On Path diagrams and Stirling permutations

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Abstract

An ordinary permutation can be locally classified according to the four local types called peaks, valleys, double rises and double falls, and there is a corresponding classification of binary increasing trees according to the four node types. Moreover, by the bijection between permutations, binary increasing trees, and suitably defined path diagrams induced by Motzkin paths, one can obtain a continued fraction representation of the ordinary generating function of local types. The aim of this article is to extend the notion of local types to k-Stirling permutations, establish a relation of these local types with node types of $(k+1)$-ary increasing trees, and to obtain a continued fraction type representation of the generating function of these local types through a bijection with suitably defined path diagrams. In the case of the classical Stirling permutations, we give an alternative continued fraction type representation of ordinary generating function, by using the correspondence between Stirling permutations, plane-ordered recursive trees, and a different family of path diagrams.

Keywords: Path diagrams, Stirling permutations, Increasing trees, local types, Łukasiewicz paths, formal power series

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1 Introduction

Any ordinary permutation $\tau = \tau_1 \dots \tau_n$ of size n can be locally be classified according to four local types called peaks (maxima), valleys (minima), double rises and double falls, depending on the relative order of τ_j to its neighbors (see Flajolet [7], or Conrad and Flajolet [5] and the references therein). By the bijection between ordinary permutations and binary increasing trees [7, 1], these local types correspond with node types of binary increasing trees [7, 5]. Through a bijection between permutations and path diagrams induced by Motzkin paths, Flajolet [7] presented a continued fraction representation of the ordinary generating function of permutations with respect to the local types (see also Françon and Viennot [9]).

The main aim of this article is to extend the notion of local types to a generalized version of permutations called the k-Stirling permutations, establish a relation of these local types with node types of $(k+1)$-ary increasing trees, and to obtain a continued fraction type representation of the generating function of these local types through a bijection with suitably defined path diagrams.

Stirling permutations were defined by Gessel and Stanley [10] in relation to the Stirling numbers of first and second kind and their connection to properties of Eulerian numbers. A Stirling permutation $\sigma = \sigma_1 \dots \sigma_{2n}$ is a permutation of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ such that for each i, $1 \le i \le n$,

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the elements between the two occurrences of i are larger than i. Recently, this class of combinatorial objects have generated some interest: Bona [2] studied the distribution of descents in Stirling permutation, and Janson [14] showed the connection between Stirling permutations and plane-oriented recursive trees, and proved a joint normal limit law for the parameters considered by Bona.

A natural generalization of Stirling permutations is to consider permutations of a more general multiset $\{1^k, 2^k, \ldots, n^k\}$, with $k \in \{1, 2, \ldots\}$, with the restriction that for each i, $1 \le i \le n$, the elements between two consecutive occurrences of i are greater than i, which we call k-Stirling permutations. Such generalized Stirling permutations have already previously been considered by Brenti [3, 4], and also by Park [19, 20, 21] under the name of k-multipermutations. The case $k = 1$ corresponds to ordinary permutations, and the case $k = 2$ corresponds to Stirling permutations. Recently, Janson et al. [15] studied several parameters in k-Stirling permutations, related to the studies [2, 14], extending the results of [2, 14] concerning the distribution of descents and related statistics. An important result of [15] is the natural bijection between k-Stirling permutations and $(k+1)$-ary increasing trees, which was already known to Gessel (see Park [19]).

This article is organized as follows. In section 2, we define k-Stirling permutations and increasing trees, and give a bijection between k-Stirling permutations and $(k+1)$-ary increasing trees.

In section 3, we extend the notion of local types to k-Stirling permutations, and establish a bijection between these local types and node types of $(k+1)$-ary increasing trees.

In section 4, we define path diagrams induced by a subset of Łukasiewicz paths restricted to k rise vectors, and establish a bijection between the path diagrams with appropriate possibility function and k-Stirling permutations. We then give a continued fraction type representation of the ordinary generating function of the k-Stirling permutations with respect to the local types.

Finally, in section 5, we go back to the classical Stirling permutations, presenting an alternative continued fraction type representation using unrestricted Łukasiewicz paths and the correspondence between Stirling permutations and plane-oriented recursive trees [14]. We further provide statistics in Stirling permutations and in ternary increasing trees which are combinatorially equivalent to the statistics of the number of nodes of outdegree j in a plane-oriented recursive tree of size n.

2 Increasing trees and generalized Stirling permutations

2.1 Generalized Stirling permutations

Definition 1. Let $\{1^r, 2^r, \ldots, n^k\}$ denote a multiset where each element in $\{1, 2, \ldots, n\}$ is represented k times. A *k-Stirling permutation of size n* is a permutation $\tau_1\tau_2\ldots\tau_{kn}$ of elements in the multiset $\{1^r, 2^r, \ldots, n^k\}$ with the condition that for $i < j < k$, if $\tau_i = \tau_k$, then $\tau_i \le \tau_j \ge \tau_k$.

For example, 224442113331 is a 3-Stirling permutation of size 3.

Let $Q_n(k)$ denote the number of k-Stirling permutations of size n. Then

$$Q_n(k) = \prod_{i=0}^{n-1}(ki+1) = k^{n-1}\frac{\Gamma(n+1/k)}{\Gamma(1/k)}, \tag{1}$$

by recursion on n: a k-Stirling permutation of size n can be obtained from a k-Stirling permutation of size $(n-1)$ by inserting the k copies of n as a substring into any of the $k(n-1)+1$ positions between the existing elements, including the first and the last position; see for example [19, 15].

For example, in the case $k = 3$, there is only one 3-Stirling permutation of size 1, given by 111 ; there are four 3-Stirling permutations of size 2, given by 111222, 112221, 122211, 222111; etc. And in particular, the number of 2-Stirling permutations is $Q_n(2) = (2n-1)!!$.

The correspondence between ordinary permutations and binary trees has been useful in uncovering the internal structure of permutations, and giving alternative combinatorial models to such quantities that relate to the local types of permutations, like the Eulerian numbers that count the number of permutations with a certain number of descents. Likewise, the correspondence between 2-Stirling permutations and plane-oriented recursive trees proved useful in studying the distribution of descents in Stirling permutations [14]. In the next section, we establish the correspondence for the general case of k-Stirling permutations and $(k+1)$-ary increasing trees.

2.2 Families of Increasing trees

We introduce a general family of increasing trees based on earlier considerations of Bergeron et al. [1] and Panholzer and Prodinger [18], of which $(k+1)$-ary increasing trees and plane-oriented recursive trees are special cases. These tree families and their combinatorial description are quite well known; we collect some relevant results of [1, 18, 15] for the reader's convenience.

Informally, an increasing tree of size n is a rooted ordered tree with n nodes, where the nodes are labeled by distinct integers of the set $\{1, \dots, n\}$ such that each sequence of labels along any path starting at the root is increasing. These are the simple families of increasing trees introduced in [1]; the underlying unlabeled tree model is the so-called simply generated tree [17].

Formally, a class $\mathcal{T}$ of a simple family of increasing trees can be defined in the following way. A sequence of non-negative numbers $(\varphi_\ell)_{\ell\geq 0}$ with $\varphi_0 > 0$, called the *degree-weight sequence*, is used to define the weight $w(T)$ of any ordered tree T by $w(T) := \prod_v \varphi_{\deg^+(v)}$, where v ranges over all vertices of T, and $\deg^+(v)$ is the out-degree of the vertex v. Let $\mathcal{L}(T)$ denote the set of increasing labellings of the vertexes of the ordered tree T with distinct integers $\{1, 2, \dots, |T|\}$, where $|T|$ denotes the size of the tree T. Then the simple family of increasing trees $\mathcal{T}$ consists of all ordered trees T together with their weights $w(T)$ and the increasing labelling $\lambda \in \mathcal{L}(T)$. The simple family of increasing trees $\mathcal{T}$ associated with a degree-weight generating function $\varphi(t) := \sum_{\ell\geq 0} \varphi_\ell t^\ell$ can be described by the formal recursive equation

$$\mathcal{T} = ① \times \Big(\varphi_0 \cdot \{\epsilon\} \,\dot{\cup}\, \varphi_1 \cdot \mathcal{T} \,\dot{\cup}\, \varphi_2 \cdot \mathcal{T} * \mathcal{T} \,\dot{\cup}\, \varphi_3 \cdot \mathcal{T} * \mathcal{T} * \mathcal{T} \,\dot{\cup}\, \cdots\Big) = ① \times \varphi(\mathcal{T}), \tag{2}$$

where ① denotes the node labeled by 1, $\times$ the cartesian product, $\dot{\cup}$ the disjoint union, $*$ the partition product for labeled objects; $\varphi(\mathcal{T})$ the substituted structure (e. g., see [27, 8]). For a given degree-weight sequence, we define the total weight of size n increasing trees by $T_n := \sum_{|T|=n} w(T) \cdot L(T)$, where $L(T) := \big|\mathcal{L}(T)\big|$ is the number of distinct increased labelling on the ordered tree T. It follows from the recursive structure of increasing trees that the exponential generating function $T(z) := \sum_{n\geq 1} T_n \frac{z^n}{n!}$ satisfies the autonomous first order differential equation

$$T'(z) = \varphi\big(T(z)\big), \quad T(0) = 0. \tag{3}$$

We obtain the families of $(k+1)$-ary increasing trees and plane-oriented recursive trees by choosing appropriate degree-weight sequences $(\varphi_\ell)_{\ell\geq 0}$.

Example 1. The family of $(k+1)$-ary increasing trees, with integer $k \in \mathbb{N}$, is the family of increasing trees where each node has $k+1$ (labeled) positions for children, going from left to right. The vacant positions are usually denoted by external nodes (see Figure 1 for an illustration of ternary increasing trees). Thus each node can have $0 \leq \ell \leq k+1$ children, with $\binom{k+1}{\ell}$ different ways to attach them (see Figure 2). Thus the appropriate degree-weight sequence to specify the weight of the nodes is $\varphi_\ell = \binom{k+1}{\ell}$ for $0 \leq \ell \leq k+1$, $\varphi_\ell = 0$ for $k+1 < \ell$. Consequently, the degree weight generating function is $\varphi(t) = \sum_{\ell\geq 0} \varphi_\ell t^\ell = (1+t)^{k+1}$, which lets us derive the exponential generating function $T(z) = T(z, k)$ of $(k+1)$-ary increasing trees by solving the corresponding differential equation (3).

The number $T_n = T_n(k)$ of $(k+1)$-ary increasing trees of size n can be obtained from the generating function, or by recursion on n:

$$T(z) = \frac{1}{(1-kz)^{\frac{1}{k}}} - 1, \qquad T_n = \prod_{\ell=1}^{n} (k(\ell-1)+1) = k^{n-1} \frac{\Gamma(n+1/k)}{\Gamma(1/k)}, \quad n \geq 1.$$

The case $k = 1$ is the family of binary increasing trees, and the case $k = 2$ is the family of ternary increasing trees. Note that $T_n = Q_n$, the number of k-Stirling permutations of size n (1).

Example 2. The family of plane-oriented recursive trees consists of rooted ordered increasing trees with no restriction on the out-degrees of the nodes. A new vertex may be joined to an existing vertex v in exactly $\deg^+(v) + 1$ positions, where $\deg^+(v)$ denotes the out-degree of node v. These $\deg^+(v) + 1$ positions can be represented by external nodes (see Figure 1). Consequently, the total number of positions available to the $(n+1)$-st node being attached to a tree of size n is given by $\sum_{j=1}^{n} (\deg^+(j) + 1) = 2n - 1$, independent of the actual shape of the tree. There is exactly one tree of size 1, and for $n \geq 1$, there are $T_n = \prod_{\ell=1}^{n-1} (2\ell - 1) = (2n-3)!!$ distinct plane-oriented recursive trees of size n. From the formal description of increasing trees, since there are no restrictions on node out-degrees, we set the degree-weights to $\varphi_\ell = 1$ for $\ell \geq 0$, so the degree-weight generating function is given by $\varphi(t) = \sum_{\ell \geq 0} \varphi_\ell t^\ell = \frac{1}{1-t}$. By solving the differential equation (3), we get the exponential generating function $T(z)$ of plane-oriented recursive trees:

$$T(z) = 1 - \sqrt{1-2z}, \quad \text{and} \quad T_n = \prod_{\ell=1}^{n-1} (2\ell - 1) = (2n-3)!!, \ \text{ for } n \geq 1.$$

Note that $T_{n+1} = (2n-1)!!$, equal to the number of 2-Stirling permutations of size n and the number of ternary trees of size n.

Remark 1. Both the family of $(k+1)$-ary increasing trees and the family of plane-oriented recursive trees can be generated according to tree evolution processes, as represented in Figure 1. For a comprehensive discussion, we refer the interested reader to the work of Panholzer and Prodinger [18].

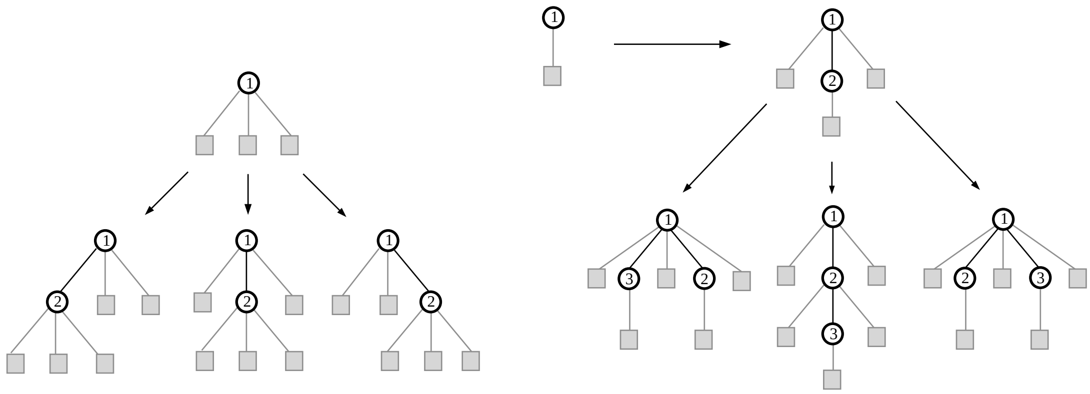

Figure 1: Ternary increasing trees of size one and two and plane-oriented increasing trees of size one, two and three, respectively. The positions where new nodes can be attached are denoted by external nodes.

2.3 Bijection of k-Stirling permutations and $(k+1)$-ary increasing trees

Janson [14] has shown that plane-oriented recursive trees of size $n+1$ are in bijection with 2-Stirling permutations of size n, and Janson et al. [15] gave a bijection between ternary increasing trees of size n and plane-oriented recursive trees of size $n+1$. More generally:

Theorem 1 (Gessel [19]; see also [15])**.** *For $k \in \mathbb{N}$, the family of $(k+1)$-ary increasing trees of size n is in a natural bijection with k-Stirling permutations of size n.*

As shown in [15], the bijection behind Theorem 1 allows to study parameters in k-Stirling permutations via the corresponding parameters in $(k+1)$-ary increasing trees. The bijection is stated explicitly below.

Bijection 1. Starting with a $(k+1)$-ary increasing tree T of size n, we construct a k-Stirling permutation of size n as follows. We consider the representation of the tree T where each labeled node has exactly $k+1$ ordered children, with unlabeled children marked by external nodes. For each node labeled by an integer, we place k copies of that integer between the $k+1$ children of the node. We then collect these copies into a string through a contour walk around the tree starting at the root and going left. (Equivalently, we construct the string by performing a depth-first walk, where we concatenate the integer v to the string every time we visit the node labeled by v, except for the first and the $(k+1)$-st visit.) This process results in a unique string of $k \cdot n$ integers $\tau = \tau_1\tau_2\dots\tau_{kn}$, where each of the integers $1,\dots,n$ appears exactly k times. Since the tree T is increasing, it guarantees that $\tau_i \le \tau_j \ge \tau_k$ whenever $\tau_i = \tau_k$ and $i < j < k$. Therefore, τ is a k-Stirling permutation of size n.
Note that through this process, the external nodes of the tree T correspond to the gaps in the string τ; adding the $(n+1)$-st labeled node to T at one of its $kn+1$ external nodes corresponds to inserting the k-tuple $(n+1)^k$ into the string τ at one of its $kn+1$ gaps.
Conversely, starting with a k-Stirling permutation σ of size n, we construct the unique corresponding $(k+1)$-ary increasing tree through the following recursive procedure. First, we decompose the permutation as $\sigma = \sigma_1 1 \sigma_2 1 \dots \sigma_k 1 \sigma_{k+1}$. The σ_i's are either empty strings, or (with proper relabelling) they are k-Stirling permutations of size smaller than n. We label the root node with integer 1. For $1 \le i \le k+1$, we label the i-th child as follows: if σ_i is empty, then the i'th child is an external node; else the i-th child is labeled by the smallest element in σ_i. We repeat this process recursively for each non-empty σ_i.

3 Local types in k-Stirling permutations

Ordinary permutations can be classified according to four local types (e.g., see [7, 5]) :

Definition 2. Let $\tau = \tau_1 \dots \tau_n$ be a permutation of size n, and let $\tau_0 = \tau_{n+1} = -\infty$. Then for $1 \le j \le n$, index j is called a *peak* if $\tau_{j-1} < \tau_j > \tau_{j+1}$, a *valley* if $\tau_{j-1} > \tau_j < \tau_{j+1}$, a *double rise* if $\tau_{j-1} < \tau_j < \tau_{j+1}$, and a *double fall* if $\tau_{j-1} > \tau_j > \tau_{j+1}$.

Note that sometimes the border condition $\tau_{n+1} = +\infty$ is used [5]; however, the border condition $\tau_{n+1} = -\infty$ is more consistent with respect to the bijection between ordinary permutations and binary increasing trees. Since a permutation $\tau = \tau_1 \dots \tau_n$ has a corresponding binary increasing tree [7], the correspondence between the local type of the index j and the node type of the j-th labeled node in the binary increasing tree is as follows.

Local type	Peak	Valley	Double rise	Double Fall
Condition	$\tau_{j-1} < \tau_j > \tau_{j+1}$	$\tau_{j-1} > \tau_j < \tau_{j+1}$	$\tau_{j-1} < \tau_j < \tau_{j+1}$	$\tau_{j-1} > \tau_j > \tau_{j+1}$
Node type	Leaf	Double node	Right-branching node	Left-branching node.

In extending the notion of local types to k-Stirling permutations, it is desirable to preserve the analogue of the correspondence between local types and node types. Thus we first clarify the node types of (k+1)-ary increasing trees. By definition of $(k+1)$-ary increasing trees, every node has exactly $k+1$ (labeled) positions for children. Some of these positions may be occupied by labeled nodes, while others may be vacant, represented by external nodes. We propose the following definition.

Definition 3. In a $(k+1)$-ary increasing tree T of size n, the *node type* of the node labeled i, for $1 \leq i \leq n$, is defined as the string $G_i(T) = g_{i,1} \ldots, g_{i,k+1} \in \{0,1\}^{k+1}$ of length $k+1$, where $g_{i,j} = 1$ if the j-th child is a labeled node, and $g_{i,j} = 0$ if the j-th child is vacant.

In other words, the sequence $G_i(T)$ specifies which of the $(k+1)$ children are not empty.

Example 3. In the case $k = 2$ of ternary increasing trees, we have $8 = 2^3$ different types of nodes. The sequence 111 corresponds to a triple node, 101 to a (left,right)-branching node, 110 to a (left,center)-branching node, 011 to a (center,right)-branching node, 100 to a left-branching node, 010 to a center-branching node, 001 to a right-branching node, and 000 to a leaf, respectively. See Figure 2 for an illustration.

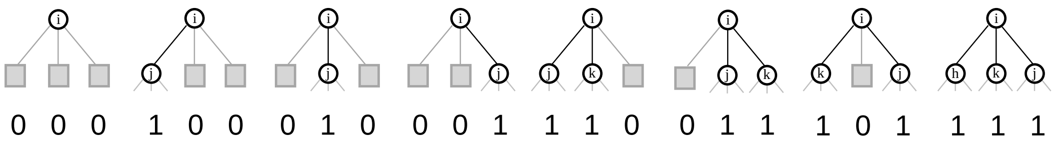

Figure 2: The eight different node types in ternary increasing trees, assuming that $j, h, k > i \geq 1$.

Example 4. In the case $k = 1$ of binary increasing trees, we have $4 = 2^2$ different types of nodes. The sequence 11 corresponds to a double node, the sequence 10 to a left-branching node, the sequence 01 to a right-branching node, and the sequence 00 to a leaf.

This motivates an alternative definition for local types for ordinary permutations, which would lend to extension over the k-Stirling permutations.

Definition 4. Given an ordinary permutation $\tau = \tau_1 \ldots \tau_n \in \mathcal{S}_n$ and entry i, with $1 \leq i \leq n$, let j_i denote the index such that $\tau_{j_i} = i$. Set the border conditions to $\tau_0 = \tau_{n+1} = -\infty$. The local type $L_i(\tau) = \ell_{i,1}\ell_{i,2}$ of the entry i in τ is a string of length 2 with elements in $\{0,1\}$, defined as follows:

$$\ell_{i,1} = \begin{cases} 1 & \text{if } \tau_{j_i-1} > i, \\ 0 & \text{otherwise;} \end{cases} \qquad \text{and} \qquad \ell_{i,2} = \begin{cases} 1 & \text{if } \tau_{j_i+1} > i, \\ 0 & \text{otherwise.} \end{cases}$$

The string $L_i(\tau)$ specifies which of the neighbors of i, going from left to right, are larger than i. Thus i is a peak if $L_i(\tau) = 00$, a valley if $L_i(\tau) = 11$, a double rise if $L_i(\tau) = 01$, and a double fall if $L_i(\tau) = 10$.

Example 5. The ordinary permutation $\tau = 2534716$ of size seven has the following local types: $L_1(\tau) = 11$, $L_2(\tau) = 01$, $L_3(\tau) = 11$, $L_4(\tau) = 01$, $L_5(\tau) = 00$, $L_6(\tau) = 00$, and $L_7(\tau) = 00$.

This new definition readily extends to the general case of k-Stirling permutation for $k \geq 1$. We again define the local type of i as a string of elements in $\{0,1\}$ that specifies which neighbors of i in the permutation, going form left to right, are larger than i.

Definition 5. Given a k-Stirling permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_{kn}$ of size n, and an entry i, with $1 \le i \le n$, let $1 \le j_{i,1} < \dots < j_{i,k} \le kn$ be the indices such that $\sigma_{j_{i,h}} = i$. The local type $L_i(\sigma) = \ell_{i,1} \dots \ell_{i,k+1}$ of the entry i is a string of length $k+1$, with elements in $\{0,1\}$, generated according to relative magnitudes of the instances of i to their neighbors by the following rules (assuming border conditions $\sigma_0 = \sigma_{nk+1} = -\infty$):

$$\ell_{i,1} = \begin{cases} 1 & \text{if } \sigma_{j_{i,1}-1} > i, \\ 0 & \text{otherwise;} \end{cases} \qquad \text{and} \qquad \ell_{i,h} = \begin{cases} 1 & \text{if } \sigma_{j_{i,h}+1} > i, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } 1 < h \le k+1.$$

Example 6. The 3-Stirling permutation $\sigma = 112233321445554666$ of size six has the following local types $L_1(\sigma) = 0011$, $L_2(\sigma) = 0010$, $L_3(\sigma) = 0000$, $L_4(\sigma) = 0011$, $L_5(\sigma) = 0000$, $L_6(\sigma) = 0000$.

Since there are exactly 2^{k+1} different possible local types, we obtain the following result.

Proposition 1. *A k-Stirling permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_{kn}$ of size n of the multiset $\{1^k, 2^k, \dots, n^k\}$ can be classified according to 2^{k+1} different local types, with respect to the local rules in Definition 5.*

It remains to prove that the local types of k-Stirling permutations correspond to the node types of $(k+1)$-ary increasing trees.

Theorem 2. *By Bijection 1, the local types $L_i(\sigma)$ in a k-Stirling permutation $\sigma = \sigma_1 \dots \sigma_{kn}$ of size n coincide with the node types $G_i(T)$ of the corresponding $(k+1)$-ary increasing trees T of size n: $L_i(\sigma) = G_i(T)$ for $1 \le i \le n$.*

Proof. We use the bijection from Theorem 1 between k-Stirling permutations and $(k+1)$-ary increasing trees, based on a depth-first walk. We start the depth-first walk at the root of a given $(k+1)$-ary increasing tree T of size n with node types $G_1(T), \dots, G_n(T)$, and construct the corresponding k-Stirling permutation $\sigma = \sigma(T)$ of size n according to Bijection 1. Let $1 \le j_{i,1} < \dots < j_{i,k} \le kn$ be the indices such that $\sigma_{j_{i,h}} = i$. We show that the local order type $L_i(\sigma) = \ell_{i,1} \dots \ell_{i,k+1}$ equals the node type $G_i(T) = g_{i,1} \dots g_{i,k+1}$ of the node labeled i, for all $1 \le i \le n$.
Let's consider the first position out of the $k+1$ positions for children of the node i. If the first position is vacant, the first element of the node degree type is $g_{i,1} = 0$. By Bijection 1 the vacancy implies that $i = \sigma_{j_{i,1}} > \sigma_{j_{i,1}-1} = m$, and consequently the first element of the local type is $\ell_{i,1} = 0$, since a smaller number $m < i$ must have been observed earlier according to the depth-first walk and the property that the tree is increasingly labeled. (If i is the root, then we use the border conditions $m = -\infty$.)
On the other hand, if the first position is not vacant, then $g_{i,1} = 1$. By the depth-first walk and the definition of increasing trees we have $i = \sigma_{j_{i,1}} < \sigma_{j_{i,1}-1}$, so $\ell_{i,1} = 1$.

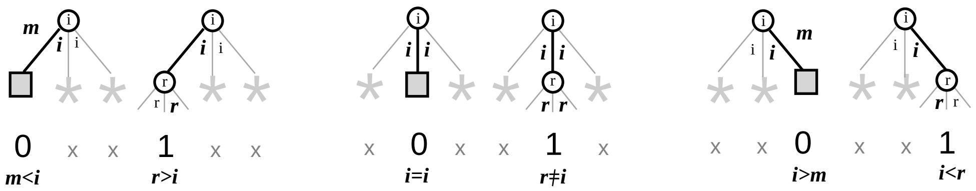

Figure 3: A schematic representation of the correspondence between local order types in 2-Stirling permutations and local node types in ternary increasing trees according to Bijection 1: the tree is traversed according to a depth-first walk, assuming that $r > i$ and $m < i$.

Analogous rationale holds for the case of the $(k+1)$-st position of the node i.
Now let's consider the rest of the positions for the children of the node i. For $1 \le h \le k$, note that $g_{i,h} = 0$ implies that the indices $j_{i,h}$ and $j_{i,h+1}$ satisfy $j_{i,h} + 1 = j_{i,h+1}$. Consequently, $\sigma_{j_{i,h}+1} = i$,and

thus $\ell_{i,h} = 0$. Conversely, if $g_{i,h} = 1$, then by the construction in Bijection 1 there is a substring between the h-th and $(h+1)$-st occurrences of i, and this substring is composed of elements greater than i. Consequently, $\sigma_{j_{i,h}+1} > i$,and thus $\ell_{i,h} = 1$.

□

4 Path diagrams and local types of k-Stirling permutations

4.1 Path diagrams and k-Stirling permutations

In their description via local types, ordinary permutations correspond to path diagrams related to Motzkin paths, as shown in the works of Françon and Viennot [9] and Flajolet [7]. In this section, we extend their ideas to give a bijection between k-Stirling permutations and a more general set of path diagrams related to a restricted version of Łukasiewicz paths. We adapt the definitions of [7] to present the path diagrams.

The paths we wish to consider are positive paths on the x-y plane, which consist only of the following $k+2$ types of steps: rise vectors $a_\ell = (1, \ell)$ for $1 \leq \ell \leq k$, a fall vector $b = (1, -1)$, and a level vector $c = (1, 0)$. To each word $u = u_1 \dots u_n$ on the alphabet $\mathcal{A} = \{a_1, \dots, a_k, b, c\}$ is associated a sequence of points $M_0 M_1 \dots M_n$ in the x-y plane, where $M_0 = (0, 0)$, and $M_j = M_{j-1} + u_j$, for all $1 \leq j \leq n$. The number n is the length of u, and for each j, the y-coordinate of the point M_j is the height of M_j. For our purposes, we only consider *positive* paths: paths whose corresponding sequence of points have only non-negative heights, and whose last point has height 0.

We now define *labeled* paths as the positive paths in which each step is indexed with the height of the point from which it starts. Specifically, for a positive path associated to the word $u = u_1 \dots u_n$ with corresponding sequence of points $M_0 M_1 \dots M_n$ with $M_i = (x_i, y_i)$, the labeling $\lambda(u)$ is a word of length n over the infinite alphabet X,

$$X = \{b_0, b_1, b_2 \dots\} \cup \{c_0, c_1, c_2 \dots\} \cup \bigcup_{\ell=1}^{k} \{a_{0,\ell}, a_{1,\ell}, a_{2,\ell} \dots\},$$

where $\lambda(u) = v_1 \dots v_n$ is determined by the following rules:

$$\begin{aligned} (i) &\quad \text{if } u_j = a_\ell, \quad \text{then } v_j = a_{y_{j-1},\ell}, \\ (ii) &\quad \text{if } u_j = b, \quad \text{then } v_j = b_{y_{j-1}}, \\ (iii) &\quad \text{if } u_j = c, \quad \text{then } v_j = c_{y_{j-1}}. \end{aligned}$$

We now recall the definition of path diagrams (see i.e. [9], [7] and the references therein).

Definition 6. A system of path diagrams is defined by a function $\mathrm{pos} : X \to \mathbb{N}$ called a possibility function; a path diagram is a couple (v, s), where $v = v_1 \dots v_n$ is a labeled path, and s is a sequence of integers $s = s_1 \dots s_n$ such that for all j, $0 \leq s_j < \mathrm{pos}(v_j)$.

Visually, a path diagram can be represented on an x-y plane as the path corresponding to v together with marked coordinates $(j-1, s_j)$ for $1 \leq j \leq n$. Path diagrams can also be described as words over a labeled alphabet

$$Y = \{a_{i,\ell}^{(j)}, b_i^{(j)}, c_i^{(j)}\}_{i,j \geq 0, \text{ and } 1 \leq \ell \leq k},$$

where letter $x_i^{(j)}$ represents the j-th possibility relative to the letter $x_i \in X$.

Now we are ready to state the connection between path diagrams, k-Stirling permutations and $(k+1)$-ary increasing trees.

Theorem 3. *The class of k-Stirling permutations of size $n+1$ (equivalently, the family of $(k+1)$-ary increasing trees of size $n+1$) is in bijection with path diagrams of length n, with possibility function* pos(.) *given by*

$$\mathrm{pos}(a_{j,\ell}) = \binom{k+1}{\ell+1}(j+1), \quad 1 \le \ell \le k, \quad \mathrm{pos}(b_j) = j+1, \quad \mathrm{pos}(c_j) = (k+1)(j+1),$$

with respect to the labeled paths induced by the family of $k+2$ step vectors $a_1, \dots, a_k, b, c$, with rise vectors $a_\ell = (1, \ell)$ for $1 \le \ell \le k$, fall vector $b = (1, -1)$, and level vector $c = (1, 0)$.

Remark 2. For $k = 1$ this reduces to the classical correspondence of Françon and Viennot [9]. Note that one may interpret c as a rise vector, corresponding to the case $\ell = 0$, which would simplify the presentation. However, due to the importance of the case $k = 1$ we opted not to do so, in order to be coherent with the presentations of [9], [7].

Proof. Given a labeled path $v = v_1 \dots v_n$, we give an algorithm for constructing a $(k+1)$-ary tree with $n+1$ internal nodes. At each step, there is a number of possible ways to carry out the instructions; this number corresponds to the defined possibility function. Thus each of the path diagrams induced by the path v can be assigned a unique $(k+1)$-ary tree that can be constructed using v. Furthermore, the algorithm can be reversed: given a $(k+1)$-ary tree with $n+1$ internal nodes, one can construct a unique labeled path of length n. Thus one can't construct the same tree using two different paths. In this way, the bijection will be established between $(k+1)$-ary trees with $n+1$ internal nodes and path diagrams of length n with the possibility function given above.

The algorithm for constructing a $(k+1)$-ary tree using a positive path $v = v_1 \dots v_n$ is as follows. We begin with one placeholder for an internal node. For $1 \le k \le n$: at k-th step of the path, choose one of the available place-holders, and replace it with a node labeled by k. If $v_k = b_j$, there are no further instructions. However, if $v_k = a_{j,\ell}$, then out of the $k+1$ possible child positions for the node, choose $\ell + 1$ of them as placeholders for internal nodes; and if $v_k = c_j$, then out of the $k+1$ possible child positions, choose one of them as a placeholder for an internal node.

Note that in each of the cases, the net change in the number of placeholders at the k-th step of the construction is equal to the change in height at the k-th step of the path. Since we begin the construction with one placeholder, at the beginning of the k-th construction step the number of placeholders is one more than the height of the point M_{k-1} before the k-th step of the path. After the n-th step, the height the point M_n is 0, so there is only one placeholder left. To complete the tree, replace that placeholder with a node labeled by $n+1$.

Knowing the number of placeholders at each step allows us to compute the number of possible ways each construction step can be implemented, and thus determine the appropriate possibility function. If $v_k = b_j$, there are $j+1$ possible placeholders to choose from for the new node labeled by k, so $\mathrm{pos}(b_j) = j+1$. If $v_k = c_j$, taking into account both the number of existing placeholders to choose from, and the possibilities for choosing a new placeholder out of the node's $k+1$ possible child positions, determines that $\mathrm{pos}(c_j) = (k+1)(j+1)$. Similarly, if $v_k = a_{j,\ell}$, the number of possibilities is $\mathrm{pos}(a_{j,\ell}) = \binom{k+1}{\ell+1}(j+1)$.

This establishes a one-to-one correspondence between the set of path diagrams associated with the path v, where the position function is specified as above, and the set of $(k+1)$-ary trees that can be constructed by using the above algorithm and the path v.

The algorithm can be reversed: given a $(k+1)$-ary tree T with $n+1$ internal nodes, construct an unlabeled path $u = u_1 \dots u_n$ as follows. For $1 \le k \le n$, at k-th step: if the node labeled by k has out-degree of 0, then the k-th step of the path is $u_k = b$; if the node has out-degree of 1, then $u_k = c$; and if the node has out-degree of $2 \le \ell \le k+1$, then $u_k = a_{\ell-1}$. The corresponding labeled path is $v = \lambda(u)$. Note the lack of choice in this reversed construction, which guarantees that the path is uniquely constructed. The rationale that the path obtained by this construction is a positive path

is as follows. At the end of the k-th step of the path u, the height is equal to $\sum_{j=1}^{k}(\deg^+(j)-1)$, where $\deg^+(j)$ denotes the out-degree of the node labeled by j. Since the tree T is connected and increasing, $\sum_{j=1}^{k}\deg^+(j)\geq k$, since each of the nodes labeled by $2,\dots,k+1$ contribute once to the sum of out-degrees of the nodes labeled by $1,\dots,k$. Therefore, this height is non-negative. Moreover, at the end of the n-th step, the height is 0, since $\sum_{j=1}^{n}\deg^+(j)=|\{2,\dots,n+1\}|=n$, since the nodes labeled by $2,\dots,n+1$ account for all out-degrees.
From this reversed version the algorithm, we establish that every $(k+1)$-ary tree can be constructed using a unique positive path. Combining this with the previously established one-to-one correspondence between the set of path diagrams induced by a labeled positive path v and the set of $(k+1)$-ary trees that can be constructed using v, we get the one-to-one correspondence between path diagrams of length n, with the possibility function specified above, and the family of $(k+1)$-ary trees with $n+1$ labeled nodes. □

Example 7. Consider the case $k=2$, corresponding to ternary increasing trees, and equivalently Stirling permutations. Below we illustrate the procedure on constructing a ternary tree with seven labeled nodes, using the path $v=a_{0,2}a_{2,1}b_3b_2c_1b_1$. By using for $k=2$ the bijection between $(k+1)$-

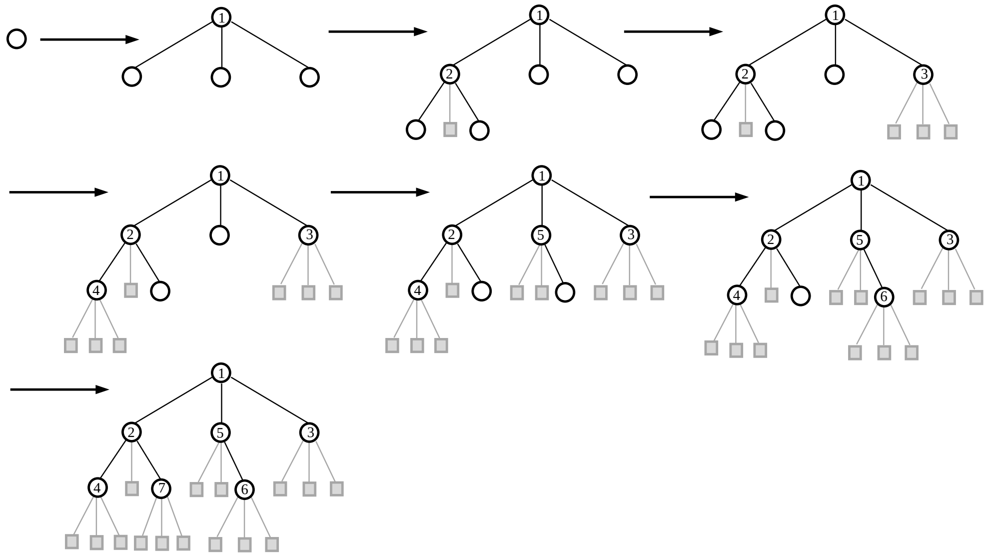

Figure 4: An illustration of the recursive construction of a ternary increasing trees, with respect to the path $v=a_{0,2}a_{2,1}b_3b_2c_1b_1$.

ary increasing trees and k-Stirling permutations, we immediately obtain the corresponding Stirling permutation σ of size seven, $\sigma=44227715566133$. Note that we can also directly construct the Stirling permutation, since the local types of the outdegree of the nodes in the ternary increasing tree correspond to the local types of the numbers in the permutation. One may think of this procedure as some kind of "flattening of the tree to a line" (see below and compare with the sequence of trees in Figure 4).

$$\circ\to\circ1\circ1\circ\to\circ22\circ1\circ1\circ\to\circ22\circ1\circ133\to4422\circ1\circ133\to4422\circ155\circ133$$
$$\to442277155\circ133\to44227715566133.$$

4.2 Generating function of local types

Flajolet [7] used the correspondence between path diagrams and formal power series to obtain continued fraction representations of the generating functions of many parameters in ordinary permutations. More precisely, in the context of permutations and binary increasing trees he derived, amongst many other results, a continued fraction representation of the generating function of local types in permutations, or equivalently of node types in binary increasing trees. We will use the methods of [7] and the previously proven path diagram representation of k-Stirling permutations and $(k+1)$-ary increasing trees to obtain a continued fraction type representation of the generating function of the local types, and consequently also of node types. First we have to recall some more definitions of the work [7] concerning formal power series. Let $C\langle\!\langle X\rangle\!\rangle$ denote the monoid algebra of formal power series $s = \sum_{u\in X^*} s_u \cdot u$ on the set of non-commutative variables (alphabet) X with coefficients in the field of complex numbers, with sums and Cauchy products defined in the usual way

$$s + t = \sum_{u\in X^*} (s_u + t_u)\cdot u, \qquad s\cdot t = \sum_{u\in X^*} \Bigl(\sum_{vw=u} s_v t_w\Bigr)\cdot u.$$

In order to define the convergence of a series, one introduces the valuation of a series $\mathrm{val}(s)$, defined by

$$\mathrm{val}(s) = \min\{|u| : s_u \neq 0\},$$

where $|u|$ denotes the length of the word $u \in X^*$. A sequence of elements $(s_n)_{n\in\mathbb{N}}$, $s_n \in C\langle\!\langle X\rangle\!\rangle$, converges to a limit $s \in C\langle\!\langle X\rangle\!\rangle$ if

$$\lim_{n\to\infty} \mathrm{val}(s - s_n) = \infty.$$

Multiplicative inverses exist for series having a constant term different from zero; for example $(1-u)^{-1} = \sum_{\ell\geq 0} u^\ell$, where $(1-u)^{-1}$ is known as the quasi-inverse of u. Note that we will subsequently use the notation $(u|v)/w = uw^{-1}v$. The characteristic series $\mathrm{char}(S)$ of $S \subset X^*$ is defined as

$$\mathrm{char}(S) = \sum_{u\in S} u.$$

Finally, following [7] we use for subsets E, F of X^* the alternative notations $E + F$ for the union $E \cup F$, $E \cdot F$ for the extension to sets of the catenation operation on words, and let $E^* = \epsilon + E + E\cdot E + E\cdot E\cdot E + \dots$, with ϵ denoting the empty word. Moreover, we will use a Lemma (Lemma 1 of Flajolet [7]), which allows to translate operations on sets of words into corresponding operations on series, provided certain non-ambiguity conditions are satisfied.

Lemma 1. *Let E, F be subsets of X^*. Then*

1. $\mathrm{char}(E+F) = \mathrm{char}(E) + \mathrm{char}(F)$ *provided* $E \cap F = \emptyset$*,*
2. $\mathrm{char}(E\cdot F) = \mathrm{char}(E)\cdot\mathrm{char}(F)$ *provided that* $E\cdot F$ *has the unique factorization property,* $\forall u, u' \in E\ \forall v, v' \in F\ uv = u'v'$ *implies* $u = u'$ *and* $v = v'$*,*
3. $\mathrm{char}(E^*) = (1-\mathrm{char}(E))^{-1}$ *provided the following two condition hold:* $E^j \cap E^k = \emptyset\ \forall j,k$ *with* $j \neq k$*, each* E^k *has the unique factorization property.*

With the help of Lemma 1, one can translate operations on sets of words into corresponding operations on series, provided that the non-ambiguity conditions are satisfied.

Let $C_i^{[h]} = C_i^{[h]}(k)$ be defined as the characteristic series of all labeled paths with step vectors given by $a_1, \dots, a_k, b, c$ starting and ending at the level i, with $i \geq 0$, never going below level i and above level $i + h$, with $h \geq 0$. We assume that formal convention $C_i^{[h]} = 0$ if $h < 0$. Moreover, let $C^{[h]} = C_0^{[h]}$.

We introduce the notation $\left\langle C_i^{[h]}\right\rangle_1 := (a_{i,1}|b_{i+1})C_{i+1}^{[h-1]}$, $\left\langle C_i^{[h]}\right\rangle_2 := ((a_{i,2}|b_{i+2})\cdot C_{i+2}^{[h-2]}|b_{i+1})\cdot C_{i+1}^{[h-1]}$, and in general for integer $1\le \ell\le k$ let $\left\langle C_i^{[h]}\right\rangle_\ell$ be defined by

$$\left\langle C_i^{[h]}\right\rangle_\ell = (\dots((a_{i,\ell}|b_{i+\ell})C_{i+\ell}^{[h-\ell]}|b_{i+\ell-1})C_{i+\ell-1}^{[h-(\ell-1)]}\dots|b_{i+1})C_{i+1}^{[h-1]}.$$

Proposition 2. *The characteristic series $C_i^{[h]} = C_i^{[h]}(k)$ of all labeled paths with step vectors given by $a_1,\dots,a_k,b,c$ starting and ending at the level i, with $i\ge 0$, never going below level i and above level $i+h$, with $h\ge 0$, satisfies*

$$C_i^{[h]} = \frac{1}{1-c_i-\sum_{\ell=1}^{k}\left\langle C_i^{[h]}\right\rangle_\ell}.$$

The double sequence $(C_i^{[h]})_{i,h\ge 0}$ converges for $h\to\infty$. Its limit $(C_i)_{i\ge 0}$ is given as follows:

$$C_i = \frac{1}{1-c_i-\sum_{\ell=1}^{k}\left\langle C_i\right\rangle_\ell}.$$

In particular, $C=C_0$ equals the characteristic sequence of all labeled paths $\mathcal{P}$, starting and ending at the x-axis, never going below the y-axis, with step vectors given by $a_1,\dots,a_k,b,c$.

Remark 3. The case $k=1$, treated by Flajolet [7], corresponds to binary increasing trees and ordinary permutation.

Proof. For the sake of simplicity we only present the proof of the special case $k=2$, corresponding to Stirling permutations and ternary increasing trees. We prove that

$$C_0^{[h]} = \frac{1}{1-c_0-\sum_{\ell=1}^{2}\left\langle C_0^{[h]}\right\rangle_\ell} = \frac{1}{1-c_0-(a_{0,1}|b_1)C_1^{[h-1]}-((a_{0,2}|b_2)C_2^{[h-2]}|b_1)C_1^{[h-1]}}$$

equals the characteristic series of the set $\mathcal{P}^{[h]}$ of all labeled paths with step vectors a_1,a_2,b,c, starting and ending at level zero with height bounded by h. More generally, for $i\ge 0$

$$C_i^{[h]} = \frac{1}{1-c_i-\sum_{\ell=1}^{2}\left\langle C_i^{[h]}\right\rangle_\ell} = \frac{1}{1-c_i-(a_{i,1}|b_{i+1})C_{i+1}^{[h-1]}-((a_{i,2}|b_{i+2})C_{i+2}^{[h-2]}|b_{i+1})C_{i+1}^{[h-1]}}$$

equals the characteristic series of all labeled paths starting and ending at level i with height bounded by h. Note that by our previous notation $(u|v)/w = uw^{-1}v$ and $(1-u)^{-1}=\sum_{\ell\ge 0}u^\ell$ regarding quasi-inverse series, we have for instance

$$\begin{aligned}(a_{i,1}|b_{i+1})C_{i+1}^{[h-1]} &= a_{i,1}\big(C_{i+1}^{[h-1]}\big)^{-1}b_{i+1}\\ &= a_{i,1}\sum_{\ell\ge 0}\Big(c_{i+1}+(a_{i+1,1}|b_{i+2})C_{i+2}^{[h-2]}+((a_{i+1,2}|b_{i+3})C_{i+3}^{[h-3]}|b_{i+2})C_{i+2}^{[h-2]}b_{i+2}\Big)^\ell b_{i+1}.\end{aligned}$$

For the first few values of $h=1,2,3$ we obtain

$$\begin{aligned}\mathcal{P}^{[0]} &= (c_0)^*\\ \mathcal{P}^{[1]} &= (c_0+a_{0,1}c_1^*b_1)^*\\ \mathcal{P}^{[2]} &= (c_0+a_{0,1}(c_1+a_{1,1}c_2^*b_2)^*b_1+a_{0,2}c_2^*b_2(c_1+a_{1,1}c_2^*b_2)^*b_1)^*\\ \mathcal{P}^{[3]} &= \Big(c_0+a_{0,1}\big(c_1+a_{1,1}(c_2+a_{2,1}c_3^*b_3)^*b_2+a_{1,2}c_3^*b_3(c_2+a_{2,1}c_3^*b_3)^*b_2\big)^*b_1\\ &\quad +a_{0,2}(c_2+a_{2,1}c_3^*b_3)^*b_2\big(c_1+a_{1,1}(c_2+a_{2,1}c_3^*b_3)^*b_2+a_{1,2}c_3^*b_3(c_2+a_{2,1}c_3^*b_3)^*b_2\big)^*b_1\Big)^*.\end{aligned}$$

In order to simplify the recursive description of $\mathcal{P}^{[h]}$ we introduce the refined sets $\mathcal{P}_i^{[h]}$ consisting of the paths starting and ending at level i with height bounded by h, where $\mathcal{P}_0^{[h]} = \mathcal{P}^{[h]}$. Note that $\mathcal{P}_i^{[h]}$ can easily be obtained from $\mathcal{P}_0^{[h]} = \mathcal{P}^{[h]}$ by shifting the index encoding the level by i. We can write $\mathcal{P}^{[h]}$ in the following way.

$$\begin{aligned}
\mathcal{P}^{[0]} &= (c_0)^* \\
\mathcal{P}^{[1]} &= (c_0 + a_{0,1}\mathcal{P}_1^{[0]}b_1)^* \\
\mathcal{P}^{[2]} &= (c_0 + a_{0,1}\mathcal{P}_1^{[1]}b_1 + a_{0,2}\mathcal{P}_2^{[0]}b_2\mathcal{P}_1^{[1]}b_1)^* \\
\mathcal{P}^{[3]} &= \Big(c_0 + a_{0,1}\mathcal{P}_1^{[2]}b_1 + a_{0,2}\mathcal{P}_2^{[1]}b_2\mathcal{P}_1^{[2]}b_1 + a_{0,3}\mathcal{P}_3^{[0]}b_3\mathcal{P}_2^{[1]}b_2\mathcal{P}_1^{[2]}b_1\Big)^*.
\end{aligned}$$

By induction one can prove that the following unambiguous description of $\mathcal{P}^{[h]}$.

$$\mathcal{P}^{[h]} = \Big(c_0 + a_{0,1}\mathcal{P}_1^{[h-1]}b_1 + a_{0,2}\mathcal{P}_2^{[h-2]}b_2\mathcal{P}_1^{[h-1]}b_1 + \cdots + a_{0,h}\mathcal{P}_h^{[0]}b_h \dots \mathcal{P}_1^{[h-1]}b_1\Big)^*.$$

More generally, we have

$$\mathcal{P}_i^{[h]} = \Big(c_i + a_{i,1}\mathcal{P}_{i+1}^{[h-1]}b_{i+1} + a_{i,2}\mathcal{P}_{i+2}^{[h-2]}b_{i+2}\mathcal{P}_{i+1}^{[h-1]}b_{i+1} + \cdots + a_{i,h}\mathcal{P}_{i+h}^{[0]}b_{i+h} \dots \mathcal{P}_{i+1}^{[h-1]}b_{i+1}\Big)^*.$$

Since $\mathcal{P}_i^{[h]}$ is obtained from $\mathcal{P}_0^{[h]} = \mathcal{P}^{[h]}$ by an index shift, we recursively obtain the stated description of $C_0^{[h]}$ by replacing the operations $+, \cdot, *$ on the sets of words by the series operations $+, |$, and quasi-inverse. Moreover, the characteristic series of the refined sets $\mathcal{P}_i^{[h]}$ is simply given by $C_i^{[h]}$. One observes the inclusion

$$\mathcal{P}^{[0]} \subset \mathcal{P}^{[1]} \subset \mathcal{P}^{[2]} \subset \cdots \subset \mathcal{P},$$

or more generally

$$\mathcal{P}_i^{[0]} \subset \mathcal{P}_i^{[1]} \subset \mathcal{P}_i^{[2]} \subset \cdots \subset \mathcal{P}_i \quad i \geq 0.$$

Since paths of height h have at least length greater or equal $\lceil \frac{h}{k} \rceil$, with $k = 2$ in the presented case corresponding to ternary increasing trees and Stirling permutations, we have

$$\operatorname{val}(C_i - C_i^{[h-1]}) \geq \lceil \frac{h}{k} \rceil,$$

and consequently

$$\lim_{h\to\infty} C_i^{[h]} = C_i.$$

The idea of the proof of the general case $k > 2$ is similar, but the implementation is more involved. □

Subsequently, we will enumerate k-Stirling permutations, keeping track of the 2^{k+1} different local types. Recall that each local type is a string of length $k+1$ on the alphabet $\{0, 1\}$. For our purposes, we arrange the local types first by the number of 1's, and then by lexicographic order. For a k-Stirling permutation, denote the number of instances of the local type with i 1's and j-th position in the lexicographic order by $m_{i,j}$. Let $P_{\mathbf{m}_0,\dots,\mathbf{m}_{k+1}}$ denote the number of k-Stirling permutations whose local types are counted according to $\mathbf{m}_i = (m_{i,1}, \dots, m_{i,\binom{k+1}{i}})$, with $0 \leq i \leq k+1$. The generating function $P(\mathbf{z}_0, \dots, \mathbf{z}_{k+1}, t)$ of k-Stirling permutations with respect to the 2^{k+1} local types is defined by

$$P(\mathbf{z}_0, \dots, \mathbf{z}_{k+1}, t) = \sum_{\mathbf{m}_0,\dots,\mathbf{m}_{k+1}} P_{\mathbf{m}_0,\dots,\mathbf{m}_{k+1}} \mathbf{z_0}^{\mathbf{m}_0} \dots \mathbf{z_{k+1}}^{\mathbf{m}_{k+1}} t^n,$$

where $n = \sum_{i=1}^{k+1} \sum_{j=1}^{\binom{k+1}{i}} m_{i,j}$ is the length of the k-Stirling permutations. To simplify notation, we use z_0 to stand for $z_{0,1}$, the variable that keeps track of the local type with all zeroes (equivalently, a leaf).

Now we can state the main result of this section: the continued fraction representation of the generating function of local types in k-Stirling permutations.

Theorem 4. *The generating function $P(\mathbf{z}_0, \dots, \mathbf{z}_{k+1})$ of k-Stirling permutations, or equivalently $(k+1)$-ary increasing trees, with respect to the 2^{k+1} local types,*

$$P(\mathbf{z}_0, \dots, \mathbf{z}_{k+1}, t) = \sum_{\mathbf{m}_0, \dots, \mathbf{m}_{k+1}} P_{\mathbf{m}_0, \dots, \mathbf{m}_{k+1}} \mathbf{z_0}^{\mathbf{m}_0} \dots \mathbf{z_{k+1}}^{\mathbf{m}_{k+1}} t^n$$

is given by

$$P(\mathbf{z}_0, \dots, \mathbf{z}_{k+1}, t) = \cfrac{z_0 t}{1 - 1 \cdot t \sum_{i=1}^{k+1} z_{1,i} - \cfrac{1 \cdot 2 \cdot t^2 z_0 \sum_{i=1}^{\binom{k+1}{2}} z_{2,i}}{1 - 2 \cdot t \sum_{i=1}^{k+1} z_{1,i} - \cfrac{2 \cdot 3 \cdot t^2 z_0 \sum_{i=1}^{\binom{k+1}{2}} z_{2,\ell}}{\dots} - \dots} - \dots - \cfrac{(k+1)! t^{k+1} z_0^k z_{k+1,1}}{\dots}}.$$

Corollary 1. *An expansion of the generating function $\sum_{n \geq 0} k^n \frac{\Gamma(n+1+\frac{1}{k})}{\Gamma(\frac{1}{k})} t^n$ is obtained from the generating function $P(\mathbf{z}_0, \dots, \mathbf{z}_{k+1}, t)$ by setting $\mathbf{z}_\ell = (1, \dots, 1)$, $0 \leq \ell \leq k+1$, and dividing by t. In particular, we obtain for $k = 2$ the identity*

$$\begin{aligned}\sum_{n \geq 0} (2n+1)!!\, t^n &= \cfrac{1}{1 - 1 \cdot \binom{3}{1} t - \cfrac{1 \cdot 2 \cdot \binom{3}{2} t^2}{1 - 2 \cdot \binom{3}{1} t - \cfrac{2 \cdot 3 \cdot \binom{3}{2} t^2}{1 - 3 \cdot \binom{3}{1} t \dots} - \cfrac{2 \cdot 1 \cdot 2 \cdot 3 \cdot \binom{3}{3} t^3}{1 - 4 \cdot \binom{3}{1} t \dots}} - \cfrac{1 \cdot 2 \cdot 3 \cdot \binom{3}{3} t^3}{\left(1 - 3 \cdot \binom{3}{1} t \dots\right)\left(1 - 2 \cdot \binom{3}{1} t \dots\right)}} \\ &= 1 + 3t + 15t^2 + 105t^3 + 945t^4 + \dots\end{aligned}$$

Remark 4. Below each fraction bar in the continued fraction type representation of the formal power series there are $k+1$ terms, starting with 1. As mentioned earlier the case $k = 1$ is a result of Flajolet [7].

Proof. According to Theorem 2, we can use the same representation for the local types of k-Stirling permutations and node types of $(k+1)$-ary trees. In particular, the local types with i 1's correspond to the node types with i internal children.

Recall the construction from the proof of Theorem 3 of a $(k+1)$-ary tree of size $n+1$, given a labeled path v of length n. From the construction, it is clear that the step $a_{j,\ell}$ corresponds to a node with $\ell + 1$ internal children, the step c_j corresponds to a node with one internal child, and the step b_j corresponds to a leaf node. Finally, at the end of the construction one more leaf node was created.

Taking these correspondences into account, and considering the number of possibilities described by the construction, we define the morphism $\mu : C\langle\langle X \rangle\rangle \to C[[\mathbf{z}]]$ by

$$\begin{aligned}\mu(a_{j,\ell}) &= (j+1)t \sum_{i=1}^{\binom{k+1}{\ell+1}} z_{\ell+1,i} \quad \text{for } 1 \leq \ell \leq k, \\ \mu(c_j) &= (j+1)t \sum_{i=1}^{k+1} z_{1,i} \\ \mu(b_j) &= (j+1)t z_0.\end{aligned}$$

Taking into account the creation of one more leaf node at the end of the construction, we get the generating function of $(k+1)$-ary trees with respect to the node types from the characteristic series C of labeled paths: $P(\mathbf{z}_0, \ldots, \mathbf{z}_{k+1}, t) = z_0 t\mu(C)$.

□

5 Classical Stirling permutations

5.1 Another representation of generating function of Stirling permutations

In the special case $k = 2$ Janson [14] showed that the class of 2-Stirling permutations of size n is in bijection with the class of plane-oriented recursive trees of size $n+1$. A bijection between ternary increasing tree of size n and plane-oriented recursive trees of size $n+1$ was given in [15]. We will provide a bijection between path diagrams with an infinite number of rise vectors (known as Łukasiewicz paths), plane-oriented recursive trees and Stirling permutations.

We define path diagrams induced by Łukasiewicz paths in exactly the same manner as in section 4.1, except that we allow the set of step vectors to consist of an infinite number of rise vectors $\mathbf{a} = (a_\ell)_{\ell \in \mathbb{N}}$, with $a_\ell = (1, \ell)$ for $\ell \in \mathbb{N}$, a fall vector $b = (1, -1)$, and a level vector $c = (1, 0)$, and accordingly modify the alphabet X for the labeled paths.

Theorem 5. *The class of plane-oriented recursive trees of size $n+1$ is in bijection with path diagrams of length n, with possibility function* pos *given by*

$$\mathrm{pos}(a_{j,\ell}) = j+1, \quad \ell \in \mathbb{N}, \quad \mathrm{pos}(b_j) = j+1, \quad \mathrm{pos}(c_j) = j+1,$$

with respect to labeled paths induced by Łukasiewicz paths.

Proof. The proof is similar to that of Theorem 3, so it is sufficient to sketch the necessary construction for a plane-oriented recursive tree of size $n+1$, given a labeled path $v = v_1 \ldots v_n$. We begin the construction with one placeholder for an internal node. Then for $1 \le k \le n$, at k-th step: replace any existing placeholder with the node labeled by k; if $v_k = a_{j,\ell}$, create $\ell + 1$ children for the node k, each labeled by a placeholder; if $v_k = c_j$, create one child for the node k labeled by a placeholder; and if $v_k = b_j$, the node k is a leaf node. Note that in each case, the net number of placeholders changes by the same amount as the height of the path; since the construction began with one placeholder, at the beginning of the k-th construction step the number of placeholders is one more than the height of the starting point M_{k-1} of the k-th vector of the path v. Therefore the number of possibilities for carrying out the k-th step is $j+1$, which defines the corresponding possibility function for the path diagrams. At the end of the n-th step, there is one more placeholder, which we replace by a leaf labeled by $n+1$. □

Let $\mathcal{F}_n$ denote the set of path diagrams induced by Łukasiewicz paths of length n, with possibility function defined by $\mathrm{pos}(a_{j,\ell} = \mathrm{pos}(b_j) = \mathrm{pos}(c_j) = j+1$, and let $f(t) = \sum_{n \ge 0} |\mathcal{F}_n| t^n$ be the ordinary generating function of these path diagrams. Let $\mathcal{G}_n$ denote the set of path diagrams induced by paths with steps a_1, a_2, b, c, with possibility function defined by $\mathrm{pos}(a_{j,1}) = \mathrm{pos}(c_j) = 3(j+1)$ and $\mathrm{pos}(a_{j,2}) = \mathrm{pos}(b_j) = j+1$, and let $g(t) = \sum_{n \ge 0} |\mathcal{G}_n| t^n$ be the ordinary generating function of these path diagrams.

Theorem 6. *Let $S_n = (2n-1)!!$ denote the number of Stirling permutations of size n, and $S(t) = \sum_{n \ge 1} S_n t^n$ be the generating function of Stirling polynomials. Then $1 + S(t) = f(t) = 1 + tg(t)$, and* *can therefore be represented by two different continuous fraction types:*

$$\begin{aligned}1+S(t) &= \cfrac{1}{1-t-\cfrac{2\cdot 1\cdot t^2}{1-2\cdot t-\frac{3\cdot 2\cdot t^2}{1-3\cdot t-\cdots}}-\cfrac{3\cdot 2\cdot 1\cdot t^3}{\left(1-2\cdot t-\frac{3\cdot 2\cdot t^2}{1-3\cdot t-\cdots}-\cdots\right)\left(1-3\cdot t-\frac{4\cdot 3\cdot t^2}{1-4\cdot t-\cdots}-\cdots\right)}-\cfrac{4\cdot 3\cdot 2\cdot 1\cdot t^3}{(\cdots)(\cdots)(\cdots)}-\cdots} \\ &= 1+\cfrac{t}{1-1\cdot\binom{3}{1}t-\cfrac{1\cdot 2\cdot\binom{3}{2}t^2}{1-2\cdot\binom{3}{1}t-\frac{2\cdot 3\cdot\binom{3}{2}t^2}{1-3\cdot\binom{3}{1}t\ldots}-\frac{2\cdot 1\cdot 2\cdot 3\cdot\binom{3}{3}t^3}{1-4\cdot\binom{3}{1}t\ldots}}-\cfrac{1\cdot 2\cdot 3\cdot\binom{3}{3}t^3}{\left(1-3\cdot\binom{3}{1}t\ldots\right)\left(1-2\cdot\binom{3}{1}t\ldots\right)}}\end{aligned} \tag{4}$$

Proof. The set $\mathcal{S}_n$ of Stirling permutations of size n are in bijection with the set of path diagrams $\mathcal{G}_{n-1}$ by Theorem 3. Also, since $\mathcal{S}_n$ is in bijection with the set of plane-oriented recursive paths of size $n+1$ (see Janson [14]), which in turn is in bijection with the set of path diagrams $\mathcal{F}_n$ by Theorem 5, we get the identity $1+S(t)=f(t)=1+tg(t)$.

The continued fraction representation for $g(t)$ is given in Corollary 1. The continued fraction representation of $f(t)$ is known as a Łukasiewicz fraction; its form is obtained through the characteristic polynomial of Łukasiewicz paths in a manner similar to that in section 4.2, taking into account that there is no restriction in the rise steps (for more on Łukasiewicz paths, see the works of Roblet [24] and Viennot [26], and also [25]).

□

5.2 Equivalent statistics on plane-oriented recursive trees, ternary trees, and Stirling permutations

Let $X_{n,j}$ denote the number of nodes of outdegree j in a random plane-oriented recursive tree of size n. We relate the distribution of outdegrees to suitably defined statistics in ternary increasing trees and Stirling permutations. Any ternary increasing tree can be decomposed by deleting all center edges into trees having only left or right edges. Let $X_{n,j}^{[LR]}$ denote the number of size j left-right trees in a random ternary increasing tree of size n.

Concerning Stirling permutations $\sigma=\sigma_1\ldots\sigma_{2n}$, we introduce block structures as follows. A block in a Stirling permutation σ is a substring $\sigma_p\cdots\sigma_q$ with $\sigma_p=\sigma_q$ that is maximal, i.e. not contained in any larger such substring [15]. There is obviously at most one block for every $i=1,\ldots,n$, extending from the first occurrence of i to the last; we say that i forms a block when this substring is not contained in a string $\ell\cdots\ell$ for some $\ell<i$. The decomposition $\sigma=[B_1][B_2]\ldots[B_j]$ is a block structure of σ. Removing from each of the blocks the leftmost and the rightmost number, we are left with possibly empty substrings, which after an order-preserving relabeling form (sub-)Stirling permutations. We recursively determine the block structure in these (sub)-Stirling permutations. The Stirling permutation σ has a block structure of size j if either σ or any of the recursively obtained (sub)-Stirling permutations decompose into j blocks. Let $X_{n,j}^{[B]}$ denotes the number of block structures of size j in a random Stirling permutation of size n.

Example 8. The Stirling permutation $\sigma=221553367788614499$ of size nine has block decomposition $\sigma=[22][155336778861][44][99]$ of size 4. After removal of the leftmost and the rightmost entries in the blocks, the only non-empty (sub-)Stirling permutation is given by 5533677886. After an order-preserving relabeling we get $\sigma'=2211344553$. We have $\sigma'=[22][11][344553]$, a block decomposition of size 3; consequently we obtain the (sub-)Stirling permutation $\sigma''=1122$, which has block decomposition $\sigma''=[11][22]$ of size 2. Hence, $X_{9,4}^{[B]}(\sigma)=1$, $X_{9,3}^{[B]}(\sigma)=1$ and $X_{9,2}^{[B]}(\sigma)=1$.

Theorem 7. *For $j>2$, the distribution of the number of nodes $X_{n+1,j}$ of outdegree j in a random plane-oriented recursive tree of size $n+1$ coincides with the distribution of the number $X_{n,j-1}^{[LR]}$ of size $j-1$ left-right trees in a random ternary increasing tree of size n, and with the distribution of*

the number $X_{n,j}^{[B]}$ of sub-Stirling permutations with number of blocks equal to j in a random Stirling permutation of size n.
Moreover, the nodes of outdegree two in plane-oriented recursive tree of size $n+1$ correspond to the number of nodes in ternary increasing trees of size n having exactly one child, connected by a center edge, where the child is a leaf node.

The proof of the the result consists of a simple application of the bijection stated in [15], and is therefore omitted.

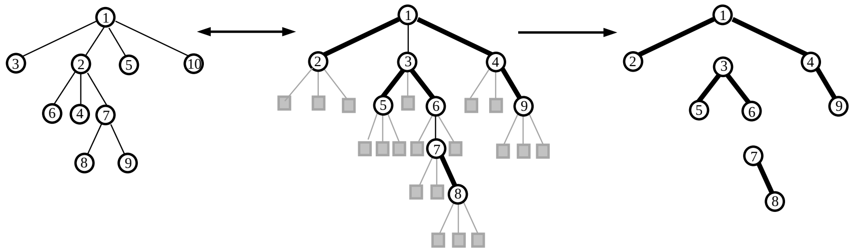

Figure 5: A plane-oriented increasing tree of size 10, the corresponding size 9 ternary increasing trees together with its left-right tree decomposition.

Example 9. The Stirling permutation σ of size nine corresponding to the trees in Figure 5, obtained either using the bijection with plane-oriented recursive tree [14] or with ternary increasing trees [15], is given by $\sigma = 221553367788614499$. As observed before we have $X_{9,4}^{[B]}(\sigma) = 1$, $X_{9,3}^{[B]}(\sigma) = 1$ and $X_{9,2}^{[B]}(\sigma) = 1$, corresponding to the number of nodes with outdegrees given by four, three and two in the corresponding plane-oriented recursive trees, and with the sizes of the left-right trees in ternary increasing trees.

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