# ANALYSIS OF A GENERALIZED FRIEDMAN'S URN WITH MULTIPLE DRAWINGS

MARKUS KUBA, HOSAM MAHMOUD, AND ALOIS PANHOLZER

ABSTRACT. We study a generalized Friedman's urn model with multiple drawings of white and blue balls. After a drawing, the replacement follows a policy of opposite reinforcement. We give the exact expected value and variance of the number of white balls after a number of draws, and determine the structure of the moments. Moreover, we obtain a strong law of large numbers, and a central limit theorem for the number of white balls. Interestingly, the central limit theorem is obtained combinatorially via the method of moments and probabilistically via martingales. We briefly discuss the merits of each approach. The connection to a few other related urn models is briefly sketched.

## 1. INTRODUCTION

In the classic theory of urn models, balls are drawn one at a time. In these classical models, square ball replacement matrices underly the random structures, and their eigenvalues play a significant role in the formulation of asymptotic results. For background see [9], and [11]–[13]. In recent years, several new theoretical studies and applications required the consideration of models with multiple drawing (drawing multiple balls each time). The theoretical studies included [2] and [7]. The applications include modeling logic circuits; see [1] and [15]. For these applications, the underlying ball replacement matrices are rectangular, and eigenvalue techniques are harder to formulate.

In this article, we consider a generalization to Friedman's urn, a classic urn first introduced in [4], which covered a range of combinatorial aspects (see [3] for an asymptotic theory). The classic Friedman's urn model is an urn containing up to two colors (say white, $W$, and blue, $B$), and at each time epoch one ball is drawn, then placed back in the urn together with a ball of the opposite color. We call the actions taken *opposite reinforcement*. We look at a generalized Friedman's urn, from which samples of a given size (say $s \geq 1$ balls) are taken out of the urn, and the colors of the balls in the sample are noted. A drawn sample is returned to the urn, and opposite reinforcement takes place: For every white ball in the sample, the urn is reinforced with $C \in \mathbb{N}$ blue balls, and vice versa, for every blue ball in the sample, the urn is reinforced with $C$ white ball. This is to be contrasted with Chen and Wei's semblance reinforcement [2], in which balls of the same color are added. The classic Friedman's urn is the case $C = s = 1$. Let $W_n$ be the number of white balls in the urn after $n$ (multiple) draws. As each draw adds $Cs$ balls, $T_n$, the total number of balls after $n$ draws is given by

$$T_n = Csn + T_0, \qquad \text{for } n \geq 0. \tag{1}$$

How is the sample taken? The basic sampling techniques are to take samples without replacement or with replacement. We shall take up in detail the model of sampling without replacement, and we shall return in a later section of the paper to the case of sampling with replacement; we shall see that the

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growth of a generalized Friedman's urn under either sampling technique is essentially equivalent in some exact and all asymptotic results. We emphasize here the meaning of sampling a set of $s$ balls without replacement. This means that the $s$ balls are obtained randomly, one at a time, and a ball taken out is kept outside until all the other members of a sample draw are taken from the urn. In other words, in the $n$th drawing, the sample is obtained by picking a ball at random from among the $T_{n-1}$ balls in the urn and set aside, then a second ball is obtained at random from among the remaining $T_{n-1} - 1$ balls in the urn and set aside, and so forth until a sample of size $s$ is completed, and that is when we put the sample back in the urn with opposite reinforcement.

Alternatively, under sampling with replacement, in the $n$th drawing, the sample is obtained by picking a ball at random from among the $T_{n-1}$ balls, and the ball is put back in the urn, then a second ball is obtained at random from among the $T_{n-1}$ balls in the urn and the ball is put back in the urn, and so forth until $s$ balls are drawn (and put back in the urn), and that is when we enact the opposite reinforcement.

Under either sampling technique (or whatever else that can be applied to sampling), the reinforcement step is the same: If, say, $s - b$ white balls and $b$ blue balls appear in the sample, $0 \le b \le s$, the drawn balls are returned to the urn together with additional $Cb$ white balls and $C(s - b)$ blue balls.

A concise description of the actions taken is captured by a ball replacement matrix $\mathbf{A} := [a_{i,j}]$ of $s+1$ rows and two columns. The rows are indexed by the number of blue balls that appear in the sample, and the two columns are indexed with $W$ and $B$: The $b$th row corresponds to a pair $(s - b, b)$ of white and blue balls in the sample, and the entry $a_{b,W}$ is the number of white balls added to the urn, if a sample with $b$ blue balls is withdrawn (which is $Cb$), whereas $a_{b,B}$ is the number of blue balls added to the urn, if a sample with $b$ blue balls is withdrawn (which is $C(s - b)$). We thus have

$$\mathbf{A} = \begin{bmatrix} 0 & Cs \\ C & (s-1)C \\ \vdots & \vdots \\ (s-1)C & C \\ Cs & 0 \end{bmatrix}. \tag{2}$$

For the reader's convenience, we introduce some notation used throughout this paper. The Stirling numbers of second kind (i.e., the number of ways to partition a set of $n$ elements into $k$ nonempty subsets) are denoted by $\left\{ {m \atop k} \right\}$, whereas the signless Stirling number of the first kind (i.e., the number of permutations of $\{1, 2, \ldots, m\}$ with exactly $k$ cycles) are denoted by $\left[ {m \atop k} \right]$. The $m$th order harmonic numbers are defined by $H_n^{(m)} := \sum_{j=1}^{n} \frac{1}{j^m}$; as common we also set $H_n := H_n^{(1)}$. Furthermore, we use $x^{\underline{m}} := x(x - 1) \cdots (x - m + 1)$ to denote the falling factorials.

The rest of the paper appears in sections organized as follows. In Section 2 a basic stochastic recurrence for the number of white balls is set up. In Section 3 the stochastic recurrence is used to derive recurrences for the moments. Three Subsections (3.1–3.3) are then devoted to solving these recurrences, respectively for the cases of the first (mean), second, and generally higher moments. The structure of the moments follows those of asympstotic normal distributions (a main result of this paper presented in Subsection 3.3). In the remaining subsection of Section 3 we use the first few moments to argue a strong law. In Section 4 a martingale underlying the number of white balls is formulated and used to reprove the central limit theorem. It is part of our purpose in this paper to compare the merits of the method of moments to martingales in the context of urns. Therefore, we say a few words about each method after its use. Section 5 concludes the paper with connections to other urns that have similar moments structure, and thus a similar analysis can be conducted. Specifically, generalized Friedman's urns growing under sampling with replacement are discussed in Subsection 5.1, and urns underlying logic circuits are discussed in Subsection 5.2.

## 2. A STOCHASTIC RECURRENCE

Until otherwise is stated, the reader should assume the treatment is for a generalized Friedman's urn under sampling without replacement. The dynamics of replacement are such that the number of white balls after $n$ draws is what it was after $n-1$ draws plus the number of blue balls in the $n$th sample (which is $s$ minus the conditionally hypergeometrically distributed number of white balls in the sample), and we can write

$$W_n = W_{n-1} + C(s - \xi_n), \tag{3}$$

where, given $\mathcal{F}_{n-1}$, the $\sigma$-field generated by the first $n-1$ draws, the random variable $\xi_n$ has a hypergeometric distribution:

$$\mathbb{P}\big(\xi_n = k \,|\, \mathcal{F}_{n-1}\big) = \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}};$$

the binomial coefficients are as usual interpreted to be 0, when the lower index is negative or higher than the upper index.

## 3. MOMENT STRUCTURE

Our starting point is the stochastic recurrence for $W_n^r$, which we obtain by raising both sides of (3) to the $r$th power. We then take the conditional expectation, with respect to $\mathcal{F}_{n-1}$, and obtain

$$\begin{aligned}\mathbb{E}(W_n^r \,|\, \mathcal{F}_{n-1}) \;&=\; W_{n-1}^r + \sum_{\ell=1}^{r}\binom{r}{\ell} W_{n-1}^{r-\ell} \\ &\qquad \times C^\ell \sum_{k=0}^{s} (s-k)^\ell \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}}.\end{aligned} \tag{4}$$

In the following we will simplify the inner sums in the last expression; the cases $r=1$ and $r=2$ lead then to the mean and variance of $W_n$.

### 3.1. **The mean.**

For the first power ($r=1$), equation (4) takes the form

$$\mathbb{E}(W_n \,|\, \mathcal{F}_{n-1}) = W_{n-1} + C\sum_{k=0}^{s}(s-k)\frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}}.$$

Let hypergeo$(T, s, w)$ be a hypergeometric random variable counting the number of white balls that appear in a sample of size $s$ taken from an urn containing $w$ white and $T-w$ blue balls. Then, the sum

$$\sum_{k=0}^{s} \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}}$$

is 1 (being the sum of all probabilities of hypergeo$(T_{n-1}, s, W_{n-1})$), and

$$\sum_{k=0}^{s} k\frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}}$$

is $sW_{n-1}/T_{n-1}$ (being the expectation of hypergeo$(T_{n-1}, s, W_{n-1})$); let us remark that from a combinatorial point of view these simplifications follow from an application of the Vandermonde convolution formula, see Subsection 3.2. We thus have

$$\mathbb{E}(W_n \,|\, \mathcal{F}_{n-1}) = W_{n-1} + Cs - \frac{Cs}{T_{n-1}} W_{n-1}, \tag{5}$$

with expectation

$$\begin{aligned}\mathbb{E}(W_n) &= \Big(1 - \frac{Cs}{T_{n-1}}\Big)\mathbb{E}(W_{n-1}) + Cs\\ &= \frac{T_0 + Cs(n-2)}{T_0 + Cs(n-1)}\mathbb{E}(W_{n-1}) + Cs.\end{aligned}$$

Iterating this recurrence, written conveniently in the form

$$(T_0 + Cs(n-1))\mathbb{E}(W_n) = (T_0 + Cs(n-2))\mathbb{E}(W_{n-1}) + Cs(T_0 + Cs(n-1)), \quad \text{for } n \ge 1,$$

with initial value $\mathbb{E}(W_0) = W_0$, leads to the mean in exact form (and asymptotics, as $n \to \infty$, follow easily):

$$\begin{aligned}\mathbb{E}(W_n) \;&=\; \frac{C^2 s^2 n(n-1) + 2CsT_0 n + 2(T_0 - Cs)W_0}{2(Cs(n-1) + T_0)} &(6)\\ &=\; \frac{1}{2}Csn + \frac{1}{2}T_0 + \mathcal{O}(1). &(7)\end{aligned}$$

3.2. **The variance.** Substituting $r = 2$ in (4) gives

$$\begin{aligned}\mathbb{E}(W_n^2 \,|\, \mathcal{F}_{n-1}) &= W_{n-1}^2 + 2CW_{n-1}\sum_{k=0}^{s}(s-k)\frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}}\\ &\quad + C^2\sum_{k=0}^{s}\big((s-k)^{\underline{2}} + (s-k)\big)\frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}}\\ &= W_{n-1}^2 + 2CW_{n-1}(T_{n-1} - W_{n-1})\sum_{k=0}^{s-1}\frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}-1}{s-1-k}}{\binom{T_{n-1}}{s}}\\ &\quad + C^2(T_{n-1} - W_{n-1})^{\underline{2}}\sum_{k=0}^{s-2}\frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}-2}{s-2-k}}{\binom{T_{n-1}}{s}}\end{aligned}$$

$$+ C^2(T_{n-1} - W_{n-1}) \sum_{k=0}^{s-1} \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}-1}{s-1-k}}{\binom{T_{n-1}}{s}}.$$

The sums appearing in the latter equation can be simplified by applying the Vandermonde convolution formula (see, e.g., [5]):

$$\sum_{k=0}^{m} \binom{x}{k}\binom{y}{m-k} = \binom{x+y}{m},$$

yielding after simple manipulations

$$\begin{aligned}\mathbb{E}(W_n^2 \,|\, \mathcal{F}_{n-1}) &= W_{n-1}^2 + 2CW_{n-1}(T_{n-1} - W_{n-1})\frac{\binom{T_{n-1}-1}{s-1}}{\binom{T_{n-1}}{s}} + C^2(T_{n-1} - W_{n-1})^{\underline{2}}\frac{\binom{T_{n-1}-2}{s-2}}{\binom{T_{n-1}}{s}} \\ &\quad + C^2(T_{n-1} - W_{n-1})\frac{\binom{T_{n-1}-1}{s-1}}{\binom{T_{n-1}}{s}} \\ &= \Big(1 - \frac{2Cs}{T_{n-1}} + \frac{C^2 s^{\underline{2}}}{T_{n-1}^{\underline{2}}}\Big)W_{n-1}^2 + \Big(2Cs - \frac{C^2 s(2s-1)}{T_{n-1}} - \frac{C^2 s^{\underline{2}}}{T_{n-1}^{\underline{2}}}\Big)W_{n-1} + C^2 s^2.\end{aligned}$$

Taking expectations leads to the following recurrence for the second moment of $W_n$:

$$\begin{aligned}\mathbb{E}(W_n^2) &= \underbrace{\Big(1 - \frac{2Cs}{T_{n-1}} + \frac{C^2 s^{\underline{2}}}{T_{n-1}^{\underline{2}}}\Big)}_{=:\, g_n} \mathbb{E}(W_{n-1}^2) \\ &\qquad + \underbrace{\Big(2Cs - \frac{C^2 s(2s-1)}{T_{n-1}} - \frac{C^2 s^{\underline{2}}}{T_{n-1}^{\underline{2}}}\Big) \mathbb{E}(W_{n-1}) + C^2 s^2}_{=:\, h_n},\end{aligned} \tag{8}$$

for $n \geq 1$, with $\mathbb{E}(W_0^2) = W_0^2$. The first-order linear recurrence (8) can be solved by standard means leading to the explicit solution

$$\mathbb{E}(W_n^2) = \Big(\prod_{i=1}^{n} g_i\Big)\Big(W_0^2 + \sum_{j=1}^{n} \frac{h_j}{\prod_{i=1}^{j} g_i}\Big). \tag{9}$$

We will get a somewhat simpler expression for the second moment by considering the factorization of $g_n$:

$$g_n = 1 - \frac{2Cs}{T_{n-1}} + \frac{C^2 s^{\underline{2}}}{T_{n-1}^{\underline{2}}} = \frac{(n+\lambda_1)(n+\lambda_2)}{(n-1+\frac{T_0}{Cs})(n-1+\frac{T_0-1}{Cs})},$$

with

$$\lambda_{1,2} = -2 + \frac{2T_0 - 1 \pm \sqrt{1 + 4Cs(C-1)}}{2Cs}. \tag{10}$$

We obtain then

$$\prod_{i=1}^{n} g_i = \frac{\binom{n+\lambda_1}{n}\binom{n+\lambda_2}{n}}{\binom{n-1+\frac{T_0}{Cs}}{n}\binom{n-1+\frac{T_0-1}{Cs}}{n}}. \tag{11}$$

Furthermore, plugging the explicit expression (6) for the mean into (8), we eventually get

$$h_n = C^2 s^2 n + \frac{1}{2}Cs\big(2T_0 - C(2s-1)\big) + \frac{\alpha_1}{T_{n-1}} + \frac{\alpha_2}{T_{n-1}^{\underline{2}}} + \frac{\alpha_3}{T_{n-1}^{\underline{2}}T_{n-2}}, \tag{12}$$

with

$$\alpha_1 = \frac{1}{2}Cs\big(4W_0(T_0-Cs) - C(s-1) + 2T_0(Cs-T_0)\big),$$
$$\alpha_2 = \frac{1}{2}C^2s\big(2W_0(T_0-Cs) - s + 1 + T_0(Cs-T_0)\big),$$
$$\alpha_3 = \frac{1}{2}C^2s^2(T_0-2W_0)(C-1)(Cs-T_0).$$

Combining the results (9) and (11) gives an explicit expression for $\mathbb{E}(W_n^2)$ (and thus one also follows for the variance $\mathbb{V}(W_n) = \mathbb{E}(W_n^2) - (\mathbb{E}(W_n))^2$):

$$\mathbb{E}(W_n^2) = \frac{\binom{n+\lambda_1}{n}\binom{n+\lambda_2}{n}}{\binom{n-1+\frac{T_0}{Cs}}{n}\binom{n-1+\frac{T_0-1}{Cs}}{n}}\left(W_0^2 + \sum_{j=1}^{n}\frac{\binom{j-1+\frac{T_0}{Cs}}{j}\binom{j-1+\frac{T_0-1}{Cs}}{j}}{\binom{j+\lambda_1}{j}\binom{j+\lambda_2}{j}}h_j\right),$$

with $\lambda_{1,2}$ and $h_n$ given by (10) and (12), respectively. The asymptotic behaviour of $\mathbb{E}(W_n^2)$ and also of the variance $\mathbb{V}(W_n)$ could be obtained easily from the last explicit result; however, we omit these computations here, since we will discuss the asymptotics of the higher moments in more detail in the next subsection.

### 3.3. **Asymptotics of the centered moments.**

For the central limit theorem we require the asymptotic behaviour of the variance of $W_n$, for $n \to \infty$, and for the strong law of large numbers as given in Subsection 3.4, we even need an estimate of the asymptotics of the fourth centered moment of $W_n$, as $n \to \infty$. However, we will even show how the fundamental stochastic recurrence (3) for $W_n$ can be used to give a detailed specification of the asymptotic behaviour of the centered moments of arbitrary order.

From (6) we know that $\mathbb{E}(W_n) = \frac{1}{2}Csn + \frac{1}{2}T_0 + \mathcal{O}\left(\frac{1}{n}\right) = \frac{1}{2}T_n + \mathcal{O}\left(\frac{1}{n}\right)$. The following computations will simplify considerably if we "shift by the asymptotic mean," i.e., if we introduce

$$W_n^* := W_n - \frac{1}{2}T_n.$$

Note that the centered moments of $W_n$, i.e., the moments of $W_n - \mathbb{E}(W_n)$, and the corresponding moments of $W_n^*$ have, owing to $W_n - \mathbb{E}(W_n) = W_n^* + \mathcal{O}\left(\frac{1}{n}\right)$, the same asymptotic behaviour.

We start with the stochastic recurrence (3) and subtract $\frac{1}{2}T_n$ on both sides, which gives

$$W_n - \frac{1}{2}T_n = W_{n-1} - \frac{1}{2}T_{n-1} + C(s-\xi_n) + \frac{1}{2}(T_{n-1}-T_n) = W_{n-1} - \frac{1}{2}T_{n-1} + C\Big(\frac{s}{2}-\xi_n\Big),$$

and thus

$$W_n^* = W_{n-1}^* + C\Big(\frac{s}{2}-\xi_n\Big).$$

Taking the $r$th power then gives

$$\mathbb{E}\big((W_n^*)^r \mid \mathcal{F}_{n-1}\big) = (W_{n-1}^*)^r + \sum_{\ell=1}^{r}\binom{r}{\ell}(W_{n-1}^*)^{r-\ell}C^\ell\sum_{k=0}^{s}\big(s-k-\frac{s}{2}\big)^\ell\frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}}. \quad (13)$$

To simplify the sums appearing in (13) we will apply the Vandermonde convolution formula; however, in order to do that we will first express the powers of $(\frac{s}{2}-k)$ as linear combinations of the falling factorials of $(s-k)$. In particular, we use here and later on the well-known relations (see, e.g., [5]) involving the Stirling numbers:

$$x^m = \sum_{k=0}^{m} \left\{ m \atop k \right\} x^{\underline{k}}, \qquad \text{and} \qquad x^{\underline{m}} = \sum_{k=0}^{m} \left[ m \atop k \right] (-1)^{m-k} x^k.$$

We first the following obtain from (13)

$$\begin{aligned}
\mathbb{E}\big((W_n^*)^r \,|\, \mathcal{F}_{n-1}\big) &= (W_{n-1}^*)^r + \sum_{\ell=1}^{r} \binom{r}{\ell} (W_{n-1}^*)^{r-\ell} C^\ell \sum_{q=0}^{\ell} \binom{\ell}{q} (-1)^{\ell-q} \left(\frac{s}{2}\right)^{\ell-q} \\
&\quad \times \sum_{k=0}^{s} (s-k)^q \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}} \\
&= (W_{n-1}^*)^r + \sum_{\ell=1}^{r} \binom{r}{\ell} (W_{n-1}^*)^{r-\ell} C^\ell \sum_{q=0}^{\ell} \binom{\ell}{q} (-1)^{\ell-q} \left(\frac{s}{2}\right)^{\ell-q} \\
&\quad \times \sum_{k=0}^{s} \sum_{t=0}^{q} (s-k)^{\underline{t}} \left\{ q \atop t \right\} \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{s-k}}{\binom{T_{n-1}}{s}} \\
&= (W_{n-1}^*)^r + \sum_{\ell=1}^{r} \binom{r}{\ell} (W_{n-1}^*)^{r-\ell} C^\ell \sum_{q=0}^{\ell} \binom{\ell}{q} (-1)^{\ell-q} \left(\frac{s}{2}\right)^{\ell-q} \\
&\quad \times \sum_{t=0}^{q} \left\{ q \atop t \right\} (T_{n-1}-W_{n-1})^{\underline{t}} \sum_{k=0}^{s-t} \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}-t}{s-t-k}}{\binom{T_{n-1}}{s}} \\
&= (W_{n-1}^*)^r + \sum_{\ell=1}^{r} \binom{r}{\ell} (W_{n-1}^*)^{r-\ell} C^\ell \sum_{q=0}^{\ell} \binom{\ell}{q} (-1)^{\ell-q} \left(\frac{s}{2}\right)^{\ell-q} \\
&\quad \times \sum_{t=0}^{q} \left\{ q \atop t \right\} (T_{n-1}-W_{n-1})^{\underline{t}} \frac{\binom{T_{n-1}-t}{s-t}}{\binom{T_{n-1}}{s}}.
\end{aligned}$$

We continue by expressing the remaining appearance of $W_{n-1}$ by $W_{n-1}^* + \frac{1}{2}T_{n-1}$ and expanding with respect to powers of $W_{n-1}^*$:

$$\begin{aligned}
\mathbb{E}\big((W_n^*)^r \,|\, \mathcal{F}_{n-1}\big) &= (W_{n-1}^*)^r + \sum_{\ell=1}^{r} \binom{r}{\ell} (W_{n-1}^*)^{r-\ell} C^\ell \sum_{q=0}^{\ell} \binom{\ell}{q} (-1)^{\ell-q} \left(\frac{s}{2}\right)^{\ell-q} \\
&\quad \times \sum_{t=0}^{q} \left\{ q \atop t \right\} \big(\frac{1}{2}T_{n-1} - W_{n-1}^*\big)^{\underline{t}} \, \frac{s^{\underline{t}}}{T_{n-1}^{\underline{t}}} \\
&= (W_{n-1}^*)^r + \sum_{\ell=1}^{r} \binom{r}{\ell} (W_{n-1}^*)^{r-\ell} C^\ell \sum_{q=0}^{\ell} \binom{\ell}{q} (-1)^{\ell-q} \left(\frac{s}{2}\right)^{\ell-q} \\
&\quad \times \sum_{t=0}^{q} \left\{ q \atop t \right\} \frac{s^{\underline{t}}}{T_{n-1}^{\underline{t}}} \sum_{m=0}^{t} \left[ t \atop m \right] (-1)^{t-m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \Big(\frac{T_{n-1}}{2}\Big)^{m-k} (W_{n-1}^*)^k
\end{aligned}$$

$$
\begin{aligned}
&= (W^*_{n-1})^r + \sum_{\ell=1}^{r} \binom{r}{\ell} (W^*_{n-1})^{r-\ell} C^{\ell} \sum_{k=0}^{\ell} (W^*_{n-1})^{k} (-1)^{k} \sum_{m=k}^{\ell} \binom{m}{k} \Big(\frac{T_{n-1}}{2}\Big)^{m-k} \\
&\quad \times \sum_{t=m}^{\ell} \genfrac{[}{]}{0pt}{}{t}{m} (-1)^{t-m} \frac{s^{\underline{t}}}{T_{n-1}^{\underline{t}}} \sum_{q=t}^{\ell} \binom{\ell}{q} \genfrac{\{}{\}}{0pt}{}{q}{t} (-1)^{\ell-q} \Big(\frac{s}{2}\Big)^{\ell-q}.
\end{aligned}
$$

To get a form suitable for our purpose we substitute $k := \ell - k$ in the latter equation and proceed as follows:

$$
\begin{aligned}
\mathbb{E}\big((W^*_n)^r \,|\, \mathcal{F}_{n-1}\big) &= (W^*_{n-1})^r + \sum_{\ell=1}^{r} \binom{r}{\ell} (W^*_{n-1})^{r-\ell} C^{\ell} \sum_{k=0}^{\ell} (W^*_{n-1})^{\ell-k} (-1)^{\ell-k} \\
&\quad \times \sum_{m=\ell-k}^{\ell} \binom{m}{\ell-k} \big(\frac{T_{n-1}}{2}\big)^{m-\ell+k} \sum_{t=m}^{\ell} \genfrac{[}{]}{0pt}{}{t}{m} (-1)^{t-m} \frac{s^{\underline{t}}}{T_{n-1}^{\underline{t}}} \\
&\quad \times \sum_{q=t}^{\ell} \binom{\ell}{q} \genfrac{\{}{\}}{0pt}{}{q}{t} (-1)^{\ell-q} \Big(\frac{s}{2}\Big)^{\ell-q} \\
&= \sum_{k=0}^{r} (W^*_{n-1})^{r-k} \sum_{\ell=k}^{r} \binom{r}{\ell} C^{\ell} (-1)^{\ell-k} \sum_{m=\ell-k}^{\ell} \binom{m}{\ell-k} \Big(\frac{1}{2}\Big)^{m-\ell+k} T_{n-1}^{m-\ell+k} \\
&\quad \times \sum_{t=m}^{\ell} \genfrac{[}{]}{0pt}{}{t}{m} (-1)^{t-m} \frac{s^{\underline{t}}}{T_{n-1}^{\underline{t}}} \sum_{q=t}^{\ell} \binom{\ell}{q} \genfrac{\{}{\}}{0pt}{}{q}{t} (-1)^{\ell-q} \Big(\frac{s}{2}\Big)^{\ell-q} \\
&= (W^*_{n-1})^r \left( \sum_{\ell=0}^{r} \binom{r}{\ell} C^{\ell} (-1)^{\ell} \frac{s^{\underline{\ell}}}{T_{n-1}^{\underline{\ell}}} \right) + \sum_{k=1}^{r} (W^*_{n-1})^{r-k} \sum_{\ell=k}^{r} \binom{r}{\ell} C^{\ell} (-1)^{\ell-k} \\
&\quad \times \sum_{m=\ell-k}^{\ell} \binom{m}{\ell-k} \Big(\frac{1}{2}\Big)^{m-\ell+k} T_{n-1}^{m-\ell+k} \sum_{t=m}^{\ell} \genfrac{[}{]}{0pt}{}{t}{m} (-1)^{t-m} \frac{s^{\underline{t}}}{T_{n-1}^{\underline{t}}} \\
&\quad \times \sum_{q=t}^{\ell} \binom{\ell}{q} \genfrac{\{}{\}}{0pt}{}{q}{t} (-1)^{\ell-q} \Big(\frac{s}{2}\Big)^{\ell-q}.
\end{aligned}
$$

Taking expectations of the latter leads thus to the following first-order linear recurrence for the $r$th moment of $W^*_n$:

$$
\mathbb{E}\big((W^*_n)^r\big) = g_n^{[r]}\, \mathbb{E}\big((W^*_{n-1})^r\big) + h_n^{[r]}, \quad \text{for } n \ge 1, \tag{14}
$$

with

$$
g_n^{[r]} = \sum_{\ell=0}^{r} \binom{r}{\ell} C^{\ell} (-1)^{\ell} \frac{s^{\underline{\ell}}}{T_{n-1}^{\underline{\ell}}}, \quad \text{and} \quad h_n^{[r]} = \sum_{k=1}^{r} f_k^{[r]}(n)\, \mathbb{E}\big((W^*_{n-1})^{r-k}\big), \tag{15}
$$

and

$$
\begin{aligned}
f_k^{[r]}(n) &= \sum_{\ell=k}^{r} \binom{r}{\ell} C^{\ell} (-1)^{\ell-k} \sum_{m=\ell-k}^{\ell} \binom{m}{\ell-k} \big(\frac{1}{2}\big)^{m-\ell+k} T_{n-1}^{m-\ell+k} \sum_{t=m}^{\ell} \genfrac{[}{]}{0pt}{}{t}{m} (-1)^{t-m} \frac{s^{\underline{t}}}{T_{n-1}^{\underline{t}}} \\
&\quad \times \sum_{q=t}^{\ell} \binom{\ell}{q} \genfrac{\{}{\}}{0pt}{}{q}{t} (-1)^{\ell-q} \big(\frac{s}{2}\big)^{\ell-q}.
\end{aligned} \tag{16}
$$

At a first glance the recurrence (14) might seem to be too involved to be useful. However, it turns out that the asymptotic behaviour of $\mathbb{E}((W_n^*)^r)$ can be obtained quite easily from it. The first step is to obtain the explicit solution of this recurrence for the $r$th moment of $W_n^*$ in terms of the lower order moments, which follows immediately from (14) and the initial condition $\mathbb{E}((W_0^*)^r) = (W_0 - \frac{1}{2}T_0)^r$ by standard techniques. The solution is stated in the following proposition.

**Proposition 1.** *The $r$th moment of $W_n^* = W_n - \frac{1}{2}T_n$ is given as follows:*

$$\begin{aligned}\mathbb{E}\big((W_n^*)^r\big) &= \Big(\prod_{i=1}^{n} g_i^{[r]}\Big)\Big(\big(W_0 - \frac{1}{2}T_0\big)^r + \sum_{j=1}^{n}\frac{h_j^{[r]}}{\prod_{i=1}^{j} g_i^{[r]}}\Big)\\ &= \Big(\prod_{i=1}^{n} g_i^{[r]}\Big)\big(W_0 - \frac{1}{2}T_0\big)^r + \sum_{j=1}^{n}\Big(\prod_{i=j+1}^{n} g_i^{[r]}\Big)\, h_j^{[r]}, \end{aligned} \tag{17}$$

*with $g_n^{[r]}$ and $h_n^{[r]}$ defined in (15).*

We state now the theorem concerning the asymptotic behaviour of the $r$th moments of $W_n^*$.

**Theorem 1.** *The asymptotic behaviour of the $r$th integer moment $\mathbb{E}((W_n^*)^r)$ of $W_n^* = W_n - \frac{1}{2}T_n$ (and thus also the asymptotic behaviour of the $r$th centered moment $\mathbb{E}((W_n - \mathbb{E}(W_n))^r)$ of $W_n$) is, for $n \to \infty$, given as follows:*

$$\mathbb{E}\big((W_n^*)^r\big) = \kappa_{r'} n^{r'} + \mathcal{O}(n^{r'-1}), \quad \text{for } r = 2r' \ge 0 \text{ even}, \quad \text{with} \quad \kappa_{r'} = \Big(\frac{C^2 s}{12}\Big)^{r'}\frac{(2r')!}{2^{r'} r'!},$$
$$\mathbb{E}\big((W_n^*)^r\big) = \mathcal{O}(n^{r'}), \quad \text{for } r = 2r'+1 \text{ odd and } r' \ge 1, \quad \text{and} \quad \mathbb{E}(W_n^*) = \mathcal{O}\Big(\frac{1}{n}\Big).$$

*Proof.* We will show the theorem by induction with respect to $r$. For $r = 0$ and $r = 1$ the claim is obviously true. Let us now consider $r \ge 2$ and let us further assume that the theorem holds for all $\mathbb{E}((W_n^*)^p)$, with $0 \le p < r$. We now examine the asymptotic behaviour of the functions $g_n^{[r]}$ and $h_n^{[r]}$ defined in (15) and appearing in the exact solution stated in Proposition 1. Owing to to the relation $T_n = T_0 + Csn$, for $g_n^{[r]}$ we get

$$g_n^{[r]} = \sum_{\ell=0}^{r}\binom{r}{\ell}C^{\ell}(-1)^{\ell}\frac{s^{\ell}}{T_{n-1}^{\ell}} = 1 - \frac{rCs}{T_{n-1}} + \mathcal{O}\Big(\frac{1}{n^2}\Big) = 1 - \frac{r}{n} + \mathcal{O}\Big(\frac{1}{n^2}\Big).$$

Using the well-known asymptotic expansion of the first and second order harmonic numbers:

$$H_n = \ln n + \gamma + \mathcal{O}\Big(\frac{1}{n}\Big), \qquad H_n^{(2)} = \frac{\pi^2}{6} + \mathcal{O}\Big(\frac{1}{n}\Big),$$

with $\gamma = 0.5772\ldots$ being the Euler-Masceroni constant, we further obtain

$$\begin{aligned}\prod_{i=j+1}^{n} g_i^{[r]} &= \exp\Big(\sum_{i=j+1}^{n}\ln g_i^{[r]}\Big)\\ &= \exp\Big(\sum_{i=j+1}^{n}\ln\Big(1 - \frac{r}{i} + \mathcal{O}(i^{-2})\Big)\Big)\\ &= \exp\Big(\sum_{i=j+1}^{n}\Big(-\frac{r}{i} + \mathcal{O}(i^{-2})\Big)\Big)\end{aligned}$$

$$\begin{aligned}
&= \exp\Big( -r(H_n - H_j) + \mathcal{O}(H_n^{(2)} - H_j^{(2)})\Big) \\
&= \exp\big(-r(\ln n - \ln j) + \mathcal{O}(j^{-1})\big) \\
&= \frac{j^r}{n^r} \exp\big(\mathcal{O}(j^{-1})\big) \\
&= \frac{j^r}{n^r} \big(1 + \mathcal{O}(j^{-1})\big), \qquad (18)
\end{aligned}$$

where the $\mathcal{O}$-bound holds uniformly for all $1 \le j \le n$. As a first consequence of (18) we get the bound

$$\Big(\prod_{i=1}^{n} g_i^{[r]}\Big) \Big(W_0 - \frac{1}{2}T_0\Big)^r = \mathcal{O}\Big(\frac{1}{n^r}\Big), \qquad (19)$$

which will turn out to be asymptotically negligible compared to the remaining part of (17).

In order to describe the asymptotic behaviour of $h_n^{[r]}$ we have to consider the asymptotic behaviour of the functions $f_k^{[r]}(n)$ given in (16), for $n \to \infty$ and $k$, $r$ fixed. We observe that the only appearance of $n$ in this expression is coming from the terms $\frac{T_{n-1}^{m-\ell+k}}{T_{n-1}^{\underline{t}}}$. Since $\ell \ge k$ and $t \ge m$ it immediately follows that $\frac{T_{n-1}^{m-\ell+k}}{T_{n-1}^{\underline{t}}} = \mathcal{O}(1)$, which implies the simple but useful bound

$$f_k^{[r]}(n) = \mathcal{O}(1), \quad \text{for all } 1 \le k \le r. \qquad (20)$$

It even follows the that asymptotic expansion $f_k^{[r]}(n) = c_k^{[r]} + \mathcal{O}(n^{-1})$, where the constant term $c_k^{[r]}$ can be computed easily. Namely, the only terms of $f_k^{[r]}(n)$ contributing to the constant term $c_k^{[r]}$ occur for $\ell = k$ and $t = m$. It turns out to be important to consider the particular instances $k = 1$ and $k = 2$ in more detail. For $k = 1$, the constant term vanishes

$$c_1^{[r]} = rC \sum_{m=0}^{1} \frac{s^{\underline{m}}}{2^m} \sum_{q=m}^{1} \binom{1}{q} {q \brace m} (-1)^{1-q} \Big(\frac{s}{2}\Big)^{1-q} = rC\Big(-\frac{s}{2} + \frac{s}{2}\Big) = 0.$$

Thus, we have

$$f_1^{[r]}(n) = \mathcal{O}\Big(\frac{1}{n}\Big). \qquad (21)$$

For $k = 2$, we obtain

$$c_2^{[r]} = \binom{r}{2} C^2 \sum_{m=0}^{2} \frac{s^{\underline{m}}}{2^m} \sum_{q=m}^{2} \binom{2}{q} {q \brace m} (-1)^{2-q} \Big(\frac{s}{2}\Big)^{2-q} = \binom{r}{2} C^2 \frac{s}{4},$$

and thus

$$f_2^{[r]}(n) = \binom{r}{2} \frac{C^2 s}{4} + \mathcal{O}\Big(\frac{1}{n}\Big). \qquad (22)$$

In order to proceed we have to distinguish whether $r$ is even or odd. First let us consider the case $r = 2r'$ even, with $r' \ge 1$. Using the induction hypothesis and the bounds (20)–(22) on $f_k^{[r]}(n)$ computed above we get

$$\mathbb{E}\big((W_{n-1}^*)^{r-1}\big) f_1^{[r]}(n) = \mathcal{O}(n^{r'-1})\mathcal{O}(n^{-1}) = \mathcal{O}(n^{r'-2}),$$

$$\sum_{k=3}^{r} \mathbb{E}\big((W_{n-1}^*)^{r-k}\big) f_k^{[r]}(n) = \mathcal{O}\Big(\mathbb{E}\big((W_{n-1}^*)^{r-3}\big) f_3^{[r]}(n)\Big) = \mathcal{O}(n^{r'-2})\mathcal{O}(1) = \mathcal{O}(n^{r'-2}),$$

$$\begin{aligned}\mathbb{E}\big((W_{n-1}^{*})^{r-2}\big)f_{2}^{[r]}(n) &= \big(\kappa_{r'-1}n^{r'-1}+\mathcal{O}(n^{r'-2})\big)\left(\binom{2r'}{2}\frac{C^{2}s}{4}+\mathcal{O}(n^{-1})\right)\\ &= \kappa_{r'-1}\binom{2r'}{2}\frac{C^{2}s}{4}n^{r'-1}+\mathcal{O}(n^{r'-2}),\end{aligned}$$

and thus the following asymptotic expansion of $h_n^{[r]}$ holds:

$$h_n^{[r]}=\sum_{k=1}^{r}\mathbb{E}\big((W_{n-1}^{*})^{r-k}\big)f_{k}^{[r]}(n)=\kappa_{r'-1}\binom{2r'}{2}\frac{C^{2}s}{4}n^{r'-1}+\mathcal{O}(n^{r'-2}).$$

Together with the already computed expansion (18) this yields the following asymptotic expansion, which holds uniformly for $1\le j\le n$:

$$\Big(\prod_{i=j+1}^{n}g_i^{[r]}\Big)h_j^{[r]}=\kappa_{r'-1}\binom{2r'}{2}\frac{C^{2}s}{4}\times\frac{j^{3r'-1}}{n^{2r'}}\big(1+\mathcal{O}(j^{-1})\big).$$

Using the crude asymptotic expansion

$$\sum_{j=1}^{n}j^{p}=\frac{n^{p+1}}{p+1}+\mathcal{O}(n^{p}),\quad \text{for a fixed integer } p\ge 1 \text{ and } n\to\infty, \tag{23}$$

we further get

$$\sum_{j=1}^{n}\Big(\prod_{i=j+1}^{n}g_i^{[r]}\Big)h_j^{[r]}=\kappa_{r'-1}\binom{2r'}{2}\frac{C^{2}s}{4}\times\frac{n^{r'}}{3r'}+\mathcal{O}(n^{r'-1})=\kappa_{r'-1}\frac{C^{2}s}{12}(2r'-1)n^{r'}+\mathcal{O}(n^{r'-1}).$$

Together with (17) and (19) this already shows, for $r=2r'$ even, the asymptotic expansion

$$\mathbb{E}\big((W_n^{*})^{r}\big)=\kappa_{r'}n^{r'}+\mathcal{O}(n^{r'-1}),\quad \text{with}\quad \kappa_{r'}=\kappa_{r'-1}\frac{C^{2}s}{12}(2r'-1).$$

Using the induction hypothesis on the constant $\kappa_{r'-1}$, we obtain

$$\kappa_{r'}=\frac{C^{2}s}{12}(2r'-1)\Big(\frac{C^{2}s}{12}\Big)^{r'-1}\frac{(2r'-2)!}{2^{r'-1}(r'-1)!}=\Big(\frac{C^{2}s}{12}\Big)^{r'}\frac{(2r')!}{2^{r'}r'!},$$

as stated in Theorem 1, which finishes the proof of the theorem for $r$ even.

For $r=2r'+1$ odd, with $r'\ge 1$, we get from the induction hypothesis and (20)–(21):

$$\begin{aligned}\mathbb{E}\big((W_{n-1}^{*})^{r-1}\big)f_{1}^{[r]}(n)&=\mathcal{O}(n^{r'})\mathcal{O}(n^{-1})=\mathcal{O}(n^{r'-1}),\\ \sum_{k=2}^{r}\mathbb{E}\big((W_{n-1}^{*})^{r-k}\big)f_{k}^{[r]}(n)&=\mathcal{O}\Big(\mathbb{E}\big((W_{n-1}^{*})^{r-2}\big)f_{2}^{[r]}(n)\Big)=\mathcal{O}(n^{r'-1})\mathcal{O}(1)=\mathcal{O}(n^{r'-1}),\end{aligned}$$

and thus the following asymptotic bound for $h_n^{[r]}$ holds:

$$h_n^{[r]}=\sum_{k=1}^{r}\mathbb{E}\big((W_{n-1}^{*})^{r-k}\big)f_{k}^{[r]}(n)=\mathcal{O}(n^{r'-1}).$$

Together with (18) this implies the following bound, which holds uniformly for $1\le j\le n$:

$$\Big(\prod_{i=j+1}^{n}g_i^{[r]}\Big)h_j^{[r]}=\mathcal{O}\Big(\frac{j^{3r'}}{n^{2r'+1}}\Big).$$

Using (23) and summing up gives then

$$\sum_{j=1}^{n}\Big(\prod_{i=j+1}^{n} g_i^{[r]}\Big)h_j^{[r]} = \mathcal{O}(n^{r'}),$$

and thus the required bound for $r = 2r' + 1$ odd:

$$\mathbb{E}\big((W_n^*)^r\big) = \mathcal{O}(n^{r'}),$$

which finishes the proof of the theorem. □

As a direct consequence of Theorem 1 we can describe the asymptotic behaviour of the variance and of the fourth centered moment of $W_n$ as required in the following sections. We see that

$$\mathbb{V}(W_n) \sim \mathbb{E}\big((W_n^*)^2\big) = \frac{C^2 s}{12} n + \mathcal{O}(1),$$
$$\mathbb{E}\big((W_n - \mathbb{E}(W_n))^4\big) \sim \mathbb{E}\big((W_n^*)^4\big) = 3\Big(\frac{C^2 s}{12}\Big)^2 n^2 + \mathcal{O}(n) \sim 3\big(\mathbb{V}(W_n)\big)^2.$$

Moreover, by an application of the theorem of Fréchet and Shohat (see [12], Page 187). we immediately obtain from Theorem 1 a central limit theorem for $W_n$ (with convergence of all moments), which is a main result of this paper (stated below); however, later in Section 4 we will reprove the central limit law in a much less computational way by using the martingale central limit theorem.

The theorem of Fréchet and Shohat is basically an appeal to claiming limit distributions by discovering that all the moments converge to those of some distribution, provided that such a distribution is uniquely determined by its moments, which is the case for the normal distribution we found. The idea of shifting by the asymptotic mean is due to Chern and Hwang [8], and recent experience indicates huge success in the area of random structures and algorithms.

**Theorem 2.** *Let $W_n$ be the number of white balls after $n$ draws from a generalized Friedman's urn grown under sampling without replacement. Then,*

$$\frac{W_n - \frac{1}{2}Csn}{\sqrt{n}} \xrightarrow{D} \mathcal{N}\Big(0, \frac{1}{12}C^2 s\Big),$$

*as $n \to \infty$.*

In this approach to central limit theorem, we also identify the rate of convergence in each moment.

3.4. **Strong law of large numbers.** The fourth moment is relatively small to give us a strong law of large numbers, presented next.

**Proposition 2.**

$$\frac{W_n}{n} \xrightarrow{a.s.} \frac{Cs}{2}, \qquad n \to \infty.$$

*Proof.* As we have shown in Subsection 3.3 the fourth centered moment satisfies the expansion

$$\mathbb{E}\big((W_n - \mathbb{E}(W_n))^4\big) \sim 3(\mathbb{V}(W_n))^2.$$

Next we use a general form of the Markov inequality (also known as Chebyshev's inequality for higher moments) to obtain

$$\begin{aligned}\mathbb{P}\Big\{\Big|\frac{W_n}{n} - \frac{\mathbb{E}(W_n)}{n}\Big| > \varepsilon\Big\} &\le \frac{1}{\varepsilon^4 n^4}\mathbb{E}(|W_n - \mathbb{E}(W_n)|^4)\\ &\sim \frac{3}{\varepsilon^4 n^4}(\mathbb{V}(W_n))^2\\ &\sim \frac{C^4 s^2}{48\varepsilon^4 n^2}\\ &\to 0,\end{aligned}$$

valid for all $\varepsilon > 0$. Hence, we have

$$\sum_{n=1}^{\infty}\mathbb{P}\Big\{\Big|\frac{W_n}{n} - \frac{\mathbb{E}(W_n)}{n}\Big| > \varepsilon\Big\} \le \sum_{n=1}^{\infty}\frac{C^4 s^2}{48\varepsilon^4 n^2} < \infty.$$

By the Borel-Cantelli Lemma we have

$$\mathbb{P}\Big\{\Big|\frac{W_n}{n} - \frac{\mathbb{E}(W_n)}{n}\Big| > \varepsilon \quad \text{infinitely often}\Big\} = 0.$$

This, being true for any $\varepsilon > 0$, implies that

$$\frac{W_n}{n} - \frac{\mathbb{E}(W_n)}{n} \xrightarrow{a.s.} 0.$$

However, we also have $\frac{\mathbb{E}(W_n)}{n} \to \frac{1}{2}CS$. The result as stated follows according to the laws of addition of sequences of almost surely convergent random variables. □

**Corollary 1.**

$$W_n = \frac{1}{2}Csn + o_{\mathcal{L}_1}(n).$$

*Proof.* The random variables $W_n/n$ are uniformly bounded, as can be seen from

$$\frac{W_n}{n} = \frac{W_n}{T_n} \times \frac{T_n}{n} \le \frac{T_n}{n} = \frac{Csn + T_0}{n} \le Cs + T_0.$$

This uniform bound, together with Proposition 2 shows that

$$\frac{W_n}{n} \xrightarrow{\mathcal{L}_1} \frac{1}{2}Cs.$$

□

**Corollary 2.**

$$W_n^2 = \frac{1}{4}C^2 s^2 n^2 + o_{\mathcal{L}_1}(n^2).$$

## 4. Discrete Martingales and a central limit theorem

The random variables $W_n$ are not directly a martingale. However, transformations of these variables are. We can find values $\varphi_n$ and $\psi_n$ such that $M_n = \varphi_n W_n + \psi_n$ is a martingale.

**Lemma 1.** *The random variables*

$$M_n = \frac{T_{n-1}}{T_0} W_n - Cs \sum_{k=0}^{n-1} \frac{T_k}{T_0}$$

*are a martingale with respect to the filtration* $\{\mathcal{F}_n\}_{n=0}^{\infty}$.

*Proof.* We use the *ansatz* $M_n = \varphi_n W_n + \psi_n$, and seek suitable values for $\varphi_n$ and $\psi_n$ that render $M_n$ a martingale.[1] For $M_n$ to be a martingale, we must have

$$\begin{aligned}
\mathbb{E}(M_n \,|\, \mathcal{F}_{n-1}) &= \mathbb{E}(\varphi_n W_n + \psi_n \,|\, \mathcal{F}_{n-1}) \\
&= \varphi_n \mathbb{E}(W_n \,|\, \mathcal{F}_{n-1}) + \psi_n \\
&= \varphi_n \mathbb{E}(W_n \,|\, \mathcal{F}_{n-1}) + \psi_n \\
&= \varphi_n \Big( \Big(1 - \frac{Cs}{T_{n-1}}\Big) W_{n-1} + Cs \Big) + \psi_n \\
&= \varphi_{n-1} W_{n-1} + \psi_{n-1};
\end{aligned}$$

the penultimate line was obtained from (5). This is possible if the coefficients of $W_{n-1}^r$are matched, for $r = 0, 1$, that is,

$$\varphi_{n-1} = \Big(1 - \frac{Cs}{T_{n-1}}\Big)\varphi_n, \qquad \text{and} \qquad \psi_{n-1} = Cs\varphi_n + \psi_n.$$

The recurrence

$$\varphi_n = \Big(\frac{T_{n-1}}{T_{n-1} - Cs}\Big)\varphi_{n-1} = \frac{T_{n-1}}{T_{n-2}}\varphi_{n-1}$$

has the solution $\varphi_n = \frac{T_{n-1}}{T_0}\varphi_0$, for arbitrary $\varphi_0$ (which we take to be 1), and the recurrence

$$\psi_n = \psi_{n-1} - Cs\varphi_n$$

has the solution $\psi_n = -Cs \sum_{k=0}^{n-1} \frac{T_k}{T_0} + \psi_0$, for arbitrary $\psi_0$ (which we take to be 0). $\square$

To prove a central limit theorem for the number of white balls, it suffices to check the conditional Lindeberg condition and the conditional variance condition for suitably normalized differences of the martingale $M_n$; see [6], Page 58.

Let $\triangledown M_j = M_j - M_{j-1}$ denote the backward differences of the martingale. The success of this method hinges on having small martingale differences, relative to an appropriate scale. Let us first look at the raw differences:

$$\triangledown M_j = \varphi_j W_j - \varphi_{j-1} W_{j-1} + \psi_j - \psi_{j-1}.$$

We have

$$\psi_j - \psi_{j-1} = -Cs\frac{T_{j-1}}{T_0} = -Cs\varphi_j,$$

[1]Such coefficients are not unique.

and

$$\varphi_j = \frac{T_{j-1}}{T_0} = \varphi_{j-1} + \frac{Cs}{T_0}.$$

We further obtain, using (3),

$$\begin{aligned}
\triangledown M_j &= \varphi_j(W_j - W_{j-1}) + \frac{Cs}{T_0}W_{j-1} - Cs\varphi_j \\
&= \frac{Cs}{T_0}W_{j-1} - C\varphi_j\xi_j \\
&= \frac{Cs}{T_0}W_{j-1} - C\frac{T_{j-1}}{T_0}\xi_j.
\end{aligned}$$

Hence, by the bounds $W_{j-1} \le T_{j-1}$ and $\xi_j \le Cs$ we get

$$|\triangledown M_j| \le \frac{2Cs}{T_0}T_{j-1}. \tag{24}$$

We can now check the conditions for the martingale central limit theorem. We use the notation $\mathbb{I}(\mathcal{E})$ for the indicator function that assumes the value 1, if $\mathcal{E}$ occurs, and assumes the value 0 otherwise.

**Lemma 2.** *The martingale $M_n$ satisfies Lindeberg's conditional condition: For any fixed $\varepsilon > 0$,*

$$U_n := \sum_{j=1}^{n} \mathbb{E}\Big(\Big(\frac{\triangledown M_j}{n^{\frac{3}{2}}}\Big)^2 \mathbb{I}\Big(\Big|\frac{\triangledown M_j}{n^{\frac{3}{2}}}\Big| > \varepsilon\Big) \,\Big|\, \mathcal{F}_{j-1}\Big) \xrightarrow{P} 0,$$

*as $n \to \infty$.*

*Proof.* The absolute differences in (24) are $O(n)$. Thus, the sets

$$\Big\{\Big|\frac{\triangledown M_j}{n^{\frac{3}{2}}}\Big| > \varepsilon\Big\}$$

are empty for large enough $n \ge n_0$. Hence, the sum $U_n$ is truncated at $n_0$, yielding

$$\begin{aligned}
U_n &= \sum_{j=1}^{n_0} \mathbb{E}\Big(\Big(\frac{\triangledown M_j}{n^{\frac{3}{2}}}\Big)^2 \mathbb{I}\Big(\Big|\frac{\triangledown M_j}{n^{\frac{3}{2}}}\Big| > \varepsilon\Big) \,\Big|\, \mathcal{F}_{j-1}\Big) \\
&\le \frac{1}{n^3}\sum_{j=1}^{n_0} \mathbb{E}\big((\triangledown M_j)^2 \,\big|\, \mathcal{F}_{j-1}\big) \\
&= \frac{1}{n^3}\sum_{j=1}^{n_0} \frac{4C^2s^2T_{j-1}^2}{T_0^2} \\
&\le \frac{4C^2s^2T_{n_0-1}^2 n_0}{T_0^2 n^3} \\
&\to 0.
\end{aligned}$$

Lindeberg's conditional condition has been verified. $\square$

**Lemma 3.** *The martingale $M_n$ satisfies the conditional variance condition:*

$$V_n := \sum_{j=1}^{n} \mathbb{E}\Big(\Big(\frac{\nabla M_j}{n^{\frac{3}{2}}}\Big)^2 \,\Big|\, \mathcal{F}_{j-1}\Big) \xrightarrow{P} \frac{C^4 s^3}{12 T_0^2}.$$

*as $n \to \infty$.*

*Proof.* In view of the absolute differences in (24), we have

$$\begin{aligned} V_n &= \frac{1}{n^3} \sum_{j=1}^{n} \frac{C^2}{T_0^2} \mathbb{E}\big((sW_{j-1} - T_{j-1}\xi_j)^2 \,|\, \mathcal{F}_{j-1}\big) \\ &= \frac{C^2}{T_0^2 n^3} \sum_{j=1}^{n} \mathbb{E}\big(s^2 W_{j-1}^2 + T_{j-1}^2 \xi_j^2 - 2sT_{j-1}W_{j-1}\xi_j \,|\, \mathcal{F}_{j-1}\big). \end{aligned}$$

Using the known (conditional) mean and variance for the hypergeometric random variable $\xi_j$, we get

$$\begin{aligned} V_n &= \frac{C^2}{T_0^2 n^3} \sum_{j=1}^{n} s^2 W_{j-1}^2 + T_{j-1}^2 \mathbb{E}(\xi_j^2 \,|\, \mathcal{F}_{j-1}) - 2sT_{j-1}W_{j-1}\mathbb{E}(\xi_j \,|\, \mathcal{F}_{j-1}) \\ &= \frac{C^2}{T_0^2 n^3} \sum_{j=1}^{n} \frac{s(s - T_{j-1})}{T_{j-1} - 1}\, W_{j-1}^2 + \frac{sT_{j-1}(T_{j-1} - s)}{T_{j-1} - 1}\, W_{j-1}. \end{aligned}$$

Appealing to the concentration property in Corollary 1, we write

$$\begin{aligned} V_n = \frac{C^2}{T_0^2 n^3} \sum_{k=1}^{n} & \frac{s(s - T_{k-1})}{T_{k-1} - 1} \Big(\Big(\frac{Cs(k-1)}{2}\Big)^2 + o_{\mathcal{L}_1}(k^2)\Big) \\ & + \frac{sT_{k-1}(T_{k-1} - s)}{T_{k-1} - 1} \Big(\Big(\frac{Cs(k-1)}{2}\Big) + o_{\mathcal{L}_1}(k)\Big). \end{aligned}$$

We now plug in the value of $T_{k-1}$ from (1) and simplify the sums; the lemma follows. □

We proceed to reprove the main result.

*Proof of Theorem 2 via martingales.* The conditions for the martingale central limit theorem have been checked in Lemmas 2–3. Accordingly

$$\sum_{k=1}^{n} \frac{\nabla M_k}{n^{\frac{3}{2}}} \xrightarrow{d} \mathcal{N}\Big(0, \frac{C^4 s^3}{12 T_0^2}\Big).$$

The sum of the differences telescopes, leaving only the difference of the last term and the initial condition, that is

$$\frac{M_n - M_0}{n^{\frac{3}{2}}} \xrightarrow{d} \mathcal{N}\Big(0, \frac{C^4 s^3}{12 T_0^2}\Big),$$

or

$$\frac{\dfrac{T_{n-1} W_n}{T_0} - \dfrac{Cs}{T_0} \displaystyle\sum_{k=1}^{n} T_{k-1}}{n^{\frac{3}{2}}} \xrightarrow{d} \mathcal{N}\Big(0, \frac{C^4 s^3}{12 T_0^2}\Big).$$

This can be written as

$$\frac{(Cs(n-1)+T_0)W_n - \Big(\frac{1}{2}C^2s^2n^2 + O(n)\Big)}{n^{\frac{3}{2}}} \xrightarrow{d} \mathcal{N}\Big(0, \frac{C^4s^3}{12}\Big).$$

The theorem follows as stated after a few adjustments via Slutsky's theorem (see [12], Page 176). □

**Remark:** *The result in* [3] *is the special case $C = s = 1$. Evidently, the approach via the modern probabilistic method of martingale is much lighter computationally than the method of moments. However, the method produces only the chief asymptotic of each moment, without the rate of convergence.*

## 5. Urn models with multiple drawing and similar moments structure

Some other urn schemes have moments structure similar to the one presented. Thus, some other urn schemes can be amenable to analysis by methods like the ones just discussed. We name two schemes below.

### 5.1. Generalized Friedman's urn growing under sampling with replacement.

A natural variation on the scheme just discussed (generalized Friedman's urn growing under sampling without replacement) is one with a different sampling mechanism. The most popular other sampling method is to take the $s$ balls out with replacement (details discussed in the introduction). Such a generalized Friedman's urn has the ball replacement matrix (2), as the scheme without replacement. After $n$ draws, the number of white balls in such an urn, $\tilde{W}_n$, satisfies a recurrence similar to (3), with the random variable $\xi_n$ substituted with $\tilde{\xi}_n$, a random variable that conditionally has a binomial distribution. Namely, the stochastic recurrence is

$$\tilde{W}_n = \tilde{W}_{n-1} + C(s - \tilde{\xi}_n),$$

where,

$$\mathbb{P}(\tilde{\xi}_n = k \,|\, \mathcal{F}_{n-1}) = \binom{s}{k}\Big(\frac{\tilde{W}_{n-1}}{T_{n-1}}\Big)^k\Big(1 - \frac{\tilde{W}_{n-1}}{T_{n-1}}\Big)^{s-k},$$

and again $T_n = T_0 + Csn$. The derivation follows through, mutatis mutandis. All the steps are the same as in the case of a generalized Friedman's urn under sampling without replacement, only with hypergeometric probabilities replaced by binomial probabilities. The starting point for the computations here is the equation

$$\mathbb{E}(\tilde{W}_n^r \,|\, \mathcal{F}_{n-1}) = \tilde{W}_{n-1}^r + \sum_{\ell=1}^{r}\binom{r}{\ell}\tilde{W}_{n-1}^{r-\ell}C^\ell\sum_{k=0}^{s}(s-k)^\ell\frac{\binom{s}{k}\tilde{W}_{n-1}^k(T_{n-1}-\tilde{W}_{n-1})^{s-k}}{T_{n-1}^s}.$$

For instance, the mean of hypergeo$(T_{n-1}, s, W_{n-1})$ coincides with that of a binomial random variable counting successes in $s$ independent identically distributed trials, with probability of success $W_{n-1}/T_{n-1}$. Therefore, the two schemes have the same recurrence for the mean number of white balls, and consequently the same mean number of white balls after $n$ draws, when the starting urns are the same. That is, $\mathbb{E}(\tilde{W}_n) = \mathbb{E}(W_n)$, and $\mathbb{E}(W_n)$ is given in (6), and (7) both exactly and asymptotically. The variance follows suit—the second moment satisfies the equation

$$\mathbb{E}(\tilde{W}_n^2) = \tilde{g}_n\mathbb{E}(\tilde{W}_{n-1}^2) + \tilde{h}_n,$$

with

$$\tilde{g}_n = 1 - \frac{2Cs}{T_{n-1}} + \frac{C^2s^2}{T_{n-1}^2},$$

$$\tilde{h}_n = \left(2Cs - \frac{C^2 s(2s-1)}{T_{n-1}}\right)\mathbb{E}(\tilde{W}_{n-1}) + C^2s^2 \tag{25}$$

$$= C^2s^2n + \frac{1}{2}Cs\big(2T_0 - C(2s-1)\big) + \frac{Cs(Cs-T_0)(T_0-2\tilde{W}_0)}{T_{n-1}} + \frac{C^2s(Cs-T_0)(T_0-2\tilde{W}_0)}{2T_{n-1}T_{n-2}}.$$

This recurrence leads then to the following explicit solution of the second moment of $\tilde{W}_n$:

$$\mathbb{E}(\tilde{W}_n^2) = \frac{\binom{n-2+\frac{T_0}{Cs}+\frac{1}{\sqrt{s}}}{n}\binom{n-2+\frac{T_0}{Cs}-\frac{1}{\sqrt{s}}}{n}}{\binom{n-1+\frac{T_0}{Cs}}{n}^2}\left[\tilde{W}_0^2 + \sum_{j=1}^{n}\frac{\binom{j-1+\frac{T_0}{Cs}}{j}^2}{\binom{j-2+\frac{T_0}{Cs}+\frac{1}{\sqrt{s}}}{j}\binom{j-2+\frac{T_0}{Cs}-\frac{1}{\sqrt{s}}}{j}}\tilde{h}_j\right],$$

where $\tilde{h}_n$ is defined in (25). From this explicit solution the asymptotic behaviour of the variance of $\tilde{W}_n$ can be deduced easily and one obtains

$$\mathbb{V}(\tilde{W}_n) = \frac{1}{12}C^2sn + \mathcal{O}(1),$$

i.e., the variance of $\tilde{W}_n$ is asymptotically equivalent to $\mathbb{V}(W_n)$. Note that from a computational point of view this simply follows from the fact that the first and second order term in the asymptotic expansion of the functions $\tilde{g}_n$ and $\tilde{h}_n$ coincides with their counterparts in the asymptotic expansion of $g_n$ and $h_n$ that were encountered in the second moment of $W_n^2$.

Finally, going through all the details of the martingale central limit theorem, we see, by a calculation (omitted) rather similar to that in the proof of Theorem 2, that

$$\frac{\tilde{W}_n - \frac{1}{2}Csn}{\sqrt{n}} \xrightarrow{D} \mathcal{N}\Big(0, \frac{1}{12}C^2s\Big).$$

We now see that a generalized Friedman's urn behaves asymptotically in essentially the same way, whether it grows under sampling with or without replacement.

5.2. **Growth models for logic circuits of gates.** Another urn scheme has lately drawn attention. It is a scheme underlying the growth of logic circuits of gates. The number of input wires (binary inputs) coming into a gate in a logic circuit is called the fan-in of the circuit, and the gate computes a predicate of these inputs and produces an output (which may be fed into other layers of gates in the circuit). If the output is not fed into another gate it is an output of the whole circuit; we call such an output a *free output*. A random circuit with fan-in $s$ grows in the following way. Initially we have a starting circuit. Of the existing gates $s$ are chosen *at random*. Random can mean a sampling scheme without replacement, in which case the number of existing gates must be at least $s$, a model introduced in [15], or can mean sampling with replacement, and the starting circuit can have one or more gates, a model introduced in [1] and [10]. The output of the chosen gates is connected as input to a new gate. It is of interest to find the number of circuit outputs (that is the number of gates with free outputs).

Urn models have been developed to model logic circuits. In [10] such an urn is described, and only a law of large numbers is given. In the recent paper [14], a central limit theorem is derived.

To model outputs, consider an urn where a white ball corresponds to a gate with a free output, and a blue ball corresponds to a gate with one or more outputs feeding into other gates. The evolution can be described by an $(s+1) \times 2$ matrix $\hat{\mathbf{A}}$, with an indexing scheme just like that of (2):

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ \vdots & \vdots \\ -(s-2) & s-1 \\ -(s-1) & s \end{bmatrix}.$$

Let $T_n$ denote the total number of balls after $n$ draws. The number of balls increases by one after each draw, and we have

$$T_n = n + T_0, \quad \text{for } n \geq 0.$$

The stochastic recurrence for the number of white balls $\hat{W}_n$ after $n$ draws is given by

$$\hat{W}_n = \hat{W}_{n-1} + 1 - \hat{\xi}_n,$$

with $\hat{\xi}_n$ having the same distribution as $\xi_n$ or $\tilde{\xi}_n$ as defined in Section 2 and Subsection 5.1, respectively: conditionally hypergeometric under sampling without replacement, and conditionally binomial under sampling with replacement. This stochastic recurrence (under either sampling technique) has a structure for the moments similar to what we derived for generalized Friedman's urn, and the central limit theorem of [14] can be rederived by the methods we used for generalized Friedman's urn.

To complement the existing results we add exact formulæ for the second moment (and thus also for the variance) of $\hat{W}_n$ under each sampling scheme; note that the exact and asymptotic formulæ for the mean as stated below already appear in [10]. For sampling without replacement one starts with the equation

$$\mathbb{E}(\hat{W}_n^r \,|\, \mathcal{F}_{n-1}) = \hat{W}_{n-1}^r + \sum_{\ell=1}^{r} \binom{r}{\ell} \hat{W}_{n-1}^{r-\ell} \sum_{k=0}^{s} (1-k)^{\ell} \frac{\binom{\hat{W}_{n-1}}{k}\binom{T_{n-1}-\hat{W}_{n-1}}{s-k}}{\binom{T_{n-1}}{s}}.$$

Taking expectations and solving the ensuing recurrences for the instances $r=1$ and $r=2$ leads to the following exact formulas:

$$\mathbb{E}(\hat{W}_n) = \frac{n+T_0}{s+1} + \frac{\binom{T_0-1}{s}}{\binom{n+T_0-1}{s}}\Big(\hat{W}_0 - \frac{T_0}{s+1}\Big),$$

$$\mathbb{E}(\hat{W}_n^2) = \frac{\binom{T_0-1}{s}\binom{T_0-2}{s}}{\binom{n+T_0-1}{s}\binom{n+T_0-2}{s}}\Bigg[\hat{W}_0^2 + \sum_{j=1}^{n} \frac{\binom{j+T_0-1}{s}\binom{j+T_0-2}{s}}{\binom{T_0-1}{s}\binom{T_0-2}{s}} \hat{h}_j\Bigg],$$

with $\hat{h}_n = \frac{2n}{s+1} + \frac{2T_0-1}{s+1} - \frac{s(s-1)}{(s+1)(n+T_0-2)} + \frac{2n+2T_0+s-4}{n+T_0-2}(\hat{W}_0 - \frac{T_0}{s+1})\frac{\binom{T_0-1}{s}}{\binom{n+T_0-1}{s}}$. From this explicit results the asymptotic behaviour of the mean and the variance can be deduced easily and one obtains

$$\mathbb{E}(\hat{W}_n) = \frac{n}{s+1} + \frac{T_0}{s+1} + \mathcal{O}(n^{-1}), \qquad \mathbb{V}(\hat{W}_n) = \frac{s^2}{(s+1)^2(2s+1)} n + \mathcal{O}(1).$$

For sampling with replacement one starts with the equation (to distinguish between both sampling strategies we use now the random variable $\check{W}_n$):

$$\mathbb{E}\big((\check{W}_n)^r \,|\, \mathcal{F}_{n-1}\big) = (\check{W}_{n-1})^r + \sum_{\ell=1}^{r} \binom{r}{\ell} (\check{W}_{n-1})^{r-\ell} \sum_{k=0}^{s} (1-k)^{\ell} \frac{\binom{s}{k}(\check{W}_{n-1})^k (T_{n-1} - \check{W}_{n-1})^{s-k}}{T_{n-1}^s}.$$

As already shown in [10] the exact mean of $\check{W}_n$ and $\hat{W}_n$ coincide. Furthermore, we have

$$\mathbb{E}\big((\check{W}_n)^2\big) = \frac{\binom{n+T_0-s-1+\sqrt{s}}{n}\binom{n+T_0-s-1-\sqrt{s}}{n}}{\binom{n+T_0-1}{n}^2}\left[(\check{W}_0)^2 + \sum_{j=1}^{n} \frac{\binom{j+T_0-1}{j}^2}{\binom{j+T_0-s-1+\sqrt{s}}{j}\binom{j+T_0-s-1-\sqrt{s}}{j}}\check{h}_j\right],$$

with $\check{h}_n = \frac{2n}{s+1} + \frac{2T_0-1}{s+1} + \frac{2n+2T_0-2-s}{n+T_0-1}(\check{W}_0 - \frac{T_0}{s+1})\frac{\binom{T_0-1}{s}}{\binom{n+T_0-2}{s}}$. It follows that $\mathbb{V}(\check{W}_n)$ and $\mathbb{V}(\hat{W}_n)$ have the same asymptotic behaviour, i.e., $\mathbb{V}(\check{W}_n) \sim \mathbb{V}(\hat{W}_n) = \frac{s^2}{(s+1)^2(2s+1)}n + \mathcal{O}(1)$.

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MARKUS KUBA, INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TECHNISCHE UNIVERSITÄT WIEN, WIEDNER HAUPTSTR. 8-10/104, 1040 WIEN, AUSTRIA – HTL WIEN 5 SPENGERGASSE, SPENGERGASSE 20, 1050 WIEN, AUSTRIA

*E-mail address*: kuba@dmg.tuwien.ac.at

HOSAM MAHMOUD, DEPARTMENT OF STATISTICS, THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, D.C. 20052, U.S.A.

*E-mail address*: hosam@gwu.edu

ALOIS PANHOLZER, INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TECHNISCHE UNIVERSITÄT WIEN, WIEDNER HAUPTSTR. 8-10/104, 1040 WIEN, AUSTRIA

*E-mail address*: Alois.Panholzer@tuwien.ac.at