functions.

We will provide for $n \in \mathbb{N}$ evaluations of the function $\xi_{k_1,\dots,k_r}(n)$ to multiple zeta functions, partially answering a question of [2]; in particular we give a short and simple proof of Ohno's result [8]. For the evaluation of the general case $\xi_{k_1,\dots,k_r}(n)$ we use a finite version of the so-called stuffle identity for multiple zeta functions. Subsequently, we will utilise a variant of (finite) multiple zeta functions, called the multiple zeta star functions or non-strict multiple zeta functions $\zeta_N^*(k_1, \dots, k_r)$, which recently attracted

2000 *Mathematics Subject Classification.* 11M06, 40B05.
Key words and phrases. Multiple zeta functions, finite multiple zeta functions, multiple zeta star functions, finite stuffle identity.
This work was supported by the Austrian Science Foundation FWF, grant S9608-N13.

some interest,[1, 8, 10, 9, 6, 7, 5, 11] where the summation indices satisfy $N \geq n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$ in contrast to $N \geq n_1 > n_2 > \cdots > n_r > 1$, as in the usual definition (1),

$$\zeta_N^*(k_1,\dots,k_r) = \sum_{N\geq n_1\geq n_2\geq\cdots\geq n_r\geq 1} \frac{1}{n_1^{k_1}n_2^{k_2}\dots n_r^{k_r}},$$

with $N, k_1, \dots, k_r \in \mathbb{N}$. The star form can be converted into ordinary finite multiple zeta functions by considering all possible deletions of commas, i.e.

$$\zeta_N^*(k_1,\dots,k_r) = \sum_{h=1}^{r} \sum_{1\leq \ell_1<\ell_2<\cdots<\ell_{h-1}<r} \zeta_N\Big(\sum_{i_1=1}^{\ell_1} k_{i_1}, \sum_{i_2=\ell_1+1}^{\ell_2} k_{i_2}, \dots, \sum_{i_h=\ell_{h-1}+1}^{r} k_{i_h}\Big); \tag{1}$$

note that the first term $h = 1$ should be interpreted as $\zeta_N(\sum_{i_1=\ell_0+1}^{r} k_{i_1})$, subject to $\ell_0 = 0$. The notation $\zeta_N^*(k_1,\dots,k_r)$ is chosen in analogy with Aoki and Ohno [1] where infinite counterparts of $\zeta_N^*(k_1,\dots,k_r)$ have been treated. First we will study the instructive case of $\xi_k(n)$, reproving the result of Ohno. Then we will state our main result concerning the evaluation of $\xi_{k_1,\dots,k_r}(n)$ into multiple zeta functions.

2. A SIMPLE EVALUATION OF $\xi_k(n)$

Ohno [8] evaluated the sum $\xi_k(n)$ for $k, n \in \mathbb{N}$ applying his generalization of the duality and sum formulas for multiple zeta functions to a result of Arakawa and Kaneko [2]. In the following we will give an alternative simple and self-contained derivation of his result, stated in Theorem 1. In order to evaluate $\xi_k(n)$ for $k, n \in \mathbb{N}$ we only use the two basic facts stated below.

$$\begin{aligned} &\frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-t\ell}dt = \frac{1}{\ell^n}, \quad \text{for} \quad \ell, n \in \mathbb{N},\\ &\sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}}{\ell^r} = \zeta_n^*(\underbrace{1,\dots,1}_{r}) = \zeta_n^*(\{1\}_r) \quad \text{for} \quad r \in \mathbb{N}. \end{aligned} \tag{2}$$

The second identity can be immediately deduced by repeated usage of the formula $\binom{n}{k} = \sum_{\ell=k}^{n}\binom{\ell-1}{k-1}$. We proceed as follows.

$$\xi_k(n) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{t^{n-1}}{e^t-1}\,\mathrm{Li}_k(1-e^{-t})dt = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-t}\sum_{m\geq 1}\frac{(1-e^{-t})^{m-1}}{m^k}dt.$$

We expand $(1-e^{-t})^{m-1}$ by the binomial theorem and interchange summation and integration. According to (2) we obtain

$$\begin{aligned} \xi_k(n) &= \sum_{m\geq 1}\frac{1}{m^k}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}\frac{(-1)^\ell}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-(\ell+1)t}dt\\ &= \sum_{m\geq 1}\frac{1}{m^k}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}\frac{(-1)^\ell}{(\ell+1)^n}. \end{aligned}$$

Since $\binom{m-1}{\ell} = \binom{m}{\ell+1}\frac{\ell+1}{m}$, we get according to (2) after an index shift the following result

Theorem 1 (Ohno [8]). *For $k, n \in \mathbb{N}$ the function $\xi_k(n)$ is given by*

$$\xi_k(n) = \sum_{m_1\geq m_2\geq\cdots\geq m_n\geq 1}\frac{1}{m_1^{k+1}m_2\dots m_n} = \zeta^*(k+1,\{1\}_{n-1}).$$

Note that one can convert the multiple star zeta function above into ordinary multiple zeta functions according to (1) (with respect to the corresponding relation for infinite series), or can directly simplify the multiple zeta star function using (cycle) sum formulas, see i.e. Ohno and Wakabayashi [10] or Ohno and Okuda [9].

3. GENERAL CASE

In the general case of $\xi_{k_1,\dots,k_r}(n)$ we will prove the following result.

Theorem 2. *For $k_1,\dots,k_r,n\in\mathbb{N}$ the function $\xi_{k_1,\dots,k_r}(n)$ is given by*

$$\xi_{k_1,\dots,k_r}(n)=\sum_{n_1\geq 1}\frac{\zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\dots,k_r)}{n_1^{k_1+1}}.$$

Furthermore, $\xi_{k_1,\dots,k_r}(n)$ can be evaluated into sums of multiple zeta functions.

The explicit evaluation of $\xi_{k_1,\dots,k_r}(n)$ will be given in Corollary 1. We have

$$\begin{aligned}\xi_{k_1,\dots,k_r}(n)&=\frac{1}{\Gamma(n)}\int_0^\infty\frac{t^{n-1}}{e^t-1}\operatorname{Li}_{k_1,\dots,k_r}(1-e^{-t})dt\\&=\frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-t}\sum_{n_1>n_2>\dots>n_r\geq 1}\frac{(1-e^{-t})^{n_1-1}}{n_1^{k_1}n_2^{k_2}\dots n_r^{k_r}}dt.\end{aligned}$$

Proceeding as before we expand $(1-e^{-t})^{n_1-1}$ by the binomial theorem and interchange summation and integration. We get

$$\xi_{k_1,\dots,k_r}(n)=\sum_{n_1>n_2>\dots>n_r\geq 1}\frac{1}{n_1^{k_1}n_2^{k_2}\dots n_r^{k_r}}\sum_{\ell=0}^{n_1-1}\binom{n_1-1}{\ell}\frac{(-1)^\ell}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-(\ell+1)t}dt.$$

According to (2) we obtain

$$\xi_{k_1,\dots,k_r}(n)=\sum_{n_1>n_2>\dots>n_r\geq 1}\frac{\zeta^*_{n_1}(\{1\}_{n-1})}{n_1^{k_1+1}n_2^{k_2}\dots n_r^{k_r}}=\sum_{n_1\geq 1}\frac{\zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\dots,k_r)}{n_1^{k_1+1}}.$$

This proves the first part of Theorem 2.

Concerning the second part we proceed as follows. We will evaluate the product S of finite multiple zeta (star) functions

$$S=S_{n_1}(n,k_2,\dots,k_2)=\zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\dots,k_r)$$

into sums of finite multiple zeta functions $\zeta_{n_1-1}(\mathbf{f})$, for some $\mathbf{f}=(f_1,\dots,f_j)$, with $f_i\in\mathbb{N}$, $1\leq i\leq j$, which will prove the second part of the stated result. By (1) we can write $\zeta^*_{n_1}(\{1\}_{n-1})$ in terms of ordinary finite multiple zeta functions

$$\zeta^*_{n_1}(\{1\}_{n-1})=\sum_{h=1}^{n-1}\sum_{1\leq\ell_1<\ell_2<\dots<\ell_{h-1}<n-1}\zeta_{n_1}\big(\ell_1,\ell_2-\ell_1,\dots,n-\ell_{h-1}-1\big);$$

for example $\zeta^*_{n_1}(\{1\}_3)=\zeta_{n_1}(3)+\zeta_{n_1}(1,2)+\zeta_{n_1}(2,1)+\zeta_{n_1}(1,1,1)$. We can convert finite zeta functions $\zeta_N(a_1,\dots,a_r)$ into finite zeta functions $\zeta_{N-1}(b_1,\dots,b_s)$ by

$$\zeta_N(a_1,\dots,a_r)=\zeta_{N-1}(a_1,\dots,a_r)+\frac{1}{N^{a_1}}\zeta_{N-1}(a_2,\dots,a_r).$$

Consequently, we can express the product $S = \zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\dots,k_r)$ of finite multiple zeta (star) functions in the following way.

$$S = \sum_{h=1}^{n-1} \sum_{1\le \ell_1<\ell_2<\dots<\ell_{h-1}<n-1} \zeta_{n_1-1}\big(\ell_1,\ell_2-\ell_1,\dots,n-\ell_{h-1}-1\big)\zeta_{n_1-1}(k_2,\dots,k_r)$$
$$+\sum_{h=1}^{n-1} \sum_{1\le \ell_1<\ell_2<\dots<\ell_{h-1}<n-1} \frac{\zeta_{n_1-1}\big(\ell_2,\dots,n-\ell_{h-1}-1\big)\zeta_{n_1-1}(k_2,\dots,k_r)}{n_1^{\ell_1}}\Big). \tag{3}$$

Now we use finite versions of so-called *stuffle identities*, see i.e. Borwein et al. [3]. Stuffle identities provide evaluation of products of multiple zeta functions $\zeta(\mathbf{k})\zeta(\mathbf{h})$ into into sums of multiple zeta functions $\zeta(\mathbf{k})\zeta(\mathbf{h}) = \sum_{\mathbf{f}\in\text{stuffle}(\mathbf{k},\mathbf{h})}\zeta(\mathbf{f})$; here $\zeta(\mathbf{k}) = \zeta(k_1,\dots,k_r)$ and $\zeta(\mathbf{h}) = \zeta(h_1,\dots,h_s)$. For our problem we need finite versions of the stuffle identities, providing evaluations of products of finite multiple zeta functions $\zeta_N(\mathbf{k})\zeta_N(\mathbf{h})$ into into sums of finite multiple zeta functions $\zeta_N(\mathbf{k})\zeta_N(\mathbf{h}) = \sum_{\mathbf{f}\in\text{stuffle}(\mathbf{k},\mathbf{h})}\zeta_N(\mathbf{f})$. This would lead then to an evaluation of $\xi_{k_1,\dots,k_r}(n)$ into single finite multiple zeta functions.

Following [3] we define for two given strings $\mathbf{k} = (k_1,\dots,k_r)$ and $\mathbf{h} = (h_1,\dots,h_s)$ the set $\text{stuffle}(\mathbf{k},\mathbf{h})$ as the smallest set of strings over the alphabet $\mathcal{A}$, defined by

$$\mathcal{A} = \{k_1,\dots,k_r,h_1,\dots,h_s, \text{“}+\text{''}, \text{“},\text{''}, \text{“}(\text{''}, \text{“})\text{''}\}$$

satisfying $(\mathbf{k},\mathbf{h}) = (k_1,\dots,k_r,h_1,\dots,h_s)\in\text{stuffle}(\mathbf{k},\mathbf{h})$ and further if a string of the form $(U,k_n,h_m,V)\in\text{stuffle}(\mathbf{k},\mathbf{h})$, then so are the strings $(U,h_m,k_n,V)\in\text{stuffle}(\mathbf{k},\mathbf{h})$ and $(U,k_n+h_m,V)\in\text{stuffle}(\mathbf{k},\mathbf{h})$. Stuffle identities arise from the definition of (finite) multiple zeta functions in terms of sums; the term *stuffle* derives from the manner in which the two upper strings are combined. Other closely related identities are due to different representations of multiple zeta functions (see for example [3]). We will use the following result

Lemma 1 (Stuffle identity; finite version)**.** *Let* $\zeta_N(\mathbf{k}) = \zeta_N(k_1,\dots,k_r)$ *and* $\zeta_N(\mathbf{h}) = \zeta_N(h_1,\dots,h_s)$*, with* $N,r,s\in\mathbb{N}$ *and* $k_i,h_j\in\mathbb{N}$*,* $1\le i\le r$*,* $1\le j\le s$*. Then,*

$$\zeta_N(\mathbf{k})\zeta_N(\mathbf{h}) = \sum_{\mathbf{f}\in\text{stuffle}(\mathbf{k},\mathbf{h})}\zeta_N(\mathbf{f}). \tag{4}$$

Remark 1. Note that the conditions on the parameters k_i,h_j can be relaxed in various ways, i.e. $k_i,h_j\in\mathbb{R}^+$. Typical examples of (finite) stuffle identities read as follows,

$$\zeta_N(r)\zeta_N(t) = \zeta_N(r,t)+\zeta_N(t+r)+\zeta_N(t,r),$$
$$\zeta_N(r,s)\zeta_N(t) = \zeta_N(r,s,t)+\zeta_N(r,s+t)+\zeta_N(r,t,s)+\zeta_N(r+t,s)+\zeta_N(t,r,s);$$

we refer to [3] concerning infinite counterparts, and also to Borwein and Girgensohn [4] where the second identity is implicitly derived. Stuffle identities for finite multiple zeta functions seem to be natural, since one does not have to exclude the cases $k_1 = 1$ or $h_1 = 1$ in contrast to infinite multiple zeta functions.

Proof of Lemma 1 (Sketch). In order prove this result in an elementary way, one can use induction with respect to the total length $|\mathbf{k}|+|\mathbf{h}|$. We do not want to give a full proof, since we believe that (4) is already known (although we did not find a suitable reference in the literature for the finite version of the stuffle identity); hence, we only sketch the simple arguments. By definition

$$\zeta_N(\mathbf{k}) = \sum_{N\ge n_1>n_2>\dots>n_r\ge 1}\frac{1}{n_1^{k_1}n_2^{k_2}\dots n_r^{k_r}} = \sum_{n_1=1}^{n}\frac{1}{n_1^{k_1}}\sum_{n_2=1}^{n_1-1}\frac{1}{n_2^{k_2}}\cdots\sum_{n_r=1}^{n_{r-1}-1}\frac{1}{n_r^{k_r}}.$$

Consequently,

$$\zeta_N(\mathbf{k})\zeta_N(\mathbf{h}) = \sum_{n_1=1}^{n} \frac{1}{n_1^{k_1}} \sum_{n_2=1}^{n_1-1} \frac{1}{n_2^{k_2}} \cdots \sum_{n_r=1}^{n_{r-1}-1} \frac{1}{n_r^{k_r}} \sum_{m_1=1}^{N} \frac{1}{m_1^{h_1}} \zeta_{m_1-1}(h_2,\ldots,h_s).$$

Since by definition of $\zeta_N(\mathbf{k})$ the summation ranges are given by $N \geq n_1 > n_2 > \cdots > n_r \geq 1$, we can simple split the summation range of m_1 into the following parts

$$\sum_{m_1=1}^{N} = \sum_{m_1=1}^{n_r-1} + \sum_{m_1=n_r} + \sum_{\ell=1}^{r-1}\Bigg(\sum_{m_1=n_{r+1-\ell}+1}^{n_{r-\ell}-1} + \sum_{m_1=n_{r-\ell}} \Bigg) + \sum_{m_1=n_1+1}^{N} .$$

The terms corresponding to single term sums of the form $m_1 = n_\ell$, $1 \leq \ell \leq r$ are merged into the corresponding sums in $\zeta_N(\mathbf{k})$; then we recursively apply the same procedure to products

$$\zeta_{m_1-1}(k_{\ell+1},\ldots,k_r)\zeta_{m_1-1}(h_2,\ldots,h_s),$$

which are of smaller length $r-\ell+s-1 < r+s$. Concerning the remaining sums we simply interchange summations with the corresponding sums in $\zeta_N(\mathbf{k})\zeta_N(\mathbf{h})$ and repeat the same procedure to the arising products of finite multiple zeta functions, which are also of smaller length. This proves the stated result. □

Remark 2. Evidently, as remarked in [3], the relative order of the two strings is preserved, but elements of the two strings may also be shoved together into a common slot (stuffing), thereby reducing the depth.

Subsequently we use the notation $\ell_n^{[1]} = (\ell_1, \ell_2-\ell_1, \ldots, n-\ell_{h-1}-1)$, $\ell_n^{[2]} = (\ell_2, , \ell 3-\ell_2, \ldots, n-\ell_{h-1}-1)$, and $\mathbf{k} = (k_2,\ldots,k_r)$. For the simplification of $S = \zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\ldots,k_r)$, as given in (3), we apply the stuffle identity of Lemma 1 to values $\zeta_{n_1-1}(\ell_n^{[1]})\zeta_{n_1-1}(\mathbf{k})$ and $\zeta_{n_1-1}(\ell_n^{[2]})\zeta_{n_1-1}(\mathbf{k})$. Hence, $\xi_{k_1,\ldots,k_r}(n)$ can be completely evaluated into finite sums of multiple zeta functions, which proves the second part of Theorem 2. We state the explicit evaluation of $\xi_{k_1,\ldots,k_r}(n)$ in the following corollary.

Corollary 1. *For $k_1,\ldots,k_r,n \in \mathbb{N}$ the function $\xi_{k_1,\ldots,k_r}(n)$ is given by*

$$\begin{aligned}\xi_{k_1,\ldots,k_r}(n) = &\sum_{h=1}^{n-1} \sum_{1\leq \ell_1<\ell_2<\cdots<\ell_{h-1}<n-1} \sum_{\mathbf{f}\in \mathrm{stuffle}(\ell_n^{[1]},\mathbf{k})} \zeta(k_1+1,\mathbf{f}) \\ &+ \sum_{h=1}^{n-1} \sum_{1\leq \ell_1<\ell_2<\cdots<\ell_{h-1}<n-1} \sum_{\mathbf{f}\in \mathrm{stuffle}(\ell_n^{[2]},\mathbf{k})} \zeta(k_1+1+\ell_1,\mathbf{f}),\end{aligned}$$

with respect to the notation $\ell_n^{[1]} = (\ell_1, \ell_2-\ell_1, \ldots, n-\ell_{h-1}-1)$, $\ell_n^{[2]} = (\ell_2, \ell_3-\ell_2, \ldots, n-\ell_{h-1}-1)$, and $\mathbf{k} = (k_2,\ldots,k_r)$.

References

[1] T. Aoki and Y. Ohno, Sum relations for multiple zeta values and connection formulas for the Gauss hypergeometric functions, *Publ. Res. Inst. Math. Sci.*, vol. 41, Number 2, 329–337, 2005.

[2] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions. *Nagoya Math. J.* 153, 189–209, 1999.

[3] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisonek, Special values of multidimensional polylogarithms, *Trans. Amer. Math. Soc.* 353, 907–941, 2001.

[4] J. M. Borwein and Roland Girgensohn, with an appendix by David J. Broadhurst, Evaluation of Triple Euler Sums, *Electronic Journal of Combinatorics* 3, no. 1, #R23, 1996.

[5] M. Eie, W.-C. Liawa and Y. L. Ong, A restricted sum formula among multiple zeta values, *Journal of Number Theory*, vol.129, 908-921, 2009.

[6] M. E. Hoffman, Multiple harmonic series. *Pacific J. Math.* vol. 152 no. 2, 275–290, 1992.

[7] S. Muneta, On some explicit evaluations of multiple star-zeta values, *J. Number Theory* 128, 2538–2548, 2008.

[8] Y. Ohno, A Generalization of the Duality and Sum Formulas on the Multiple Zeta Values, *Journal of Number Theory* 74, 39–43, 1999.

[9] Y. Ohno and J.-I. Okuda, On the sum formula for the q-analogue of non-strict multiple zeta values, *Proceedings of the AMS*, vol. 135, no. 10, 3029-3037, 2007.

[10] Y. Ohno and N. Wakabayashi, Cyclic sum of multiple zeta values, *Acta Arith.* 123, 289–295, 2006.

[11] Y. Ohno and W. Zudilin, Zeta stars, *Commun. Number Theory Phys.* 2, 325–347, 2008.

MARKUS KUBA, INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TECHNISCHE UNIVERSITÄT WIEN, WIEDNER HAUPTSTR. 8-10/104, 1040 WIEN, AUSTRIA

E-mail address: kuba@dmg.tuwien.ac.at