

LIMITING DISTRIBUTIONS FOR A CLASS OF DIMINISHING URN MODELS.

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ABSTRACT. In this work we analyze a class of 2×2 Pólya-Eggenberger urn models with ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$, $a, d \in \mathbb{N}$, and $c = p \cdot a$ with $p \in \mathbb{N}_0$. We obtain limiting distributions for this 2×2 urn model by obtaining a precise recursive description of the moments of the considered random variables, which allows us to deduce asymptotic expansions of the moments. In particular, we obtain limiting distributions for the pills problem $a = c = d = 1$, originally proposed by Knuth and McCarthy. Furthermore, we also obtain limiting distributions for the well known sampling without replacement urn, $a = d = 1$ and $c = 0$, and generalizations of it to arbitrary $a, d \in \mathbb{N}$ and $c = 0$. Moreover, we obtain a recursive description of the moment sequence for a generalized problem.

1. INTRODUCTION

1.1. Pólya-Eggenberger urn models. Pólya-Eggenberger urn models are defined in the following way. We start with an urn containing n white balls and m black balls. The evolution of the urn occurs in discrete time steps. At every step a ball is drawn at random from the urn. The color of the ball is inspected and then the ball is returned to the urn. According to the observed color of the ball there are added/removed balls due to the following rules. If a white ball has been drawn, we put into the urn a white balls and b black balls, but if a black ball has been drawn, we put into the urn c white balls and d black balls. The values $a, b, c, d \in \mathbb{Z}$ are fixed integer values and the urn model is specified by the 2×2 ball replacement matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This definition extends naturally also to higher dimensions. Urn models are simple, useful mathematical tools for describing many evolutionary processes in diverse fields of application such as analysis of algorithms and data structures, statistics and genetics. Due to their importance in applications, there is a huge literature on the stochastic behavior of urn models; see for example [11, 16]. Recently, a few different approaches have been proposed, which yield deep and far-reaching results for very general urn models; see [3, 4, 9, 10]. Most papers in the literature impose the so-called tenability condition on the ball replacement matrix, so that the process can be continued ad infinitum, or no balls of a given color being completely removed. However, in some applications, there are urn models with a very different nature, which we will refer to as diminishing urn models. We refer to [8] for a detailed description of diminishing urn models.

A well known example of a diminishing urn model is the classical example sampling without replacement urn model with transition matrix $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. In this model, balls are drawn at random one after another from an urn containing balls of two different colors and not replaced. What is the probability that k balls of one color remain when balls of the other color are all removed? Another famous diminishing urn model is the so-called OK Corral urn, which serves as a mathematical model of the historical gun fight at the OK Corral. The ball transition matrix of the OK Corral urn model is given by $M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. This problem was introduced by Williams and McIlroy in [22], and can be viewed as a mathematical model for warfare and conflicts; see [13, 14]. It was studied by several authors using different approaches, leading to very deep and interesting results; see for example Stadje [19], Kingman [12, 13], Kingman and Volkov [14] or the recent works of Flajolet et al. [18, 3], and Turner [20]. An vivid interpretation is as follows. Two groups of gunmen, group A and group B (with n and m gunmen, respectively), face each other. At every discrete time step, one gunman is chosen uniformly at random who then shoots and kills exactly one gunman of the other group. The gunfight ends when one group gets completely “eliminated”. Several questions are of interest: what is the probability that group A (group B) survives, and what is the probability that the gunfight ends with k survivors of group A (group B)? Moreover, one is also interested in the total number of survivors, regardless of the group. It turns out that the limit laws arising in the OK Corral urn model are of a different nature compared to the limit laws

2000 *Mathematics Subject Classification.* 60C05 .

Key words and phrases. Pólya-Eggenberger urns models, pills problem, sampling without replacement, diminishing urns.

This work was supported by the Austrian Science Foundation FWF, grant S9608-N13.

arising in the sampling without replacement urn model, which can easily be seen by comparing the limit laws given in [19, 12, 13, 14, 18, 3], basically normal distributions or related distributions, with the limit laws – beta distributions, exponential distributions, and geometric distributions – arising from the sampling without replacement urn model: $\mathbb{P}\{X_{m,n} = k\} = \frac{\binom{m-1+n-k}{m-1}}{\binom{m+n}{m}}$, $0 \leq k \leq n$. This explicit formula can be proven in various ways, e.g., via lattice path counting arguments or generating functions [8]. Here $X_{m,n}$ denotes the random variable counting the number of white balls, when all black balls have been drawn, starting with n white and m black balls, in the sampling without replacement urn model.

In this work we will analyze diminishing Pólya-Eggenberger urn models with ball replacement matrix M given by

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}, \quad \text{with } a, d \in \mathbb{N} \text{ and } c = p \cdot a, \quad p \in \mathbb{N}_0. \quad (1)$$

We are interested in the distribution of the random variable $X_{m,n}$, counting the number of white balls, when all black balls have been drawn, starting with n white and m black balls, respectively. We assume that the initial number of white balls is a multiple of a and the initial number of black balls is a multiple of d ; equivalently we consider the random variables $X_{dm,an}$, with $m, n \in \mathbb{N}_0$. The distribution of the random variable $X_{dm,an}$ in the context of the evolution of an urn, with ball replacement matrix given by M , may be described as follows. We have a state space $\mathcal{S} := \{(d \cdot m, a \cdot n) \mid m, n \in \mathbb{N}_0\}$, where the evolution of the urn takes place. The evolution stops at *absorbing states* $\mathcal{A} := \{(0, a \cdot n) \mid n \in \mathbb{N}_0\}$. The question is then to determine the probability $\mathbb{P}\{X_{dm,an} = k\}$, that a certain state $k \in \mathcal{A}$ is reached, starting with $a \cdot n$ white balls and $d \cdot m$ black balls. The aim of this work is the derivation of limiting distributions of the random variables $X_{dm,an}$ for diminishing urn models, when the urn evolves according to a ball replacement matrix M given by (1). We will see that different limiting distributions arise according to the growth of m and n . Note that when starting with $a \cdot n + \alpha$ white balls, where $1 \leq \alpha < a$, the urn model is no longer well defined. It may happen that at some stage only α white balls are left, but we are forced to remove a white balls, when choosing a white ball. The same problem occurs when the parameter $c \neq p \cdot a$ is not a multiple of the parameter a in the definition of the ball replacement matrix.

1.2. Motivation. Our studies of the class of diminishing urns with a ball replacement matrix given by (1) is motivated by the following problems.

The pills problem. The pills problem was originally proposed by Knuth and McCarthy in [15], p. 264; the solution appeared in [7]. An vivid interpretation of the pills problem is the following: In a bottle there are m large pills and n small pills. A large pill is equivalent to two small pills. Every day a person chooses a pill at random. If a small pill is chosen it is eaten up, but if a large pill is chosen it is broken into two halves, one half is eaten and the other half which is now considered as a small pill is returned to the bottle. The problem, proposed in [15], was to find the expected number of small pills remaining when there are no more large pills left in the bottle. Brennan and Prodinger revisited this problem in [2], where they showed how one can derive the exact moments of the pills problem (at least in principle), and computed them up to the third moment. Furthermore they also considered variations of the problem assuming, e.g., that a large pill is equal to p small pills, where they also succeeded in computing the expected value. The pills problem corresponds to the derivation of the expected value of $X_{m,n}$ for a diminishing urn model with ball replacement matrix $M = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. In the recent work of Hwang et. al. [8] the limiting distributions of the pills problem and a related model, namely $M = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$, were obtained by using generating functions. It was shown that the limiting distributions significantly differ for these two problems. The generating functions approach of [8] has the benefit that one not only obtains the limiting distributions, but also the exact distribution of $X_{m,n}$ for the two considered urn models. However, it seems difficult to extend the generating function approach to study the class with a ball replacement given by (1) in full generality. Hence, the results of [8] motivated us to analyze the class of urns with replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$.

Sampling without replacement This classical urn model corresponds to the urn with ball replacement matrix $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The distribution of the types of balls after t draws is very well known (see, e.g., [3]), but here we will focus on the limiting distributions of $X_{m,n}$. Note that this problem is often treated by introducing two absorbing axes, i.e., $\{(0, n) : n \geq 0\} \cup \{(m, 0) : m \geq 0\}$, but we rather simply use the

¹Throughout this work we use the notations $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.

absorbing axis $\mathcal{A} = \{(0, n) : n \geq 0\}$, which is fully sufficient. We will also derive limiting distributions for the generalizations $M = \begin{pmatrix} -a & 0 \\ 0 & -d \end{pmatrix}$.

1.3. Weighted lattice paths. It is useful to describe and visualize the evolution of an urn with ball replacement matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by weighted paths, which is described here in the case of urns with two types of balls. If the urn contains m black balls and n white balls and we are picking up a white ball, which appears with probability $\frac{n}{m+n}$, this corresponds to a step $(m, n) \rightarrow (m + a, n + b)$, which gets the weight $\frac{n}{m+n}$, and if we are picking up a black ball, this corresponds to a step $(m, n) \rightarrow (m + c, n + d)$, which appears with probability $\frac{m}{m+n}$ and gets thus the weight $\frac{m}{m+n}$. The weight of a path after t successive draws consists of the product of the weights of every step. For a diminishing urn we obtain that the sum of the weights of all possible paths starting at state (m, n) and ending at the absorbing state $(i, j) \in \mathcal{A}$ (which did not pass another absorbing state earlier) gives then the required probability, that when starting at (m, n) we are ending at (i, j) .

Unfortunately the weighted path approach is in general not effective for studying the behaviour of urn models. An example for a weighted path corresponding to the evolution of a diminishing urn is given in Figure 1. The steps associated with a ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$ are visualized in Figure 2.

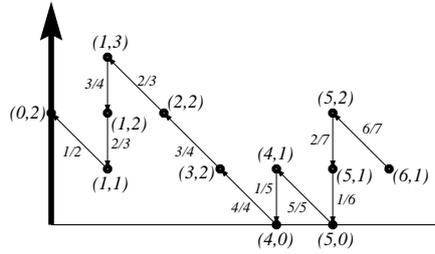


FIGURE 1. An example of a weighted path from $(6, 1)$ to the absorbing state $(0, 2)$ for the so called pills problem with ball replacement matrix $M = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and the vertical absorbing axis $\mathcal{A} = \{(0, n) : n \geq 0\}$. The illustrated path has weight $\frac{6}{7} \frac{2}{7} \frac{1}{5} \frac{1}{5} \frac{4}{4} \frac{3}{4} \frac{2}{4} \frac{3}{4} \frac{2}{3} \frac{1}{2} = \frac{3}{3920}$.

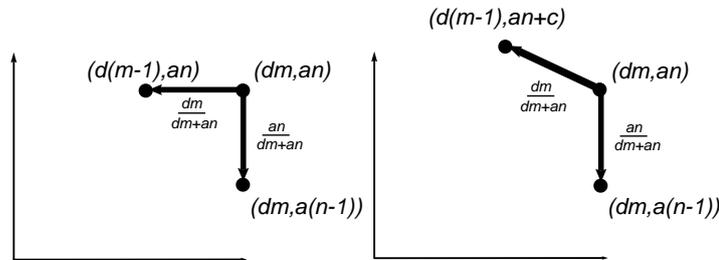


FIGURE 2. The steps associated to the transition matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$ for $c = 0$ and $c > 0$.

1.4. Goal. We will determine the structure of the moments of $X_{dm,an}$ for urn models with ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$, $a, d \in \mathbb{N}$ and $c = p \cdot a$, with $p \in \mathbb{N}_0$, as well as providing explicit formulas for the expectation and variance of $X_{dm,an}$. Moreover, we will determine limiting distributions of the random variable $X_{dm,an}$ with replacement matrix M as given in (1). As a byproduct we (re-)obtain limiting distributions for the pills problem, and also for generalizations of it.

For m fixed and n tending to infinity we can show that $X_{dm,an}/(a \cdot n)$ tends to a so-called Kumaraswamy distributed random variable. Furthermore, we show that for m tending to infinity the limiting distribution for $c \geq 0$ changes according to the quotient a/d , with $a, d \in \mathbb{N}$ and the proportion of m and n . We will also encounter Weibull distributions as limiting distributions.

1.5. Notation and plan of the paper. We denote by $X_n \xrightarrow{\mathcal{L}} X$ the weak convergence, i.e., the convergence in distribution, of the sequence of random variables X_n to a random variable X . Furthermore, we denote with $H_n := \sum_{k=1}^n 1/k$ the n -th harmonic number and with $H_n^{(2)} := \sum_{k=1}^n 1/k^2$ the n -th second order harmonic number. We denote with $\begin{bmatrix} n \\ k \end{bmatrix}$ the unsigned Stirling number of the first kind, and with $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ the Stirling numbers of the second kind, see e.g. [5]. Furthermore, we use the abbreviation $x^\ell := x(x-1)\cdots(x-\ell+1)$ and $x^{\bar{\ell}} := x(x+1)\cdots(x+\ell-1)$ for the falling and rising factorials, respectively. Moreover, we use the standard asymptotic notations, such as the big-O notation, small-o notation, and also the asymptotic equivalence of functions $f \sim g \iff \lim(f/g) = 1$.

The main results of this work are the characterization of the limiting distributions of $X_{dm,an}$ depending on the ball replacement matrix M and the initial states, which are contained in the next section. In Section 3 we will give a recursive description of the moments of $X_{dm,an}$ together with the derivation of the expectation and the variance. Section 4 is devoted to the proofs of the limiting distribution results. A generalization of the considered urn model is then discussed in Section 5.

2. RESULTS

Next we state our limiting distribution results for $X_{dm,an}$, divided into three cases, namely first $c = 0$, second $c \neq 0$ and $\frac{a}{d} \leq 1$, and third $c \neq 0$ and $\frac{a}{d} > 1$.

2.1. Limiting distributions for $c = 0$.

Theorem 1. *For the ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ 0 & -d \end{pmatrix}$, the random variable $X_{dm,an}$, counting the number of white balls when all black balls have been removed, starting with $a \cdot n$ white and $d \cdot m$ black balls, has the following limiting behaviour:*

- (1) *For fixed m and $n \rightarrow \infty$ the scaled random variable $\frac{X_{dm,an}}{an}$ converges in distribution to a limiting variable $Y(d/a, m)$:*

$$\frac{X_{dm,an}}{an} \xrightarrow{\mathcal{L}} Y\left(\frac{d}{a}, m\right),$$

with convergence of all moments, where $Y(\alpha, \beta)$ denotes a Kumaraswamy distributed random variable with parameters α and β .

- (2) *For $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$ the scaled random variable $\frac{m^{a/d} X_{dm,an}}{an}$ converges in distribution to a Weibull distributed random variable,*

$$\frac{m^{a/d} X_{dm,an}}{an} \xrightarrow{\mathcal{L}} W\left(\frac{d}{a}, 1\right).$$

- (3) *For $m, n \rightarrow \infty$ such that $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$, the random variable $X_{dm,an}$ converges to a random variable X , which is fully characterized by the sequence of its moments:*

$$X_{dm,an} \xrightarrow{\mathcal{L}} X, \quad \text{with} \quad \mathbb{E}(X^s) = a^s \sum_{\ell=1}^s \left\{ \begin{matrix} s \\ \ell \end{matrix} \right\} \rho^\ell \Gamma\left(1 + \frac{a\ell}{d}\right).$$

- (4) *For $m \rightarrow \infty$ and $n = n(m) = o(m^{a/d})$ the random variable $X_{dm,an}$ converges to a limit X , which has all its mass concentrated at zero:*

$$X_{dm,an} \xrightarrow{\mathcal{L}} 0.$$

Remark 1. For $a = d = 1$ we obtain the limit laws for the well known sampling without replacement urn model $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which seem to be mathematical folklore, although we could not find a proper reference to the literature. In particular, one obtains a Beta $B(1, m)$ limiting distribution for fixed m and $n \rightarrow \infty$, an exponential distribution for $m, n \rightarrow \infty$ such that $m = o(n)$, and geometric distributions for $m, n \rightarrow \infty$ such that $n \sim \rho m$.

2.2. Limiting distributions for $c \neq 0$ and $\frac{a}{d} \leq 1$.

Theorem 2. For the ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$, with $c = p \cdot a > 0$ with $p \in \mathbb{N}$, and $\frac{a}{d} \leq 1$ the random variable $X_{dm,an}$, counting the number of white balls when all black balls have been removed, starting with $a \cdot n$ white and $d \cdot m$ black balls, has the following limiting behaviour:

- (1) For fixed m and $n \rightarrow \infty$ the scaled random variable $\frac{X_{dm,an}}{an}$ converges in distribution to a limiting variable $Y(d/a, m)$:

$$\frac{X_{dm,an}}{an} \xrightarrow{\mathcal{L}} Y\left(\frac{d}{a}, m\right),$$

with convergence of all moments, where $Y(\alpha, \beta)$ denotes a Kumaraswamy distributed random variable with parameters α and β .

- (2) For $m \rightarrow \infty$ and arbitrary $n = n(m)$, possibly constant or a function of m , the random variable $\frac{X_{dm,an}}{g_{m,n}}$ converges in distribution to a limit $W(d/a, 1)$:

$$\frac{X_{dm,an}}{g_{m,n}} \xrightarrow{\mathcal{L}} W\left(\frac{d}{a}, 1\right), \quad (2)$$

with convergence of all moments, where $W(k, \lambda)$ denotes a Weibull distributed random variable with parameters k and λ . The normalization values $g_{m,n}$ are given as follows:

$$g_{m,n} = g_{m,n}(a, c, d) = \begin{cases} \frac{an + m \frac{cd}{d-a}}{m^{\frac{a}{d}}}, & \text{for } \frac{a}{d} < 1, \\ \frac{an}{m} + c \log m, & \text{for } \frac{a}{d} = 1. \end{cases} \quad (3)$$

Remark 2. The special case $a = c = d = 1$ of Theorem 2 was already proved by Hwang et. al. in [8]. Furthermore, the special case $a = c = 1, d = 2$ of Theorem 2 reproves the Rayleigh limiting distribution for $\sqrt{m}X_{n,2m}/(n+2m)$, also stated in Hwang et. al. [8].

2.3. Limiting distributions for $c \neq 0$ and $a/d > 1$.

Theorem 3. For the ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$, with $c = p \cdot a > 0$ with $p \in \mathbb{N}$, and $\frac{a}{d} > 1$ the random variable $X_{dm,an}$, counting the number of white balls when all black balls have been removed, starting with $a \cdot n$ white and $d \cdot m$ black balls, has the following limiting behaviour:

- (1) For fixed m and $n \rightarrow \infty$ the scaled random variable $\frac{X_{dm,an}}{an}$ converges in distribution to a limiting variable $Y(d/a, m)$:

$$\frac{X_{dm,an}}{an} \xrightarrow{\mathcal{L}} Y\left(\frac{d}{a}, m\right),$$

with convergence of all moments, where $Y(\alpha, \beta)$ denotes a Kumaraswamy distributed random variable with parameters α and β .

- (2) For $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$ the random variable $X_{dm,an}$ converges after suitable scaling to a Weibull distributed random variable,

$$\frac{m^{\frac{a}{d}} X_{dm,an}}{an} \xrightarrow{\mathcal{L}} W\left(\frac{d}{a}, 1\right).$$

- (3) For $m, n \rightarrow \infty$ such that $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$, the moments of the random variable $X_{dm,an}$ converge,

$$\mathbb{E}(X_{dm,an}^s) \rightarrow a^s \sum_{\ell=0}^s \rho^\ell \Gamma\left(1 + \frac{a\ell}{d}\right) \sum_{r=\ell}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \vartheta_{r,\ell;0,0}, \quad s \geq 1,$$

where the values $\vartheta_{s,\ell;h,g}$ satisfy a system of recurrence relations given in Proposition 1.

- (4) For $m \rightarrow \infty$ and arbitrary $n = n(m)$ satisfying $n = o(m^{a/d})$ the moments of the random variable $X_{dm,an}$ converge,

$$\mathbb{E}(X_{dm,an}^s) \rightarrow a^s \sum_{r=0}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \vartheta_{r,0;0,0}, \quad s \geq 1,$$

where the values $\vartheta_{s,\ell;h,g}$ satisfy a system of recurrence relations given in Proposition 1.

Remark 3. Unfortunately, even though we can prove convergence of the moments also for the regions $m, n \rightarrow \infty$ such that $n \sim \rho m^{a/d}$ or $n = o(m^{a/d})$, we are not able to show that the resulting moment sequence defines a unique distribution for these two cases.

3. THE STRUCTURE OF THE MOMENTS

3.1. A recurrence for the moments. By definition of Pólya-Eggenberger urn models with ball replacement matrix given by (1) the probability generating function $h_{m,n}(v) := \sum_{k \geq 0} \mathbb{P}\{X_{dm,an} = k\}v^k$ of $X_{dm,an}$ satisfies the following recurrence (recall that $c = pa$, $p \in \mathbb{N}_0$):

$$h_{m,n}(v) = \frac{an}{an + dm} h_{m,n-1}(v) + \frac{dm}{an + dm} h_{m-1,n+p}(v), \quad \text{for } n \geq 0, m \geq 1, \quad (4)$$

with initial values given by $h_{0,n}(v) = v^{an}$ for all $n \geq 0$.

Our aim is to derive limiting distributions of $X_{dm,an}$, for $\max\{m, n\} \rightarrow \infty$. To do this we will give a precise description of the moments, which enables us to determine the limiting distributions using the so-called method of moments, i.e., by applying the moment convergence theorem of Fréchet and Shohat, the second central limit theorem, see, e.g. [17]. Of course, it follows from (4) that the moments $e_{m,n}^{[s]} := \mathbb{E}(X_{dm,an}^s)$ satisfy the recurrence

$$e_{m,n}^{[s]} = \frac{an}{an + dm} e_{m,n-1}^{[s]} + \frac{dm}{an + dm} e_{m-1,n+p}^{[s]}, \quad \text{for } n \geq 0, m \geq 1, \quad (5)$$

with initial values $e_{0,n}^{[s]} = a^s n^s$, for $n \in \mathbb{N}_0$.

Our first observation for determining the structure of the moments is that $e_{m,n}^{[s]}$ is a polynomial of degree s in n , in other words the s -th moment is of the form $e_{m,n}^{[s]} = \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell$, where the numbers $\lambda_{s,\ell,m}$ are independent of n . In the following we obtain an explicit result for $\lambda_{s,s,m}$, and a recursive description of the quantities $\lambda_{s,\ell,m}$, $1 \leq \ell \leq s-1$ in terms of $\lambda_{s,i,j}$, with $\ell+1 \leq i \leq s$ and $1 \leq j \leq m$.

Lemma 1. *The s -th moments $e_{m,n}^{[s]} = \mathbb{E}(X_{dm,an}^s)$ of the random variable $X_{dm,an}$ satisfy the expansion $e_{m,n}^{[s]} = \sum_{k=0}^s \lambda_{s,k,m} n^k$, where $\lambda_{s,s,m} = \frac{a^s}{\binom{m+\frac{as}{d}}{m}}$. Furthermore the values $\lambda_{s,\ell,m}$, $1 \leq \ell \leq s-1$, satisfy the following recurrence relations:*

$$\lambda_{s,\ell,m} = \sum_{k=0}^{m-1} \frac{\binom{m}{k}}{\binom{m+\frac{a\ell}{d}}{k}} \mu_{s,\ell,m-k},$$

where

$$\mu_{s,\ell,m} := \frac{a}{a\ell + dm} \sum_{k=\ell+1}^s \binom{k}{\ell-1} (-1)^{k-\ell-1} \lambda_{s,k,m} + \frac{dm}{a\ell + dm} \sum_{k=\ell+1}^s \binom{k}{\ell} p^{k-\ell} \lambda_{s,k,m-1}.$$

For $\ell = 0$ we have

$$\lambda_{s,0,m} = \sum_{k=0}^{m-1} \mu_{s,0,k}, \quad \text{with } \mu_{s,0,m} := \sum_{k=1}^s p^k \lambda_{s,k,m}.$$

The initial values are given by $\lambda_{s,s,0} = a^s$ and $\lambda_{s,\ell,0} = 0$, for $0 \leq \ell \leq s-1$.

Proof. In order to prove the stated expansion of $e_{m,n}^{[s]}$ we start with the ansatz $e_{m,n}^{[s]} = \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell$, and get from the recurrence relation (5) the equation

$$(an + dm) \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell = an \sum_{\ell=0}^s \lambda_{s,\ell,m} (n-1)^\ell + dm \sum_{\ell=0}^s \lambda_{s,\ell,m-1} (n+p)^\ell. \quad (6)$$

By comparing the coefficients of n^ℓ , for $0 \leq \ell \leq s+1$, in equation (6) we obtain a system of $s+2$ equations:

$$\begin{aligned}\lambda_{s,s,m} &= \lambda_{s,s,m}, \\ dm\lambda_{s,\ell,m} + a\lambda_{s,\ell-1,m} &= a \sum_{k=\ell-1}^s (-1)^{k-\ell+1} \lambda_{s,k,m} \binom{k}{\ell-1} + dm \sum_{k=\ell}^s p^{k-\ell} \lambda_{s,k,m-1} \binom{k}{\ell}, \quad 1 \leq \ell \leq s, \\ \lambda_{s,0,m} &= \sum_{k=0}^s p^k \lambda_{s,k,m-1},\end{aligned}$$

with initial values $\lambda_{s,s,0} = a^s$, and $\lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s-1$, which are determined by $e_{0,n}^{[s]} = a^s n^s$. The first equation is trivially true, so there remain $s+1$ equations, which determine the values $\lambda_{s,\ell,m}$, $0 \leq \ell \leq s$. The term $\lambda_{s,\ell-1,m}$ on the left hand side cancels with the first summand of $\sum_{k=\ell-1}^s (-1)^{k-\ell+1} \lambda_{s,k,m} \binom{k}{\ell-1}$ on the right hand side, and we obtain

$$\begin{aligned}dm\lambda_{s,\ell,m} &= -a\ell\lambda_{s,\ell,m} + a \sum_{k=\ell+1}^s (-1)^{k-\ell+1} \lambda_{s,k,m} \binom{k}{\ell-1} \\ &\quad + dm\lambda_{s,k,m-1} + dm \sum_{k=\ell+1}^s p^{k-\ell} \lambda_{s,k,m-1} \binom{k}{\ell},\end{aligned}$$

$1 \leq \ell \leq s$. The key step is to note that for computing the values $\lambda_{s,\ell,m}$, for $1 \leq \ell \leq s$, only values $\lambda_{s,i,m}$ and $\lambda_{s,i,m-1}$, with $\ell+1 \leq i \leq s$, are needed, which allows to recursively describe these values. Hence, we can obtain for the values $\lambda_{s,\ell,m}$ the following recurrence relations:

$$\lambda_{s,\ell,m} = \frac{dm}{dm+a\ell} \lambda_{s,\ell,m-1} + \mu_{s,\ell,m}, \quad \text{for } 1 \leq \ell \leq s,$$

with

$$\mu_{s,\ell,m} := \frac{a}{dm+a\ell} \sum_{k=\ell+1}^s \binom{k}{\ell-1} (-1)^{k-\ell-1} \lambda_{s,k,m} + \frac{dm}{dm+a\ell} \sum_{k=\ell+1}^s \binom{k}{\ell} p^{k-\ell} \lambda_{s,k,m-1}.$$

In the case $\ell=0$ we directly obtain

$$\lambda_{s,0,m} = \lambda_{s,0,m-1} + \sum_{k=1}^s p^k \lambda_{s,k,m-1},$$

and further

$$\lambda_{s,0,m} = \lambda_{s,0,m-1} + \mu_{s,0,m-1}, \quad \text{with } \mu_{s,0,m} := \sum_{k=1}^s p^k \lambda_{s,k,m}.$$

Using induction with respect to m and n we conclude that the recurrence (5) has a unique solution for the given initial values $\lambda_{s,s,0} = a^s$, and $\lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s-1$.

Now we will compute $\lambda_{s,s,m}$. We have

$$dm\lambda_{s,s,m} + a\lambda_{s,s-1,m} = -sa\lambda_{s,s,m} + a\lambda_{s,s-1,m} + dm\lambda_{s,s,m-1},$$

leading to

$$(dm+as)\lambda_{s,s,m} = dm\lambda_{s,s,m-1}.$$

This gives

$$\lambda_{s,s,m} = \frac{dm}{dm+as} \lambda_{s,s,m-1}, \quad \text{and further } \lambda_{s,s,m} = \frac{m!}{(m+\frac{as}{d})^m} = \frac{a^s}{(m+\frac{as}{d})^m},$$

where $(m+\frac{as}{d})^m = (m+\frac{as}{d}) \dots (1+\frac{as}{d})$ is written using the falling factorials notation. \square

By Lemma 1 we can derive arbitrarily high moments of $X_{dm,an}$. In particular, we will derive the expectation of $X_{dm,an}$, and use Lemma 1 to prove our limiting distribution results.

3.2. Factorial moments and an explicit formula for the moments in the case $c = 0$. The computation of several moments using Lemma 1 suggested that an alternative description of the moments can be obtained. Let $e_{m,n}^{(s)} := \mathbb{E}\left(\frac{X_{dm,an}}{a}\right)^s$ denote the s -th factorial moment of the normalized random variable $X_{dm,an}/a$, with $e_{0,n}^{(s)} = 1$. In the following we obtain an alternative expansion of $e_{m,n}^{(s)}$ in terms of $n^{\underline{s}}$, the falling factorial powers of n .

Lemma 2. *The s -th factorial moments $e_{m,n}^{(s)} = \mathbb{E}\left(\frac{X_{dm,an}}{a}\right)^s$ of the normalized random variable $X_{dm,an}/a$ satisfies the expansion $e_{m,n}^{(s)} = \sum_{\ell=0}^s \Lambda_{s,\ell,m} n^{\underline{\ell}}$, with $\Lambda_{s,s,m} = \frac{1}{\binom{m+\frac{as}{d}}{m}}$, and $\Lambda_{s,\ell,m}$, $1 \leq \ell \leq s-1$, recursively described by*

$$\Lambda_{s,\ell,m} = \frac{1}{\binom{m+\frac{a\ell}{d}}{m}} \sum_{k=0}^{m-1} \binom{k+\frac{a\ell}{d}}{k} \sum_{j=\ell+1}^s \binom{j}{\ell} p^{j-\ell} \Lambda_{s,j,m}.$$

The initial values are given by $\Lambda_{s,s,0} = 1$ and $\Lambda_{s,\ell,0} = 0$, for $0 \leq \ell \leq s-1$.

Moreover, the values $\lambda_{s,\ell,m}$, with $e_{m,n}^{[s]} = \mathbb{E}(X_{dm,an}^s) = \sum_{\ell=0}^s n^{\underline{\ell}} \lambda_{s,\ell,m}$, arising in the expansion of the ordinary moments of $X_{dm,an}$, are related to $\Lambda_{s,j,m}$ in the following way.

$$\lambda_{s,\ell,m} = a^s \sum_{r=\ell}^s \sum_{j=\ell}^r (-1)^{j-\ell} \begin{Bmatrix} s \\ r \end{Bmatrix} \begin{bmatrix} j \\ \ell \end{bmatrix} \Lambda_{r,j,m}, \quad 0 \leq \ell \leq s,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the unsigned Stirling numbers of the first kind, and $\begin{Bmatrix} n \\ k \end{Bmatrix}$ denotes the Stirling numbers of the second kind, respectively.

As an immediate consequence of the result above we get explicit results for all moments in the case $c = 0$.

Corollary 1. *In the case $c = 0$ the s -th factorial moment $\mathbb{E}\left(\frac{X_{dm,an}}{a}\right)^s$ of the normalized random variable $X_{dm,an}/a$ is given by*

$$\mathbb{E}\left(\frac{X_{dm,an}}{a}\right)^s = \frac{n^{\underline{s}}}{\binom{m+\frac{as}{d}}{m}}.$$

Consequently, the ordinary s -th moment of $X_{dm,an}$ is given by

$$\mathbb{E}(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^{\underline{\ell}} \sum_{j=\ell}^s (-1)^{j-\ell} \frac{\begin{Bmatrix} s \\ j \end{Bmatrix} \begin{bmatrix} j \\ \ell \end{bmatrix}}{\binom{m+\frac{aj}{d}}{m}}.$$

Proof of Lemma 2. First we note that the factorial moments $e_{m,n}^{(s)}$ satisfy the same recurrence relations (5) as their ordinary counterparts $e_{m,n}^{[s]}$, only the initial condition changes to $e_{0,n}^{(s)} = \mathbb{E}\left(\frac{X_{0,an}}{a}\right)^s = n^{\underline{s}}$. We proceed as in the proof of Lemma 1 and obtain by the ansatz $e_{m,n}^{(s)} = \sum_{k=0}^s \Lambda_{s,k,m} n^{\underline{k}}$ the equation.

$$(an + dm) \sum_{\ell=0}^s \Lambda_{s,\ell,m} n^{\underline{\ell}} = an \sum_{\ell=0}^s \Lambda_{s,\ell,m} (n-1)^{\underline{\ell}} + dm \sum_{\ell=0}^s \Lambda_{s,\ell,m-1} (n+p)^{\underline{\ell}}. \quad (7)$$

Next we use the facts

$$(an + dm)n^{\underline{\ell}} = (a(n-\ell) + al + dm)n^{\underline{\ell}} = an^{\underline{\ell+1}} + al + dmn^{\underline{\ell}}, \quad n \cdot (n-1)^{\underline{\ell}} = n^{\underline{\ell+1}},$$

and the binomial theorem for falling factorials

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k},$$

in order to write (7) as

$$\sum_{\ell=0}^s \Lambda_{s,\ell,m} \left(an^{\underline{\ell+1}} + (al + dm)n^{\underline{\ell}} \right) = a \sum_{\ell=0}^s \Lambda_{s,\ell,m} n^{\underline{\ell+1}} + dm \sum_{j=0}^s n^{\underline{j}} \sum_{\ell=j}^s \binom{\ell}{j} p^{\ell-j} \Lambda_{s,\ell,m-1}.$$

The equation above simplifies to

$$\sum_{\ell=0}^s \Lambda_{s,\ell,m} (a\ell + dm) n^\ell = dm \sum_{j=0}^s n^j \sum_{\ell=j}^s \binom{\ell}{j} p^{\ell-j} \Lambda_{s,\ell,m-1}.$$

Comparing the coefficients of n^ℓ , the falling factorial powers of n , we obtain the equations

$$\begin{aligned} \Lambda_{s,s,m} (as + dm) &= dm \Lambda_{s,s,m-1}, \quad \ell = s, \\ \Lambda_{s,\ell,m} (as + dm) &= dm \Lambda_{s,\ell,m-1} + dm \sum_{j=\ell+1}^s \binom{j}{\ell} p^{j-\ell} \Lambda_{s,j,m-1}, \quad 0 \leq \ell \leq s-1. \end{aligned} \quad (8)$$

Consequently, we obtain

$$\Lambda_{s,s,m} = \frac{m}{m + \frac{as}{d}} \Lambda_{s,s,m-1}, \quad \text{and further } \Lambda_{s,s,m} = \frac{1}{\binom{m + \frac{as}{d}}{m}}.$$

Moreover, we also get from (8) the stated recurrence relation for $\Lambda_{s,\ell,m}$,

$$\Lambda_{s,\ell,m} = \frac{m}{m + \frac{a\ell}{d}} \Lambda_{s,\ell,m-1} + \frac{m}{m + \frac{a\ell}{d}} \sum_{j=\ell+1}^s \binom{j}{\ell} p^{j-\ell} \Lambda_{s,j,m-1}.$$

In order to obtain the stated relation between $\lambda_{s,\ell,m}$ and $\Lambda_{s,j,m}$, $0 \leq \ell, j \leq s$, we use the expansion of the ordinary moments into factorial moments using the Stirling numbers of the first and second kind.

$$\begin{aligned} \mathbb{E}(X_{dm,an}^s) &= a^s \mathbb{E}\left(\frac{X_{dm,an}}{a}\right)^s = a^s \sum_{r=0}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \mathbb{E}\left(\frac{X_{dm,an}}{a}\right)^r = a^s \sum_{r=0}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \sum_{j=0}^r n^j \Lambda_{r,j,m} \\ &= a^s \sum_{r=0}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \sum_{j=0}^r \Lambda_{r,j,m} \sum_{\ell=0}^j \left[\begin{matrix} j \\ \ell \end{matrix} \right] (-1)^{j-\ell} n^\ell = a^s \sum_{\ell=0}^s n^\ell \sum_{r=\ell}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \sum_{j=\ell}^r \Lambda_{r,j,m} \left[\begin{matrix} j \\ \ell \end{matrix} \right] (-1)^{j-\ell}. \end{aligned}$$

On the other hand, $\mathbb{E}(X_{dm,an}^s) = \sum_{\ell=0}^s n^\ell \lambda_{s,\ell,m}$, which proves the stated result. \square

Proof of Corollary 1. In the case of $c = a \cdot p = 0$, or equivalently $p = 0$, we obtain from Lemma 2 the result

$$e_{m,n}^{(s)} = \sum_{\ell=0}^s \Lambda_{s,\ell,m} n^\ell = n^s \Lambda_{s,s,m} = \frac{n^s}{\binom{m + \frac{as}{d}}{m}},$$

since all terms $\Lambda_{s,\ell,m}$, $0 \leq \ell \leq s-1$, are zero due to the factor p . This implies that the ordinary moments are explicitly given by

$$\mathbb{E}(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^\ell \sum_{j=\ell}^s (-1)^{j-\ell} \frac{\left\{ \begin{matrix} s \\ j \end{matrix} \right\} \left[\begin{matrix} j \\ \ell \end{matrix} \right]}{\binom{m + \frac{aj}{d}}{m}}.$$

\square

3.3. The fine structure of the moments for $a/d > 1$. Next we are going to use Lemma 2 to obtain a refinement of the description of the factorial moments.

Proposition 1. For $a/d > 1$ and $c \neq 0$ the values $\Lambda_{s,\ell,m}$, arising in the expansion of the s -th factorial moment $e_{m,n}^{(s)} = \mathbb{E}\left(\frac{X_{dm,an}}{a}\right)^s = \sum_{\ell=0}^s n^\ell \Lambda_{s,\ell,m}$ of the random variable $X_{dm,an}$, satisfy

$$\Lambda_{s,\ell,m} = \sum_{h=\ell}^s \frac{1}{\binom{m + \frac{ah}{d}}{m}} \sum_{g=0}^{h-\ell} \vartheta_{s,\ell,h,g} m^g,$$

where the values $\vartheta_{s,\ell;j,i}$ satisfy $\vartheta_{s,s;s,0} = 1$,

$$\vartheta_{s,\ell;\ell,0} = - \sum_{j=\ell+1}^s \binom{j}{\ell} p^{j-\ell} \sum_{h=j}^s \sum_{i=0}^{h-j} \vartheta_{s,j;h,i} \cdot q_i \left(\frac{ah}{d}, \frac{al}{d}\right),$$

and further for $\ell + 1 \leq h \leq s - 1$ and $0 \leq g \leq h - \ell$

$$\vartheta_{s,\ell;h,g} = \sum_{j=\ell+1}^{\min\{h,h-g+1\}} \binom{j}{\ell} p^{j-\ell} \sum_{i=\max\{0,g-1\}}^{h-j} \vartheta_{s,j;h,i} \cdot p_{i,g} \left(\frac{ah}{d}, \frac{al}{d} \right).$$

The quantities $p_{i,\ell}(X, Y)$, $0 \leq \ell \leq i + 1$, and $q_i(X, Y)$ arising here will be defined in Lemma 3 stated below.

In order to give the proof of the result above, we need the following identity.

Lemma 3. The sum $\frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} k^i$ can be expanded in the following way:

$$\frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} k^i = \sum_{\ell=0}^{i+1} \frac{m^\ell p_{i,\ell}(X, Y)}{\binom{m+X}{m}} - \frac{q_i(X, Y)}{\binom{m+Y}{m}}, \quad (9)$$

assuming that $Y + h + 1 - X \neq 0$, for $0 \leq h \leq i$, $i \geq 0$, with

$$p_{i,\ell}(X, Y) = \sum_{j=\ell}^{i+1} \binom{j}{\ell} (-1)^{j-\ell} \sum_{h=\max\{0,j-1\}}^i \frac{\binom{h+1}{j} X^{h+1-j}}{h!(Y+h+1-X)} \sum_{f=h}^i (-X)^{i-f} \begin{Bmatrix} i \\ f \end{Bmatrix} \begin{Bmatrix} f \\ h \end{Bmatrix}, \quad 0 \leq \ell \leq i+1,$$

$$q_i(X, Y) = \sum_{h=0}^i \sum_{f=h}^i (-X)^{i-f} \begin{Bmatrix} i \\ f \end{Bmatrix} \begin{Bmatrix} f \\ h \end{Bmatrix} \frac{(X-h) \binom{X}{h}}{(Y+1+h-X)}.$$

Proof. We use the identity,

$$\sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} (k+X)^j = \frac{(m+X-j) \binom{m+Y}{m} \binom{m+X}{j}}{(j+1+Y-X) \binom{m+X}{m}} - \frac{(X-j) \binom{X}{j}}{j+1+Y-X}, \quad (10)$$

with $j+1+X-Y \neq 0$, and $j \geq 0$, which can be proven using induction with respect to m . In order to apply the result above to the sum $\frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} k^i$ we expand k^i in the following way:

$$k^i = (k+X-X)^i = \sum_{f=0}^i \binom{i}{f} (-X)^{i-f} (k+X)^f = \sum_{f=0}^i \binom{i}{f} (-X)^{i-f} \sum_{h=0}^f \begin{Bmatrix} f \\ h \end{Bmatrix} (k+X)^h$$

$$= \sum_{h=0}^i (k+X)^h \sum_{f=h}^i \binom{i}{f} \begin{Bmatrix} f \\ h \end{Bmatrix} (-X)^{i-f}.$$

Consequently, we get

$$\frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} k^i = \sum_{h=0}^i \frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} (k+X)^h \sum_{f=h}^i \binom{i}{f} \begin{Bmatrix} f \\ h \end{Bmatrix} (-X)^{i-f}$$

$$= \sum_{h=0}^i \left(\frac{(m+X-h) \binom{m+X}{h}}{(h+1+Y-X) \binom{m+X}{m}} - \frac{(X-h) \binom{X}{h}}{(h+1+Y-X) \binom{m+Y}{m}} \right) \sum_{f=h}^i \binom{i}{f} \begin{Bmatrix} f \\ h \end{Bmatrix} (-X)^{i-f}.$$

This proves the stated result for $q_i(X, Y)$. In order to obtain the expressions for $p_{i,\ell}(X, Y)$, $0 \leq \ell \leq i + 1$, we expand $(m+X-h) \binom{m+X}{h}$ in terms of ordinary powers of m ,

$$(m+X-h) \binom{m+X}{h} = \frac{(m+X)^{h+1}}{h!} = \frac{1}{h!} \sum_{j=0}^{h+1} \binom{h+1}{j} m^j \cdot X^{h+1-j}$$

$$= \frac{1}{h!} \sum_{j=0}^{h+1} \binom{h+1}{j} \sum_{\ell=0}^j \binom{j}{\ell} (-1)^{j-\ell} m^\ell X^{h+1-j} = \frac{1}{h!} \sum_{\ell=0}^{h+1} m^\ell \sum_{j=\ell}^{h+1} \binom{h+1}{j} \binom{j}{\ell} (-1)^{j-\ell} X^{h+1-j}.$$

Interchanging summations leads then directly to the stated results for $p_{i,\ell}(X, Y)$. \square

Proof of Proposition 1. We proceed by induction with respect to m and ℓ . We readily observe that $\Lambda_{s,s,m}$ satisfies the stated expansion. Assuming that the values $\Lambda_{s,j,k}$ have the stated expansion for all $k < m$ and $\ell + 1 \leq j \leq s$ we get by Lemma 2 and the induction hypothesis

$$\begin{aligned}\Lambda_{s,\ell,m} &= \frac{1}{\binom{m+\frac{a\ell}{d}}{m}} \sum_{k=0}^{m-1} \binom{k+\frac{a\ell}{d}}{k} \sum_{j=\ell+1}^s \binom{j}{\ell} p^{j-\ell} \Lambda_{s,j,k} \\ &= \sum_{j=\ell+1}^s \binom{j}{\ell} p^{j-\ell} \sum_{h=j}^s \sum_{i=0}^{h-j} \vartheta_{s,j;h,i} \frac{1}{\binom{m+\frac{a\ell}{d}}{m}} \sum_{k=0}^{m-1} \binom{k+\frac{a\ell}{d}}{\binom{k+\frac{ah}{d}}{k}} k^i.\end{aligned}$$

Before we can apply Lemma 3 we need to check that the conditions $Y + 1 + g - X \neq 0$ are satisfied, for $Y = \frac{a\ell}{d}$, $X = \frac{ah}{d}$, $0 \leq g \leq i$. We have $1 \leq 1 + g \leq i + 1 \leq h - j + 1 \leq h - \ell$, with equality only in the case $j = \ell + 1$. Hence, from our assumption $a/d > 1$ we get $1 + g \leq h - \ell < \frac{a}{d}(h - \ell)$, such that $1 + g - \frac{a}{d}(h - \ell) \neq 0$, $0 \leq g \leq i$. We obtain by Lemma 3

$$\Lambda_{s,\ell,m} = \sum_{j=\ell+1}^s \binom{j}{\ell} p^{j-\ell} \sum_{h=j}^s \sum_{i=0}^{h-j} \vartheta_{s,j;h,i} \left(\sum_{g=0}^{i+1} \frac{m^g p_{i,g}(\frac{ah}{d}, \frac{a\ell}{d})}{\binom{m+\frac{ah}{d}}{m}} - \frac{q_i(\frac{ah}{d}, \frac{a\ell}{d})}{\binom{m+\frac{a\ell}{d}}{m}} \right).$$

This implies the stated result for $\vartheta_{s,\ell;\ell,0}$. Furthermore, we obtain

$$\begin{aligned}&\sum_{j=\ell+1}^s \binom{j}{\ell} p^{j-\ell} \sum_{h=j}^s \sum_{i=0}^{h-j} \vartheta_{s,j;h,i} \sum_{g=0}^{i+1} \frac{m^g p_{i,g}(\frac{ah}{d}, \frac{a\ell}{d})}{\binom{m+\frac{ah}{d}}{m}} \\ &= \sum_{h=\ell+1}^s \frac{1}{\binom{m+\frac{ah}{d}}{m}} \sum_{g=0}^{h-\ell} m^g \sum_{j=\ell+1}^{\min\{h, h-g+1\}} \binom{j}{\ell} p^{j-\ell} \sum_{i=\max\{0, g-1\}}^{h-j} \vartheta_{s,j;h,i} p_{i,g}(\frac{ah}{d}, \frac{a\ell}{d}),\end{aligned}$$

which leads to the stated results. \square

3.4. Derivation of the expected value. Next we will derive the explicit expressions for the expectation of $X_{dm,an}$ using Lemma 1.

Proposition 2. *The expectation of $X_{dm,an}$ is given as follows:*

$$\begin{aligned}\mathbb{E}(X_{dm,an}) &= \frac{an}{m+1} + cH_m, & \text{for } \frac{a}{d} = 1, \\ \mathbb{E}(X_{dm,an}) &= \frac{an}{\binom{m+\frac{a}{d}}{m}} + \frac{c}{d-a} \left(\frac{dm+a}{\binom{m+\frac{a}{d}}{m}} - a \right), & \text{for } \frac{a}{d} \neq 1.\end{aligned}$$

Proof. In order to obtain the expected value of $X_{dm,an}$ we use Lemma 1 to get

$$\mathbb{E}(X_{dm,an}) = \lambda_{1,1,m}n + \lambda_{1,0,m},$$

where the values $\lambda_{1,1,m}$, $\lambda_{1,0,m}$ are given by

$$\lambda_{1,1,m} = \frac{a}{\binom{m+\frac{a}{d}}{m}}, \quad \lambda_{1,0,m} = \lambda_{1,0,m-1} + p^1 \lambda_{1,1,m}.$$

This implies that $\lambda_{1,0,m}$ can be written as

$$\lambda_{1,0,m} = p \sum_{k=0}^{m-1} \lambda_{1,k,m} = p \cdot a \sum_{k=0}^{m-1} \frac{1}{\binom{k+\frac{a}{d}}{k}}.$$

We have to distinguish between the cases $\frac{a}{d} = 1$ and $\frac{a}{d} \neq 1$. First assume that $\frac{a}{d} = 1$. We obtain

$$\sum_{k=0}^{m-1} \frac{1}{\binom{k+\frac{a}{d}}{k}} = \sum_{k=0}^{m-1} \frac{1}{\binom{k+1}{k}} = \sum_{k=0}^{m-1} \frac{1}{k+1} = H_m,$$

and further $\mathbb{E}(X_{dm,an}) = \frac{an}{m+1} + cH_m$. In the remaining case $\frac{a}{d} \neq 1$ we use the summation formula

$$\sum_{k=0}^{m-1} \frac{1}{\binom{k+X}{X}} = \frac{m+X}{(1-X)\binom{m+X}{X}} - \frac{X}{1-X},$$

which can easily be deduced from another summation formula, see [5], p. 188,

$$\sum_{\ell=0}^k \binom{k}{\ell} \frac{(-1)^\ell}{X + \ell} = \frac{1}{X \binom{k+X}{k}},$$

using the fact that $\sum_{k=0}^{m-1} \binom{k}{\ell} = \binom{m}{\ell+1}$. We get

$$\sum_{k=0}^{m-1} \frac{1}{\binom{k+\frac{a}{d}}{k}} = \frac{dm+a}{(d-a)\binom{m+\frac{a}{d}}{m}} - \frac{a}{d-a},$$

which directly leads to the result

$$\mathbb{E}(X_{dm,an}) = \frac{an}{\binom{m+\frac{a}{d}}{m}} + \frac{c(dm+a)}{(d-a)\binom{m+\frac{a}{d}}{m}} - \frac{ca}{d-a}.$$

□

3.5. Asymptotic expansions of the expected value. Next we derive asymptotic expansions of the expected value $\mathbb{E}(X_{dm,an})$ for $\max\{m, n\} \rightarrow \infty$. These expansions serve as an indicator for the normalizations used in Theorems 1-3 for the random variables $X_{dm,an}$.

Lemma 4. *For $m \in \mathbb{N}$ fixed and $n \rightarrow \infty$, and arbitrary a, c, d , the expected value of $X_{dm,an}$, as given in Theorem 2, is asymptotically given by*

$$\mathbb{E}(X_{dm,an}) = \frac{an}{\binom{m+\frac{a}{d}}{m}} + \mathcal{O}(1).$$

For $m \rightarrow \infty$, $a/d \leq 1$ and $c \neq 0$ and arbitrary $n = n(m)$ the expected value of $X_{dm,an}$ always tends to infinity:

$$\mathbb{E}(X_{dm,an}) \sim \begin{cases} \frac{an}{m} + c \log m, & \text{for } \frac{a}{d} = 1, \\ \Gamma(1 + \frac{a}{d}) \cdot \frac{an + \frac{cd}{d-a}m}{m^{\frac{a}{d}}}, & \text{for } \frac{a}{d} < 1. \end{cases}$$

For $m \rightarrow \infty$, $a/d > 1$, or $a/d \leq 1$ together with $c = 0$, we have the following three regions in the asymptotic behaviour of the expected value of $X_{dm,an}$:

- For $m^{a/d} = o(n)$ we have

$$\mathbb{E}(X_{dm,an}) = \Gamma(1 + \frac{a}{d}) \frac{an}{m^{\frac{a}{d}}} + \mathcal{O}\left(\frac{n}{m^{\frac{a}{d}+1}} + \frac{c}{m^{\frac{a}{d}-1}} + 1\right).$$

- For $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$, we have

$$\mathbb{E}(X_{dm,an}) = a\Gamma(1 + \frac{a}{d})\rho + \frac{ca}{a-d} + \mathcal{O}\left(\frac{1}{m} + \frac{c}{m^{\frac{a}{d}-1}} + \frac{1}{m^{\frac{a}{d}}}\right).$$

- For $n = n(m)$ such that $n = o(m^{a/d})$, we have

$$\mathbb{E}(X_{dm,an}) = \frac{ca}{a-d} + \mathcal{O}\left(\frac{n}{m^{\frac{a}{d}}} + \frac{c}{m^{\frac{a}{d}-1}}\right). \quad (11)$$

Remark 4. Our results above say in principle that the asymptotic behaviour of $X_{dm,an}$ is governed by the quotient a/d , together with the (non-)positivity of c . A similar situation occurs in tenable triangular urn schemes, compare with Janson [10]. A simple explanation for the results above is as follows. For m fixed and n tending to infinity, the actual values of a, d and c are irrelevant. For m tending to infinity, such that $a/d < 1$ and $c \neq 0$, the positivity of c ensures that the random variable always tends to infinity. In the remaining cases with m tending to infinity, such that $a/d > 1$, or $c = 0$ and arbitrary a, d , the random variable $X_{dm,an}$ can be rather small, depending on the growth of $n = n(m)$ compared to m .

Proof. We use the explicit results stated in Theorem 2. For m fixed and $n \rightarrow \infty$ we use Stirling's formula for the Gamma function:

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \frac{\sqrt{2\pi}}{\sqrt{z}} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right), \quad (12)$$

and obtain for arbitrary a, c, d the expansion

$$\mathbb{E}(X_{dm,an}) = \frac{an}{\left(\frac{m+\frac{a}{d}}{m}\right)} + \mathcal{O}(1) = an \frac{\Gamma(m+1)\Gamma(1+\frac{a}{d})}{\Gamma(m+1+\frac{a}{d})} + \mathcal{O}(1) = \Gamma(1+\frac{a}{d}) \frac{an}{m^{\frac{a}{d}}} + \mathcal{O}(1).$$

Assume next that $m \rightarrow \infty$, $a/d \leq 1$ and $c \neq 0$ and arbitrary $n = n(m)$. We use the asymptotic expansion of the harmonic numbers

$$H_m = \log m + \gamma - \frac{1}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right),$$

where $\gamma = 0.5772156649$ denotes the Euler-Mascheroni constant, to get for $a/d = 1$ the result

$$\mathbb{E}(X_{dm,an}) = \frac{an}{m+1} + cH_m \sim \frac{an}{m} + c \log m.$$

For $a/d < 1$ we get

$$\mathbb{E}(X_{dm,an}) = \frac{an}{\left(\frac{m+\frac{a}{d}}{m}\right)} + \frac{c}{d-a} \left(\frac{dm+a}{\left(\frac{m+\frac{a}{d}}{m}\right)} - a \right) \sim \Gamma(1+\frac{a}{d}) \frac{an}{m^{\frac{a}{d}}} + \frac{cd}{d-a} \Gamma(1+\frac{a}{d}) m^{1-\frac{a}{d}},$$

where we have used again Stirling's formula (12). For $m \rightarrow \infty$, $a/d > 1$, or $a/d \leq 1$ together with $c = 0$, we proceed similarly to the previous cases. For example, assuming that $m^{a/d} = o(n)$, we obtain by Stirling's formula (12) the result

$$\mathbb{E}(X_{dm,an}) = \frac{an}{\left(\frac{m+\frac{a}{d}}{m}\right)} + \frac{c}{d-a} \left(\frac{dm+a}{\left(\frac{m+\frac{a}{d}}{m}\right)} - a \right) = \Gamma(1+\frac{a}{d}) \frac{an}{m^{\frac{a}{d}}} \left(1 + \mathcal{O}\left(\frac{1}{m}\right)\right) + \mathcal{O}\left(\frac{c}{m^{\frac{a}{d}-1}} + 1\right).$$

□

4. DERIVATION OF THE LIMITING DISTRIBUTIONS

In the following we will present our proofs of Theorems 1-3. First we prove simultaneously the limit laws of the Theorems 1-3 in the case of fixed m , and n tending to infinity. Then, we separately provide the remaining proofs of the Theorems 1-3 for m tending to infinity and $n = n(m)$ in Subsections 4.2, 4.3, 4.4.

4.1. The case of fixed m . We assume that m is an arbitrary but fixed natural number, and derive the limit of $X_{dm,an}$ for n tending to infinity. Using Lemma 1 we can expand the s -th moment of $X_{dm,an}$ for arbitrary values of $a, d, c \in \mathbb{N}$ in powers of n in the following way:

$$\mathbb{E}(X_{dm,an}^s) = e_{m,n}^{[s]} = \sum_{k=0}^s \lambda_{s,k,m} n^k = \lambda_{s,s,m} n^s + \sum_{k=0}^{s-1} \lambda_{s,k,m} n^k = \lambda_{s,s,m} n^s + \mathcal{O}(n^{s-1}) = \frac{a^s n^s}{\left(\frac{m+\frac{as}{d}}{m}\right)} + \mathcal{O}(n^{s-1}),$$

since we assumed that m is an arbitrary but fixed natural number. Consequently, the moments of the normalized random variable $X_{dm,an}/(an)$ satisfy the following asymptotic expansion:

$$\mathbb{E}\left(\frac{X_{dm,an}^s}{a^s n^s}\right) = \frac{e_{m,n}^{[s]}}{a^s n^s} = \frac{1}{\left(\frac{m+\frac{as}{d}}{m}\right)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) = \frac{m! \Gamma(1+\frac{as}{d})}{\Gamma(1+m+\frac{as}{d})} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Hence the s -th moment of the scaled random variable $X_{dm,an}/(a \cdot n)$ tends to the s -th moment of a Kumaraswamy distributed random variable $Y = Y(d/a, m)$ with parameters $\alpha = d/a$ and $\beta = m$, for any $s \geq 1$, in symbol $\mathbb{E}\left(\frac{X_{dm,an}^s}{a^s n^s}\right) \rightarrow \mathbb{E}(Y^s)$. Hence, by the theorem of Fréchet and Shohat, which says that the moment convergence implies the convergence in distribution if the moments sequence determines a unique distribution, we obtain the convergence in distribution of $X_{dm,an}/(a \cdot n)$ to $Y(d/a, m)$.

Note that the density $f(t) = f_Y(t)$ and the distribution function of a Kumaraswamy distributed random variable $Y = Y(\alpha, \beta)$ are given as follows:

$$f(t) = \alpha \beta t^{\alpha-1} (1-t^\alpha)^{\beta-1}, \quad \text{and} \quad \mathbb{P}\{Y \leq x\} = 1 - (1-x^\alpha)^\beta.$$

Furthermore, the s -th moment of Y is given by

$$\mathbb{E}(Y^s) = \frac{\Gamma(\beta+1)\Gamma(1+\frac{s}{\alpha})}{\Gamma(1+\beta+\frac{s}{\alpha})}, \quad \text{for } s \geq 1.$$

The Kumaraswamy distribution is closely related to the beta distribution. A Kumaraswamy distributed random variable $Y = Y(\alpha, \beta)$ can be expressed as the α -th root of a beta distributed random variable with parameters 1 and β .

4.2. Proof of Theorem 1. We use the explicit results for the moments of $X_{dm,an}$ in the case $c = 0$ and arbitrary a, d stated in Corollary 1:

$$\mathbb{E}(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^\ell \sum_{j=\ell}^s (-1)^{j-\ell} \frac{\left\{ \begin{smallmatrix} s \\ j \end{smallmatrix} \right\} \left[\begin{smallmatrix} j \\ \ell \end{smallmatrix} \right]}{\binom{m+\frac{aj}{d}}{m}}.$$

Hence, using Stirling's formula (12) we obtain for $m \rightarrow \infty$ the asymptotic expansions

$$\frac{1}{\binom{m+\frac{aj}{d}}{m}} \sim \frac{\Gamma(1 + \frac{aj}{d})}{m^{\frac{aj}{d}}}, \quad \mathbb{E}(X_{dm,an}^s) \sim a^s \sum_{\ell=0}^s n^\ell \sum_{j=\ell}^s (-1)^{j-\ell} \frac{\Gamma(1 + \frac{aj}{d}) \left\{ \begin{smallmatrix} s \\ j \end{smallmatrix} \right\} \left[\begin{smallmatrix} j \\ \ell \end{smallmatrix} \right]}{m^{\frac{aj}{d}}}. \quad (13)$$

Assume first that $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$. The dominant term in the expansion above is given by $a^s n^s \Gamma(1 + \frac{as}{d}) / m^{\frac{as}{d}}$, and we get

$$\mathbb{E}\left(\frac{X_{dm,an}^s m^{\frac{as}{d}}}{a^s n^s}\right) \sim \Gamma\left(1 + \frac{as}{d}\right).$$

Let $W = W(k, \lambda)$ denote a Weibull distributed random variable. Then, the s -th moment of W is given by

$$\mathbb{E}(W^s) = \lambda^n \Gamma\left(1 + \frac{s}{k}\right), \quad \text{for } s \geq 1.$$

Moreover, special instances of the Weibull distribution are the exponential distribution and the Rayleigh distribution, and the density $f(t) = f_W(t)$ is given as follows

$$f(t) = \frac{k}{\lambda} \left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k}, \quad \text{for } t \geq 0. \quad (14)$$

Hence, in the region $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$, we can use the moment convergence theorem of Fréchet and Shohat and obtain the convergence in distribution of $X_{dm,an} m^{a/d} / (a \cdot n)$ to $W(d/a, 1)$.

For $m, n \rightarrow \infty$ such that $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$, we obtain from (13) the expansion

$$\mathbb{E}(X_{dm,an}^s) \sim a^s \sum_{\ell=0}^s (\rho m^{a/d})^\ell \frac{\Gamma(1 + \frac{a\ell}{d}) \left\{ \begin{smallmatrix} s \\ \ell \end{smallmatrix} \right\}}{m^{\frac{a\ell}{d}}} = a^s \sum_{\ell=0}^s \left\{ \begin{smallmatrix} s \\ \ell \end{smallmatrix} \right\} \rho^\ell \Gamma\left(1 + \frac{a\ell}{d}\right).$$

Consequently,

$$\lim_{m, n \rightarrow \infty} \mathbb{E}(X_{dm,an}^s) = \eta_s, \quad \text{where } \eta_s := a^s \sum_{\ell=0}^s \left\{ \begin{smallmatrix} s \\ \ell \end{smallmatrix} \right\} \rho^\ell \Gamma\left(1 + \frac{a\ell}{d}\right).$$

In order to show that the sequence of moments $(\eta_s)_s$ defines a unique distribution, we compute the corresponding moment generating function $\varphi(z)$ associated to the moment sequence $(\eta_s)_s$:

$$\varphi(z) = \sum_{s \geq 0} \eta_s \frac{z^s}{s!} = \sum_{s \geq 0} a^s \sum_{l=0}^s \left\{ \begin{smallmatrix} s \\ l \end{smallmatrix} \right\} \rho^l \Gamma\left(1 + \frac{al}{d}\right) \frac{z^s}{s!} = \sum_{l \geq 0} \rho^l \Gamma\left(1 + \frac{al}{d}\right) \sum_{s \geq l} a^s \left\{ \begin{smallmatrix} s \\ l \end{smallmatrix} \right\} \frac{z^s}{s!}.$$

Using the bivariate generating function identity of the Stirling numbers of the second kind, see Wilf [21],

$$\sum_{n \geq 0} \sum_{k \geq 0} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \frac{z^n}{n!} u^k = e^{u(e^z - 1)},$$

we obtain further

$$\varphi(z) = \sum_{\ell \geq 0} \Gamma\left(1 + \frac{a\ell}{d}\right) \frac{\rho^\ell (e^{az} - 1)^\ell}{\ell!}.$$

Since the latter expression is the moment generating function $\psi(t)$ of a Weibull distributed random variable with parameters a/d and 1, evaluated at $t = \rho(e^{az} - 1)$, we see that $\varphi(z)$ is indeed analytic around $z = 0$, which completes the proof. Hence, by the theorem of Fréchet and Shohat $X_{dm,an}$ converges in distribution

to a random variable X with moment sequence $(\eta_s)_s$. Note that the special choice $a = d$, sampling without replacement, gives a (scaled) geometric distribution with parameter $\frac{1}{1+\rho}$, due to

$$\varphi(z) = \sum_{\ell \geq 0} \rho^\ell (e^{az} - 1)^\ell = \frac{1}{1 - \rho(e^{az} - 1)} = \frac{1}{1 + \rho - \rho e^{az}} = \frac{\frac{1}{1+\rho}}{1 - (1 - \frac{1}{1+\rho})e^{az}}.$$

In the remaining case $m \rightarrow \infty$, with arbitrary $n = n(m)$ satisfying that $n = o(m^{a/d})$, we obtain $\mathbb{E}(X_{dm,an}^s) \rightarrow 0$, for all $s \geq 1$, which proves the stated result.

4.3. Proof of Theorem 2. The limiting distributions of $X_{dm,an}$, for $a/d \leq 1$ and $c \neq 0$ and $m \rightarrow \infty$, will be obtained by giving precise estimates for the s -th moments $e_{m,n}^{[s]}$. Lemma 4 suggests the right scaling factors $g_{m,n}$ chosen according to the ratio a/d . We will provide the following estimates for the moments of $X_{dm,an}$:

$$e_{m,n}^{[s]} = \begin{cases} \Gamma(1 + \frac{as}{d}) \left(\frac{an + m \frac{cd}{d-a}}{m^{\frac{a}{d}}} \right)^s (1 + \mathcal{O}(\frac{1}{m^{a/d}})), & \text{for } \frac{a}{d} < 1, \\ s! \left(\frac{an}{m} + c \log m \right)^s (1 + \mathcal{O}(\frac{1}{\log m})), & \text{for } \frac{a}{d} = 1. \end{cases}$$

Note that the above expansions will imply the limiting distribution results by applying the method of moments:

$$\mathbb{E}\left(\frac{X_{dm,an}^s}{g_{m,n}^s}\right) = \frac{e_{m,n}^{[s]}}{g_{m,n}^s} = \begin{cases} \Gamma(1 + \frac{as}{d}) (1 + \mathcal{O}(\frac{1}{m^{\frac{a}{d}}}), & \text{for } \frac{a}{d} < 1, \\ s! (1 + \mathcal{O}(\frac{1}{\log m})), & \text{for } \frac{a}{d} = 1, \end{cases}$$

since the s -th moment of $X_{dm,an}/g_{m,n}$ converges to the s -th moment of a Weibull distribution with suitably chosen parameters.

First we consider the case $a/d < 1$. Since we want to prove the asymptotic expansion

$$e_{m,n}^{[s]} = \Gamma(1 + \frac{as}{d}) \left(\frac{an + m \frac{cd}{d-a}}{m^{\frac{a}{d}}} \right)^s (1 + \mathcal{O}(\frac{1}{m^{\frac{a}{d}}}),$$

we have to determine the asymptotic growth of the coefficients $\lambda_{s,\ell,m}$ appearing in the recursive description of the moments $e_{m,n}^{[s]}$ in Lemma 1. The shape of the s -th moment $e_{m,n}^{[s]} = \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell$ implies that we have to show the following asymptotic expansion of the numbers $\lambda_{s,\ell,m}$:

$$\lambda_{s,\ell,m} = a^\ell \Gamma(1 + \frac{as}{d}) \binom{s}{\ell} m^{s-\ell-\frac{as}{d}} \frac{(cd)^{s-\ell}}{(d-a)^{s-\ell}} \left(1 + \mathcal{O}(\frac{1}{m^{\frac{a}{d}}})\right). \quad (15)$$

To show this we will use induction with respect to ℓ and apply Euler's summation formula. The statement is true for $\ell = s$, since we know by Lemma 1 that

$$\lambda_{s,s,m} = \frac{a^s}{\binom{m+\frac{as}{d}}{m}} = a^s \Gamma(1 + \frac{as}{d}) \binom{s}{l} m^{-\frac{as}{d}} (1 + \mathcal{O}(\frac{1}{m^{\frac{a}{d}}}).$$

Using the induction hypothesis for $\ell+1$ up to $s-1$ we see that the dominant contribution to $\mu_{s,\ell,m}$ is stemming from the term $\lambda_{s,\ell+1,m-1}$ and we get

$$\mu_{s,\ell,m} \sim \binom{\ell+1}{\ell} \frac{dm}{dm + a\ell} p \lambda_{s,\ell+1,m-1}. \quad (16)$$

Due to Lemma 1 we also have

$$\lambda_{s,\ell,m} = \sum_{k=0}^{m-1} \frac{\binom{m}{k}}{\binom{m+\frac{a\ell}{d}}{k}} \mu_{s,\ell,m-k} = \frac{m!}{\Gamma(m + \frac{a\ell}{d} + 1)} \sum_{k=1}^m \frac{\Gamma(k + \frac{a\ell}{d} + 1)}{k!} \mu_{s,\ell,k}. \quad (17)$$

Using the induction hypothesis we obtain the approximation

$$\sum_{k=1}^m \frac{\Gamma(k + \frac{a\ell}{d} + 1)}{k!} \mu_{s,\ell,k} = a^{\ell+1} \Gamma(1 + \frac{as}{d}) \binom{\ell+1}{\ell} \binom{s}{\ell+1} p \frac{(cd)^{s-\ell}}{(d-a)^{s-\ell}} \sum_{k=\lfloor \log m \rfloor}^m \frac{\Gamma(k + \frac{a\ell}{d}) k^{s-\ell-\frac{as}{d}}}{k!} \left(1 + \mathcal{O}(\frac{1}{m^{\frac{a}{d}}})\right).$$

Now an application of Euler's summation formula (see, e.g., [5], p. 469) leads to

$$\begin{aligned}\lambda_{s,\ell,m} &= a^\ell c \Gamma\left(1 + \frac{as}{d}\right) \binom{\ell+1}{\ell} \binom{s}{\ell+1} m^{s-\ell-\frac{as}{d}} \frac{(cd)^{s-\ell-1}}{(d-a)^{s-\ell-1}} \int_0^1 t^{s-\ell-1-\frac{a(s-\ell)}{d}} dt \left(1 + \mathcal{O}\left(\frac{1}{m^{\frac{a}{d}}}\right)\right) \\ &= a^\ell c \Gamma\left(1 + \frac{as}{d}\right) \binom{s}{\ell+1} \binom{\ell+1}{\ell} m^{s-\ell-\frac{as}{d}} c \frac{(cd)^{s-\ell-1}}{(d-1)^{s-\ell-1} (s-\ell-\frac{a(s-\ell)}{d})} \left(1 + \mathcal{O}\left(\frac{1}{m^{\frac{a}{d}}}\right)\right) \\ &= a^\ell \Gamma\left(1 + \frac{as}{d}\right) \binom{s}{\ell} m^{s-\ell-\frac{as}{d}} \frac{(cd)^{s-\ell}}{(d-a)^{s-\ell}} \left(1 + \mathcal{O}\left(\frac{1}{m^{\frac{a}{d}}}\right)\right),\end{aligned}$$

which proves the stated result for $0 < \ell \leq s$. For $\ell = 0$ we have due to Lemma 1:

$$\lambda_{s,0,\ell} = \sum_{k=0}^{m-1} \sum_{i=1}^s p^i \lambda_{s,i,k} = p \sum_{k=0}^{m-1} \lambda_{s,1,k} \left(1 + \mathcal{O}\left(\frac{1}{m^{\frac{a}{d}}}\right)\right).$$

Using Euler's summation formula and the fact $c = a \cdot p$ gives

$$\begin{aligned}\lambda_{s,0,\ell} &= pa \Gamma\left(1 + \frac{as}{d}\right) \binom{s}{1} \frac{(cd)^{s-1}}{(d-a)^{s-1}} \int_0^m t^{s-1-\frac{as}{d}} dt \left(1 + \mathcal{O}\left(\frac{1}{m^{\frac{a}{d}}}\right)\right) \\ &= s \Gamma\left(1 + \frac{as}{d}\right) m^{s-\frac{as}{d}} \frac{c^s d^{s-1}}{(d-a)^{s-1} (s-\frac{sa}{d})} \left(1 + \mathcal{O}\left(\frac{1}{m^{\frac{a}{d}}}\right)\right) \\ &= \Gamma\left(1 + \frac{as}{d}\right) m^{s-\frac{as}{d}} \frac{(cd)^s}{(d-a)^s} \left(1 + \mathcal{O}\left(\frac{1}{m^{\frac{a}{d}}}\right)\right).\end{aligned}$$

This completes the proof of Theorem 2 for $a/d < 1$.

Now we consider the remaining case $a/d = 1$. We have to prove that

$$e_{m,n}^{[s]} = s! \left(\frac{an}{m} + c \log m\right)^s \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right). \quad (18)$$

This implies that we have to show the following asymptotic expansion of the numbers $\lambda_{s,\ell,m}$:

$$\lambda_{s,\ell,m} = s! \binom{s}{\ell} \frac{a^\ell c^{s-\ell} (\log m)^{s-\ell}}{m^\ell} \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right). \quad (19)$$

We proceed exactly as in the previous case $a/d < 1$. Using (16) and (17) we finally obtain

$$\begin{aligned}\lambda_{s,\ell,m} &= a^{\ell+1} p c^{s-\ell-1} s! \binom{\ell+1}{\ell} \binom{s}{\ell+1} \frac{m!}{(m+\ell)!} \int_1^m \frac{(\log t)^{s-\ell-1}}{t} dt \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right) \\ &= a^\ell c^{s-\ell} s! \binom{\ell+1}{\ell} \binom{s}{\ell+1} \frac{1}{m^\ell} \frac{(\log(m))^{s-\ell}}{s-\ell} \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right) \\ &= s! \binom{s}{\ell} \frac{a^\ell c^{s-\ell} (\log m)^{s-\ell}}{m^\ell} \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right),\end{aligned}$$

which proves the stated result for $0 < \ell \leq s$. The remaining case $\ell = 0$ is treated in a fully analogous manner. Hence for $m \rightarrow \infty$ the limiting distribution is given by an exponential distribution with parameter 1, which also proves the part $a/d = 1$ of Theorem 2.

4.4. Proof of Theorem 3. We use the results of Lemma 2 and Proposition 10 and study the moments

$$\mathbb{E}(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^\ell \sum_{r=\ell}^s \sum_{j=\ell}^r (-1)^{j-\ell} \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \left[\begin{matrix} j \\ \ell \end{matrix} \right] \sum_{h=j}^r \frac{1}{\binom{m+\frac{ah}{d}}{m}} \sum_{g=0}^{h-j} \vartheta_{r,j;h,g} m^g,$$

for $m \rightarrow \infty$. Interchanging summations gives

$$\mathbb{E}(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^\ell \sum_{h=\ell}^s \frac{1}{\binom{m+\frac{ah}{d}}{m}} \sum_{g=0}^{h-\ell} m^g \sum_{r=h}^s \sum_{j=g+\ell}^h (-1)^{j-\ell} \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \left[\begin{matrix} j \\ \ell \end{matrix} \right] \vartheta_{r,j;h,g}.$$

Proceeding as in Subsection 4.2 we use the expansions $1/\binom{m+\frac{ah}{d}}{m} \sim \Gamma(1 + \frac{ah}{d})/m^{\frac{ah}{d}}$, and obtain

$$\mathbb{E}(X_{dm,an}^s) \sim a^s \sum_{\ell=0}^s n^\ell \sum_{h=\ell}^s \frac{\Gamma(1 + \frac{ah}{d})}{m^{\frac{ah}{d}}} \sum_{g=0}^{h-\ell} m^g \sum_{r=h}^s \sum_{j=g+\ell}^h (-1)^{j-\ell} \begin{Bmatrix} s \\ r \end{Bmatrix} \begin{bmatrix} j \\ \ell \end{bmatrix} \vartheta_{r,j;h,g}.$$

First we consider the case $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$, and directly obtain

$$\mathbb{E}(X_{dm,an}^s) \sim a^s \frac{n^s}{m^{\frac{as}{d}}} \Gamma(1 + \frac{as}{d}), \quad \mathbb{E}\left(\frac{m^{\frac{as}{d}} X_{dm,an}^s}{a^s n^s}\right) \sim \Gamma(1 + \frac{as}{d}).$$

We use again the moment convergence theorem of Fréchet and Shohat and obtain the convergence in distribution of $X_{dm,an} m^{a/d}/(a \cdot n)$ to a Weibull distributed random variable $W(d/a, 1)$.

Next assume that $m, n \rightarrow \infty$ such that $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$. We have

$$\begin{aligned} \mathbb{E}(X_{dm,an}^s) &\sim a^s \sum_{\ell=0}^s (\rho m^{a/d})^\ell \sum_{h=\ell}^s \frac{\Gamma(1 + \frac{ah}{d})}{m^{\frac{ah}{d}}} \sum_{g=0}^{h-\ell} m^g \sum_{r=h}^s \sum_{j=g+\ell}^h (-1)^{j-\ell} \begin{Bmatrix} s \\ r \end{Bmatrix} \begin{bmatrix} j \\ \ell \end{bmatrix} \vartheta_{r,j;h,g} \\ &\sim a^s \sum_{\ell=0}^s \rho^\ell \Gamma(1 + \frac{a\ell}{d}) \sum_{r=\ell}^s \begin{Bmatrix} s \\ r \end{Bmatrix} \vartheta_{r,\ell;\ell,0}. \end{aligned}$$

It seems difficult to obtain suitable bounds on $\vartheta_{r,\ell;\ell,0}$ in order to prove that the moment sequence determines a unique distribution, which is necessary to apply the theorem of Fréchet and Shohat.

In the remaining case $n = o(m^{a/d})$ only the constant term being independent of n and m , case $\ell = h = g = 0$, in the expansion of $\mathbb{E}(X_{dm,an}^s)$ is relevant, and we get

$$\mathbb{E}(X_{dm,an}^s) \sim a^s \sum_{r=0}^s \begin{Bmatrix} s \\ r \end{Bmatrix} \vartheta_{r,0;0,0}.$$

Note that this expansion is consistent with $\rho = 0$ in the case considered before. Unfortunately, again we are not able to show that the moment sequence determines a unique distribution.

5. GENERALIZATION: A BIASED PÓLYA-EGGENBERGER URN MODEL

In the ordinary Pólya-Eggenberger urn model at every step a ball is chosen at random from the urn. E.g., if the urn contains n white and m black balls, the probability of choosing a white ball is given by $n/(m+n)$, whereas the probability of choosing a black ball is given by $m/(m+n)$. We consider now a biased Pólya-Eggenberger urn model defined as follows. Starting with an urn with ball replacement matrix $M = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$, we associate with the states of the urn a sequence P of positive real numbers $P = (p_m)_{m \in \mathbb{N}_0}$, with $p_0 = 0$ and $p_m \in \mathbb{R}^+$, where P is independent of n . For the sake of simplicity we have chosen $a = d = 1$ in M . The cases $d > 1$ or $a > 1$ (with $c = p \cdot a$) can be reobtained by properly choosing the sequence $P = (p_m)_{m \in \mathbb{N}_0} = (\frac{dm}{a})_{m \in \mathbb{N}_0}$. Assuming that the urn contains n white and m black balls, for this class of biased diminishing urns, the probability of choosing a white ball is given by $n/(n+p_m)$, whereas the probability of choosing a black ball is given by $p_m/(n+p_m)$. Let $X_{m,n}$ denote the random variable, which counts the number of white balls remaining in the urn when all black ball have been removed. By definition we have the following recurrence for $\mathbb{P}\{X_{m,n} = k\}$:

$$\mathbb{P}\{X_{m,n} = k\} = \frac{n}{n+p_m} \mathbb{P}\{X_{m,n-1} = k\} + \frac{p_m}{n+p_m} \mathbb{P}\{X_{m-1,n+c} = k\},$$

with initial values $\mathbb{P}\{X_{0,n} = n\} = 1$, for $n \in \mathbb{N}_0$. We also have the following recurrence for the moments $e_{m,n}^{[s]} = \mathbb{E}(X_{m,n}^s)$:

$$e_{m,n}^{[s]} = \frac{n}{n+p_m} e_{m,n-1}^{[s]} + \frac{p_m}{n+p_m} e_{m-1,n+c}^{[s]},$$

with initial values $e_{0,n}^{[s]} = n^s$, for $n \in \mathbb{N}_0$. Obviously, the recurrence for the moment sequence is almost identical to the previous recurrence (5). This suggests that, as before, the s -th moment is again a polynomial of degree s in n , with coefficients depending only on m . The next result makes this precise – we recursively determine the moments of $X_{m,n}$ for a given sequence $P = (p_m)_{m \in \mathbb{N}_0}$, and also obtain an alternative description for the factorial moments of $X_{m,n}$, similar to Lemmata 1-2.

Proposition 3. *The s -th moment $e_{m,n}^{[s]} = \mathbb{E}(X_{m,n}^s)$ of the random variable $X_{m,n}$ satisfies the expansion $e_{m,n}^{[s]} = \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell$. The values $\lambda_{s,\ell,m}$ are recursively given by*

$$\lambda_{s,s,m} = \prod_{l=1}^m \frac{p_l}{p_l + s}, \quad \text{and} \quad \lambda_{s,\ell,m} = \sum_{k=0}^{m-1} \mu_{s,\ell,m-k} \prod_{j=m+1-k}^m \frac{p_j}{p_j + k},$$

where

$$\mu_{s,\ell,m} := \frac{1}{p_m + \ell} \sum_{k=\ell+1}^s \binom{k}{\ell-1} (-1)^{k-\ell-1} \lambda_{s,k,m} + \frac{p_m}{p_m + \ell} \sum_{k=\ell+1}^s \binom{k}{\ell} c^{k-\ell} \lambda_{s,k,m-1}.$$

For $\ell = 0$ we have

$$\lambda_{s,0,m} = \sum_{k=0}^{m-1} \mu_{s,0,k}, \quad \text{with} \quad \mu_{s,0,m} := \sum_{k=1}^s \lambda_{s,k,m} c^k.$$

The initial values are given by $\lambda_{s,s,0} = 1$ and $\lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s-1$.

Furthermore, the s -th factorial moment $e_{m,n}^{(s)} = \mathbb{E}(X_{m,n}^{\underline{s}})$ of the random variable $X_{m,n}$ satisfies the expansion $e_{m,n}^{(s)} = \sum_{k=0}^s \Lambda_{s,k,m} n^{\underline{k}}$. The values $\Lambda_{s,k,m}$ are recursively given by $\Lambda_{s,s,m} = \prod_{l=1}^m \frac{p_l}{p_l + s}$, and

$$\Lambda_{s,\ell,m} = \left(\prod_{h=1}^m \frac{p_h}{p_h + \ell} \right) \sum_{k=0}^{m-1} \left(\prod_{h=1}^k \frac{p_h}{p_h + \ell} \right) \sum_{j=\ell+1}^s \binom{j}{\ell} c^{j-\ell} \Lambda_{s,j,k-1},$$

with initial values are given by $\Lambda_{s,s,0} = 1$ and $\Lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s-1$.

The proof of the above result is fully analogous to the proofs of Lemma 1, 2 and is therefore omitted. We refrain from studying this new generalized urn problem in full generality, and only state the following immediate consequences.

Corollary 2. *In the biased urn model with $c = 0$ the factorial moments of the random variable $X_{m,n}$ are given by*

$$\mathbb{E}(X_{m,n}^{\underline{s}}) = n^{\underline{s}} \cdot \prod_{k=1}^m \frac{p_k}{p_k + s}.$$

Consequently, for $c = 0$ and any given sequence $P = (p_m)_{m \in \mathbb{N}_0}$, satisfying $\sum_{m \geq 1} \frac{1}{p_m} < \infty$, one obtains the following limiting distribution results.

- For m fixed and $n \rightarrow \infty$ the normalized random variable $X_{m,n}/n$ converges in distribution to a random variable V_m , with convergence of all moments,

$$\frac{X_{m,n}}{n} \xrightarrow{\mathcal{L}} V_m, \quad \mathbb{E}\left(\frac{X_{m,n}^s}{n^s}\right) \rightarrow \mathbb{E}(V_m^s) = \prod_{l=1}^m \frac{p_l}{p_l + s}.$$

Moreover, V_m can be written as the exponential of a weighted sum of m independent exponential random variables ϵ_k ,

$$V_m = \exp\left(-\sum_{k=1}^m \frac{\epsilon_k}{p_k}\right).$$

- For n fixed and $m \rightarrow \infty$ the random variable $X_{m,n}$ converges in distribution to a random variable Z_n , with convergence of all moments,

$$X_{m,n} \xrightarrow{\mathcal{L}} Z_n, \quad \mathbb{E}(X_{m,n}^s) \rightarrow \mathbb{E}(Z_n^s) = \sum_{j=0}^s n^{\underline{j}} \left\{ \begin{matrix} s \\ j \end{matrix} \right\} \prod_{l=1}^m \frac{p_l}{p_l + j}.$$

- For $\min\{m, n\} \rightarrow \infty$ the normalized random variable $X_{m,n}/n$ converges in distribution to a random variable W , with convergence of all moments,

$$\frac{X_{m,n}}{n} \xrightarrow{\mathcal{L}} W, \quad \mathbb{E}\left(\frac{X_{m,n}^s}{n^s}\right) \rightarrow \mathbb{E}(W^s) = \prod_{l=1}^{\infty} \frac{p_l}{p_l + s}.$$

Moreover, W can be written as the exponential of a series of independent exponential random variables $\epsilon_k \stackrel{\mathcal{L}}{=} \text{Exp}(1)$,

$$W = \exp\left(-\sum_{k=1}^{\infty} \frac{\epsilon_k}{p_k}\right).$$

Moreover, the random variables Z_n/n , and V_m both converge in distribution to W , with convergence of all moments, for n, m tending to infinity.

Proof. In the case $c = 0$ the factorial moments $e_{m,n}^{(s)}$ of $Y_{m,n}$ are given by

$$e_{m,n}^{(s)} = \sum_{k=0}^s \Lambda_{s,k,m} n^k = n^s \Lambda_{s,s,m} = n^s \prod_{\ell=1}^m \frac{p_\ell}{p_\ell + s},$$

since the values $\Lambda_{s,k,m}$, $0 \leq k \leq s-1$, all have a factor c . The assumption $\sum_{m \geq 1} \frac{1}{p_m} < \infty$ on the sequence P ensures that the product $\prod_{\ell=1}^m \frac{p_\ell}{p_\ell + s}$ converges for $s \geq 1$ and m tending to infinity. Consequently, the limiting distributions are obtained in a straightforward way using Fréchet-Shohat's moment convergence theorem, see, e.g. [17]. In order to decompose the random variables V_m and W into sums of exponential random variables, we proceed as follows. Let $\epsilon \stackrel{\mathcal{L}}{=} \text{Exp}(\lambda)$ denote an exponentially distributed random variable with parameter λ . The Laplace transform $\mathbb{E}(e^{-t\epsilon})$ of ϵ is given by $\mathbb{E}(e^{-t\epsilon}) = \frac{1}{1+\frac{t}{\lambda}}$. Let $(\epsilon_\ell)_{\ell \in \mathbb{N}}$ be independent identically $\text{Exp}(1)$ -distributed random variables. Using the fact that $\frac{1}{\lambda} \cdot \epsilon \stackrel{\mathcal{L}}{=} \text{Exp}(\lambda)$, we obtain

$$\mathbb{E}\left(\exp\left(-t \cdot \sum_{\ell=1}^m \frac{\epsilon_\ell}{p_\ell}\right)\right) = \prod_{\ell=1}^m \mathbb{E}(e^{-t\frac{\epsilon_\ell}{p_\ell}}) = \prod_{\ell=1}^m \frac{1}{1+\frac{t}{p_\ell}}.$$

The moments of the random variables V_m and W are given by

$$\mathbb{E}(V_m^s) = \prod_{\ell=1}^m \frac{p_\ell}{p_\ell + s} = \prod_{\ell=1}^m \frac{1}{1+\frac{s}{p_\ell}}, \quad \mathbb{E}(W^s) = \prod_{\ell=1}^{\infty} \frac{p_\ell}{p_\ell + s} = \prod_{\ell=1}^{\infty} \frac{1}{1+\frac{s}{p_\ell}}.$$

Hence, we obtain the stated decompositions $V_m \stackrel{\mathcal{L}}{=} \exp\left(-\sum_{k=1}^m \frac{\epsilon_k}{p_k}\right)$, and $W \stackrel{\mathcal{L}}{=} \exp\left(-\sum_{k=1}^{\infty} \frac{\epsilon_k}{p_k}\right)$. \square

Remark 5. A particularly interesting case is the biased sampling without replacement urn $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ with sequence $P = (p_m)_{m \in \mathbb{N}_0} = (m^2)_{m \in \mathbb{N}_0}$. The factorial moments and the ordinary moments of $Y_{m,n}$ are given by

$$\mathbb{E}(Y_{m,n}^s) = n^s \prod_{k=1}^m \frac{1}{1+\frac{s}{k^2}}, \quad \mathbb{E}(Y_{m,n}^j) = \sum_{j=0}^s \left\{ \begin{matrix} s \\ j \end{matrix} \right\} n^j \prod_{k=1}^m \frac{1}{1+\frac{j}{k^2}}.$$

A closed form expression for the product $\prod_{k=1}^m \frac{1}{1+\frac{s}{k^2}}$ is readily obtained using the Euler product form of the hyperbolic sine function, together with the product form of the Gamma function

$$\sinh(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2}\right), \quad \Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \frac{e^{\frac{z}{k}}}{1 + \frac{z}{k}},$$

such that

$$\prod_{k=1}^m \frac{1}{1+\frac{s}{k^2}} = \frac{\pi\sqrt{s}\Gamma(m+1)^2}{\sinh(\pi\sqrt{s})\Gamma(m+1+i\sqrt{s})\Gamma(m+1-i\sqrt{s})}; \quad \text{moreover, } \prod_{k=1}^{\infty} \frac{1}{1+\frac{s}{k^2}} = \frac{\pi\sqrt{s}}{\sinh(\pi\sqrt{s})}.$$

Note that the random variable $W = \exp\left(-\sum_{k=1}^{\infty} \frac{\epsilon_k}{k^2}\right)$ arising in the limit $\min\{m, n\} \rightarrow \infty$ is closely related to distributions considered by Biane, Pitman and Yor [1] in the context of Brownian excursions and Theta functions. For example, one can further show that the random variable W has support $[0, 1]$, and its distribution function can be expressed in terms of the Jacobi Theta function $\Theta(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}$, $\mathbb{P}\{W \leq q\} = 1 - \Theta(q)$, $0 \leq q \leq 1$.

6. CONCLUSION AND ACKNOWLEDGEMENTS

By applying the method of moments we were able to describe in a quite precise manner the asymptotic behaviour of a class of 2×2 -urn models with replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$, $a, d, p \in \mathbb{N}$ and $c = a \cdot p$. In the table below we give a short summary of our findings, using the asymptotic small-o, and equivalence notations. It is an interesting question to ask whether the approach used for a study of 2×2 -urn models can be generalized to an analysis of certain diminishing urn models with more types of balls. Moreover, the biased variant of the considered urn models has interesting connections to distributions considered by Biane, Pitman and Yor [1].

| | $n \rightarrow \infty$: m fixed | $m \rightarrow \infty$: $m = o(n^{\frac{a}{d}})$ | $m \rightarrow \infty$: $m \sim \rho \cdot n^{\frac{a}{d}}$, $\rho \in \mathbb{R}^+$ | $m \rightarrow \infty$: $n = o(m^{\frac{a}{d}})$ |
|-------------------------------------|---------------------------------------|--|---|--|
| $a/d \leq 1$ and $c \in \mathbb{N}$ | Kuramaswamy | Weibull | Weibull | Weibull |
| $a/d > 1$ and $c \in \mathbb{N}$ | Kuramaswamy | Weibull | Moment convergence | Moment convergence |
| $a, d \in \mathbb{N}$ and $c = 0$ | Kuramaswamy | Weibull | Characterized by moments | Degenerate |

The authors want to thank Hsien-Kuei Hwang for useful and encouraging discussions on this topic, and Svante Janson for pointing out the decompositions into sums of exponentials in Corollary 2. Furthermore, the authors thank the referee for very valuable remarks, pointing out several imprecisions and improving the presentation of this work.

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