Ellipsoids and Matrix valued Valuations

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Abstract

A classification is obtained of Borel measurable, $\mathrm{GL}(n)$ covariant, symmetric matrix valued valuations on the space of n-dimensional convex polytopes. The only ones turn out to be the moment matrix corresponding to the classical Legendre ellipsoid and the matrix corresponding to the ellipsoid recently discovered by Lutwak, Yang, and Zhang.

A classical concept from mechanics is the *Legendre ellipsoid* or *ellipsoid of inertia* $\Gamma_2 K$ associated with a convex body $K \subset \mathbb{R}^n$. It can be defined as the unique ellipsoid centered at the center of mass of K such that the ellipsoid's moment of inertia about any axis passing through the center of mass is the same as that of K. If we fix a scalar product $x \cdot y$ for $x, y \in \mathbb{R}^n$, $\Gamma_2 K$ can be defined by the *moment matrix* $M_2(K)$ of K. This is the $n \times n$ matrix with coefficients

$$\int_K x_i\, x_j\, dx,$$

where we use coordinates $x = (x_1, \ldots, x_n)$ for $x \in \mathbb{R}^n$. For a convex body K with non-empty interior, $M_2(K)$ is a positive definite symmetric $n \times n$ matrix. In general, such a matrix A generates an ellipsoid E_A defined by

$$E_A = \{x \in \mathbb{R}^n : x \cdot Ax \le 1\}. \tag{1}$$

Then

$$\Gamma_2 K = \sqrt{\frac{n+2}{V(K)}}\, E_{M_2(K)^{-1}},$$

where $V(K)$ denotes the n-dimensional volume of K. An important property of the operator Γ_2 is that it is *linear*, i.e., for every convex body K

$$\Gamma_2(\phi K) = \phi\, \Gamma_2 K \quad \text{for} \quad \phi \in \mathrm{GL}(n).$$

The corresponding transformation rule for M_2 is

$$M_2(\phi K) = |\det \phi|\, \phi M_2(K) \phi^t \quad \text{for} \quad \phi \in \mathrm{GL}(n),$$

where $\det \phi$ denotes the determinant of ϕ and ϕ^t denotes the transpose of ϕ. For additional information on the Legendre ellipsoid and its important applications, see [13], [14], [27].

Recently, Lutwak, Yang, and Zhang [22] defined a new ellipsoid $\Gamma_{-2}K$ for $K \in \mathcal{K}_o^n$, the space of convex bodies containing the origin in their interiors. For a polytope P, this ellipsoid can be defined by the matrix $M_{-2}(P)$ with coefficients

$$\sum_u \frac{a(u)}{h(u)}\, u_i\, u_j$$

where we sum over all unit normals u of facets of P and where $a(u)$ is the $(n-1)$-dimensional volume of the facet with normal u and $h(u)$ is the distance from the origin of the hyperplane containing this facet. For general $K \in \mathcal{K}_o^n$, approximation shows that $M_{-2}(K)$ is defined by an integral involving the L_2-surface area measure of K (see [22]). Using (1), the *LYZ ellipsoid* is given by

$$\Gamma_{-2}K = \sqrt{V(K)}\, E_{M_{-2}(K)}.$$

This definition is natural in the framework of the L_p-Brunn-Minkowski and dual L_p-Brunn-Minkowski theory (see [18], [19], [20], [21], [24]); there the ellipsoids Γ_2 and Γ_{-2} are dual notions. Lutwak, Yang, and Zhang [23] proved that $\Gamma_{-2}K \subset \Gamma_2 K$ and noted that this is an geometrical analogue of the Cramer-Rao inequality. An important property of the operator Γ_{-2} is that is linear, i.e., for every $K \in \mathcal{K}_o^n$

$$\Gamma_{-2}(\phi K) = \phi\, \Gamma_{-2}K \quad \text{for} \quad \phi \in \mathrm{GL}(n).$$

The corresponding transformation rule for M_{-2} is

$$M_{-2}(\phi K) = |\det \phi|\, \phi^{-t} M_{-2}(K) \phi^{-1} \quad \text{for} \quad \phi \in \mathrm{GL}(n).$$

For more information on the LYZ ellipsoid, its applications, and its connection to the Fisher information from information theory, see [7], [22], [23].

In addition to these two ellipsoids, there exist many well-known ellipsoids that have been introduced and used for different purposes: the John ellipsoid, the minimal surface ellipsoid (Petty ellipsoid), the ℓ-ellipsoid, the M-ellipsoids are important examples (see [6], [28], [29], [31], for definitions and applications). However, (as we will show) only Γ_2 and Γ_{-2} are linear and have the following important property. The matrix valued functions M_2 and M_{-2} corresponding to these operators are valuations. In general, a function Z defined on $\mathcal{K}_o^n$ and taking values in an Abelian semigroup is called a *valuation*, if

$$ZK_1 + ZK_2 = Z(K_1 \cup K_2) + Z(K_1 \cap K_2)$$

for $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}_o^n$. Ever since Hadwiger [8] proved his now classical characterization of the quermassintegrals (elementary mixed volumes), the classification of valuations on the space of convex bodies and related spaces has been an important subject in geometry. For detailed information and an historical account, see [26], [25], and [12]. See also [1]-[4], [9]-[11], [17] for some of the more recent contributions.

To state our results, we fix some notation. Let $\mathcal{P}_o^n$ denote the space of convex polytopes containing the origin in their interiors and call a function defined on

$\mathcal{P}_o^n$ (Borel) *measurable* if the pre-image of every open set is a Borel set. For $P \in \mathcal{P}_o^n$, let P^* denote the polar body of P, i.e.,

$$P^* = \{y \in \mathbb{R}^n \mid x \cdot y \leq 1 \text{ for all } x \in P\}.$$

Let $\mathcal{M}^n$ denote the set of real symmetric $n \times n$ matrices and for $n = 2$, let $\psi_{\pi/2}$ denote the rotation by an angle $\pi/2$.

Theorem 1. *A function $Z : \mathcal{P}_o^n \to \mathcal{M}^n$, $n \geq 3$, is a measurable valuation such that*

$$Z(\phi P) = |\det \phi|^q \, \phi Z(P) \phi^t \tag{2}$$

holds for every $\phi \in \mathrm{GL}(n)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$Z(P) = c\, M_2(P) \quad \text{or} \quad Z(P) = c\, M_{-2}(P^*)$$

for every $P \in \mathcal{P}_o^n$. A function $Z : \mathcal{P}_o^2 \to \mathcal{M}^2$ is a measurable valuation such that (2) holds for every $\phi \in \mathrm{GL}(2)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$Z(P) = c\, M_2(P) \quad \text{or} \quad Z(P) = c\, M_{-2}(P^*) \quad \text{or} \quad Z(P) = c\, \psi_{\pi/2}^{-1} M_2(P^*)\, \psi_{\pi/2}$$

for every $P \in \mathcal{P}_o^2$.

Theorem 2. *A function $Z : \mathcal{P}_o^n \to \mathcal{M}^n$, $n \geq 3$, is a measurable valuation such that*

$$Z(\phi P) = |\det \phi^{-t}|^q \, \phi^{-t} Z(P) \phi^{-1} \tag{3}$$

holds for every $\phi \in \mathrm{GL}(n)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$Z(P) = c\, M_2(P^*) \quad \text{or} \quad Z(P) = c\, M_{-2}(P)$$

for every $P \in \mathcal{P}_o^n$. A function $Z : \mathcal{P}_o^2 \to \mathcal{M}^2$ is a measurable valuation such that (3) holds for every $\phi \in \mathrm{GL}(2)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$Z(P) = c\, M_2(P^*) \quad \text{or} \quad Z(P) = c\, M_{-2}(P) \quad \text{or} \quad Z(P) = c\, \psi_{\pi/2}^{-1}\, M_2(P)\, \psi_{\pi/2}$$

for every $P \in \mathcal{P}_o^2$.

These theorems imply that every *continuous* covariant valuation on $\mathcal{K}_o^n$, $n \geq 3$, i.e., every continuous valuation that transforms according to (2), is a multiple of $M_2(K)$ or $M_{-2}(K^*)$. However, these are not all possible examples of measurable covariant valuations on $\mathcal{K}_o^n$. Define the matrices $A_p(K)$, $p > 0$, by their coefficients

$$\int_{\operatorname{bd} K} x_i\, x_j \, d\Omega_p(K, x),$$

where $\operatorname{bd} K$ is the boundary of K and $d\Omega_p(K, x)$ is the L_p-affine surface area measure (see [19]). Then A_p is a covariant valuation on $\mathcal{K}_o^n$, which (like L_p-affine surface area) depends upper semicontinuously on K.

1 Background and Notation

We work in n-dimensional Euclidean space $\mathbb{R}^n$ with origin o and use coordinates $x = (x_1, \ldots, x_n)$ for $x \in \mathbb{R}^n$. We denote the standard scalar product of $x, y \in \mathbb{R}^n$ by $x \cdot y$. We identify the subspace with equation $x_n = 0$ with $\mathbb{R}^{n-1}$. The n-dimensional volume in $\mathbb{R}^n$ is denoted by V. The $(n-1)$-dimensional volume in $\mathbb{R}^{n-1}$ is denoted by V'. In general, we denote objects in $\mathbb{R}^{n-1}$ by the same symbol as objects in $\mathbb{R}^n$ with an additional $'$. So, for example, $m(P)$ is the moment vector of a polytope $P \in \mathcal{P}_o^n$, i.e.,

$$m(P) = \int_P x \, dx,$$

and $m'(P')$ is the moment vector of $P' \in \mathcal{P}_o^{n-1}$. We denote the convex hull of $P_1, \ldots, P_k$ by $[P_1, \ldots, P_k]$, and we denote the group of special linear transformations, i.e., of linear transformations ϕ with $\det \phi = 1$, by $\mathrm{SL}(n)$ and the group of general linear transformations, i.e., of linear transformations ϕ with $\det \phi \neq 0$, by $\mathrm{GL}(n)$. For a general reference on convex geometry, see the books by Schneider [30] or Gardner [5].

We use the following results on valuations on $\mathcal{P}_o^1$ and on Cauchy's functional equation. Let $\nu : \mathcal{P}_o^1 \to \mathbb{R}$ be a measurable valuation that is homogeneous of degree p, i.e. $\nu(t\,I) = t^p \, \nu(I)$ for $t > 0$ and $I \in \mathcal{P}_o^1$. If $p = 0$, then there are constants $a, b \in \mathbb{R}$ such that

$$\nu([-s,t]) = a \, \log(\frac{t}{s}) + b \tag{4}$$

for every $s, t > 0$, and if $p \neq 0$, then there are constants $a, b \in \mathbb{R}$ such that

$$\nu([-s,t]) = a\, s^p + b\, t^p \tag{5}$$

for every $s, t > 0$ (cf. [15], equations (3) and (4)). These results follow from the fact that every measurable solution f of Cauchy's functional equation

$$f(x+y) = f(x) + f(y) \tag{6}$$

is linear. If $f : \mathbb{R}^k \to \mathbb{R}$, then every measurable solution of (6) is of the form

$$f(x) = a \cdot x \tag{7}$$

with $a \in \mathbb{R}^k$.

The proofs of Theorems 1 and 2 use induction on the dimension and the following subsets of $\mathcal{P}_o^n$. Let $\mathcal{Q}_o(x_n)$ be the set of polytopes $Q = [P', I]$ where $P' \in \mathcal{P}_o^{n-1}$ lies in the hyperplane $x_n = 0$ and $I \in \mathcal{P}_o^1$ lies on the x_n-axis, and let $\mathcal{R}_o(x_n)$ be the set of polytopes $R = [P', u, v]$ where $P' \in \mathcal{P}_o^{n-1}$ lies in the hyperplane $x_n = 0$, $u \in \mathbb{R}^n$ has $u_n < 0$, and $v \in \mathbb{R}^n$ has $v_n > 0$. We always write polytopes from $\mathcal{R}_o(x_n)$ in such a way that $[P', u] \cup [P', v]$ is convex. Let $\mathcal{Q}_o^n$ be the set of $\mathrm{SL}(n)$-images of $Q \in \mathcal{Q}_o(x_n)$ and let $\mathcal{R}_o^n$ be the set of $\mathrm{SL}(n)$-images of $R \in \mathcal{R}_o(x_n)$. The following lemma shows that we have only to prove that a matrix valued valuation vanishes on $\mathcal{R}_o^n$ to prove the corresponding result on $\mathcal{P}_o^n$.

Lemma 1 ([15]). *Let $\mu : \mathcal{P}_o^n \to \mathbb{R}$ be a valuation. If μ vanishes on $\mathcal{R}_o^n$, then $\mu(P) = 0$ for every $P \in \mathcal{P}_o^n$.*

The following results on real valued and vector valued valuations on $\mathcal{P}_o^n$ are used in the proofs of Theorems 1 and 2.

Theorem 3 ([15]). *A functional $\mu : \mathcal{P}_o^n \to \mathbb{R}$, $n \geq 2$, is a measurable valuation such that*

$$\mu(\phi P) = |\det \phi|^q \, \mu(P)$$

holds for every $\phi \in \mathrm{GL}(n)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$\mu(P) = c \quad \text{or} \quad \mu(P) = c\, V(P) \quad \text{or} \quad \mu(P) = c\, V(P^*)$$

for every $P \in \mathcal{P}_o^n$.

Theorem 4 ([16]). *A function $z : \mathcal{P}_o^n \to \mathbb{R}^n$, $n \geq 3$, is a measurable valuation such that*

$$z(\phi P) = |\det \phi|^q \, \phi z(P) \tag{8}$$

holds for every $\phi \in \mathrm{GL}(n)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\, m(P)$$

for every $P \in \mathcal{P}_o^n$. A function $z : \mathcal{P}_o^2 \to \mathbb{R}^2$ is a measurable valuation such that (8) holds for every $\phi \in \mathrm{GL}(2)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\, m(P) \quad \text{or} \quad z(P) = c\, \psi_{\pi/2}^{-1} m(P^*)$$

for every $P \in \mathcal{P}_o^2$.

Theorem 5 ([16]). *A function $z : \mathcal{P}_o^n \to \mathbb{R}^n$, $n \geq 3$, is a measurable valuation such that*

$$z(\phi P) = |\det \phi^{-t}|^q \, \phi^{-t} z(P) \tag{9}$$

holds for every $\phi \in \mathrm{GL}(n)$ with $q \in \mathbb{R}$ if and only if $n \geq 3$ and there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\, m(P^*)$$

for every $P \in \mathcal{P}_o^n$. A function $z : \mathcal{P}_o^2 \to \mathbb{R}^2$ is a measurable valuation such that (9) holds for every $\phi \in \mathrm{GL}(2)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\, m(P^*) \quad \text{or} \quad z(P) = c\, \psi_{\pi/2}^{-1} m(P)$$

for every $P \in \mathcal{P}_o^2$.

We say that vectors and matrices that transform according to (8) and (2), respectively, are *covariant*. If they transform according to (9) and (3), we say that they are *contravariant*. Note that for $n = 2$ we have

$$m(\phi P) = |\det \phi| \, \phi \, m(P) \quad \text{and} \quad \tilde{m}(\phi P) = |\det \phi|^{-2} \, \phi \, \tilde{m}(P)$$

for $\phi \in \mathrm{GL}(2)$, where $\tilde{m}(P) = \psi_{\pi/2}^{-1} m(P^*)$. Thus for $n = 2$ there are non-trivial covariant vector valued valuations only for $q = 1$ and $q = -2$. For $n = 3$ there are such valuations only for $q = 1$.

2 Proofs

Let $Z : \mathcal{P}_o^n \to \mathcal{M}^n$ be a measurable valuation that transforms according to (2) for a fixed $q \in \mathbb{R}$. The function Z^*, defined by $Z^*(P) = Z(P^*)$ for $P \in \mathcal{P}_o^n$, is again measurable. For $P, Q, P \cup Q \in \mathcal{P}_o^n$, we have

$$(P \cup Q)^* = P^* \cap Q^* \quad \text{and} \quad (P \cap Q)^* = P^* \cup Q^*.$$

Therefore

$$\begin{aligned} Z^*(P) + Z^*(Q) &= Z(P^*) + Z(Q^*) \\ &= Z(P^* \cup Q^*) + Z(P^* \cap Q^*) \\ &= Z((P \cap Q)^*) + Z((P \cup Q)^*) \;=\; Z^*(P \cap Q) + Z^*(P \cup Q), \end{aligned}$$

i.e., Z^* is a valuation on $\mathcal{P}_o^n$. For $\phi \in \mathrm{GL}(n)$ and $P \in \mathcal{P}_o^n$, we have $(\phi\, P)^* = \phi^{-t} P^*$. Therefore $Z^*(\phi P) = Z((\phi P)^*) = Z(\phi^{-t} P^*)$ and by (2)

$$Z^*(\phi P) = |\det \phi^{-t}|^q \phi^{-t} Z(P^*) \phi^{-1} = |\det \phi^{-t}|^q \phi^{-t} Z^*(P) \phi^{-1},$$

i.e., $Z^* : \mathcal{P}_o^n \to \mathbb{R}^n$ is a measurable valuation that transforms according to (3). Thus Theorems 1 and 2 are equivalent for fixed $q \in \mathbb{R}$. This enables us to prove both theorems by first proving Theorem 1 for $q > -1$ and then Theorem 2 for $q \leq -1$.

2.1 Proof of Theorem 1 for $q > -1$

1. We begin by proving Theorem 1 for $n = 2$. For $r \neq 0$, let

$$\phi = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & r \end{pmatrix}.$$

Let $I_i \in \mathcal{P}_o^1$ be an interval on the x_i-axis. Then $[I_1, I_2] \in \mathcal{Q}_o(x_2)$ and it follows from (2) that

$$Z(\phi[I_1, I_2]) = Z([r^{-1} I_1, r\, I_2]) = \phi Z([I_1, I_2]) \phi^t.$$

Setting $z_0 = z_{11}, z_1 = z_{12} = z_{21}, z_2 = z_{22}$, we have for $k = 0, 1, 2$

$$z_k([r^{-1} I_1, r\, I_2]) = r^{-2+2k}\, z_k([I_1, I_2]). \tag{10}$$

By (2) now applied with the matrices

$$\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},$$

we obtain

$$z_k([r^{-1} I_1, r\, I_2]) = r^{-2q-2}\, z_k([I_1, r^2\, I_2])$$

and

$$z_k([r^{-1} I_1, r\, I_2]) = r^{2q+2}\, z_k([r^{-2} I_1, I_2]).$$

Combined with (10) these equations show that

$$z_k([I_1, r^2 I_2]) = r^{2(q+k)} z_k([I_1, I_2])$$

and

$$z_k([r^{-2} I_1, I_2]) = r^{-2(q+2-k)} z_k([I_1, I_2]).$$

Thus $z_k([I_1, \cdot])$ is homogeneous of degree $q+k$ and $z_k([\cdot, I_2])$ is homogeneous of degree $q+2-k$.

1.1. We consider the case $q > -1$, $q \neq 0$. Since $z_k([I_1, \cdot])$ is homogeneous of degree $q+k$, we obtain from (5) that

$$z_k([I_1, I_2]) = a_k(I_1)\, s_2^{q+k} + b_k(I_1)\, t_2^{q+k}$$

with $I_2 = [-s_2, t_2]$. The functionals $a_k, b_k : \mathcal{P}_o^1 \to \mathbb{R}$ are measurable valuations. Since $z_k([\cdot, I_2])$ is homogeneous of degree $q+2-k$, they are homogeneous of degree $q+2-k$. By (5) there are constants $a_k, b_k, c_k, d_k \in \mathbb{R}$ such that

$$z_k([I_1, I_2]) = (a_k\, s_1^{q+2-k} + b_k\, t_1^{q+2-k})\, s_2^{q+k} + (c_k\, s_1^{q+2-k} + d_k\, t_1^{q+2-k})\, t_2^{q+k} \quad (11)$$

for every $s_1, t_1, s_2, t_2 > 0$ with $I_1 = [-s_1, t_1]$.

For

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

it follows from (2) that

$$z_k(\phi[I_1, I_2]) = z_k([I_1, -I_2]) = (-1)^k z_k([I_1, I_2]) \quad (12)$$

and

$$z_k(\psi[I_1, I_2]) = z_k([-I_2, I_1]) = (-1)^k z_{2-k}([I_1, I_2]). \quad (13)$$

We use (11), compare coefficients in (12) and (13) and obtain that

$$\begin{aligned} z_0([I_1, I_2]) &= a_0\, (s_1^{q+2} + t_1^{q+2})(s_2^q + t_2^q), \\ z_1([I_1, I_2]) &= a_1\, (s_1^{q+1} - t_1^{q+1})(s_2^{q+1} - t_2^{q+1}), \\ z_2([I_1, I_2]) &= a_0\, (s_1^q + t_1^q)(s_2^{q+2} + t_2^{q+2}) \end{aligned} \quad (14)$$

for every $s_1, t_1, s_2, t_2 > 0$.

We need the following results. Lemma 2 will also be used for the case $q = 0$.

Lemma 2. *Let $Z : \mathcal{P}_o^2 \to \mathcal{M}^2$ be a measurable covariant valuation for which (14) holds. Let $R = [I_1, s\,u, t\,v]$ where $I_1 = [-s_1, t_1]$ lies on the x_1-axis, $u = (x, -1)$, $v = (y, 1)$ with $x, y \in \mathbb{R}$, $s_1, t_1, s, t > 0$. If $q > -1$, then*

$$\begin{aligned} z_1(R) &= a_1\, (s_1^{q+1} - t_1^{q+1})(s^{q+1} - t^{q+1}) - a_0\, (s_1^q + t_1^q)(x\, s^{q+2} - y\, t^{q+2}), \\ z_2(R) &= a_0\, (s_1^q + t_1^q)(s^{q+2} + t^{q+2}), \end{aligned} \quad (15)$$

and if $q = 0$, then in addition

$$z_0(R) = 2\, a_0\, (s_1^2 + t_1^2 + x^2 s^2 + y^2 t^2) - 2\, a_1\, (s_1 - t_1)(x\, s + y\, t). \quad (16)$$

Proof. First, we show that for $k = 1, 2$, $q > -1$, and $k = 0$, $q = 0$,

$$\lim_{s,t\to 0} z_k([I_1, s\,u, t\,v]) \tag{17}$$

exists. Since z_k is a valuation and since u and v lie in complementary halfplanes, we have for $s, t > 0$ suitably small, $0 < t' < t$, and $t'' > 0$ suitably large

$$z_k([I_1, s\,u, t\,v]) + z_k([I_1, -t''\,v, t'\,v]) = z_k([I_1, s\,u, t'\,v]) + z_k([I_1, -t''\,v, t\,v]). \tag{18}$$

Since $[I_1, -t''\,v, t\,v] = \phi[I_1, I_2]$ with

$$\phi = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

and $I_2 = [-t'', t]$, we obtain from (2) that

$$z_k([I_1, -t''\,v, t\,v]) = z_k(\phi[I_1, I_2]) = \sum_{l=k}^{2} \binom{2-k}{l-k} y^{l-k} z_l([I_1, I_2]). \tag{19}$$

Thus, setting $a_2 = a_0$, we obtain from (14), (18), and (19) that

$$\begin{aligned} &z_k([I_1, s\,u, t\,v]) - z_k([I_1, s\,u, t'\,v]) \\ &= \sum_{l=k}^{2} \binom{2-k}{l-k} a_l\, y^{l-k} (-1)^l (s_1^{q+2-l} + (-1)^l t_1^{q+2-l})(t^{q+l} - t'^{q+l}). \end{aligned} \tag{20}$$

Similarly, we have for $s, t > 0$ suitably small, $0 < s' < s$, and $s'' > 0$ suitably large

$$\begin{aligned} &z_k([I_1, s\,u, t'\,v]) - z_k([I_1, s'\,u, t'\,v]) \\ &= \sum_{l=k}^{2} \binom{2-k}{l-k} a_l\, (-x)^{l-k} (s_1^{q+2-l} + (-1)^l t_1^{q+2-l})(s^{q+l} - s'^{q+l}). \end{aligned} \tag{21}$$

Since $q + k \ge 0$, this implies that the limit (17) exists.

Next, we show that for $k = 1, 2$

$$\lim_{s,t\to 0} z_k([I_1, s\,u, t\,v]) = 0 \tag{22}$$

and that for $q = 0$

$$\lim_{s,t\to 0} z_0([I_1, s\,u, t\,v]) = 2\,a_0\,(s_1^2 + t_1^2). \tag{23}$$

We start by proving (22) for $k = 2$. If $k = 1$, we use that (22) holds for $k = 2$, and if $k = 0$, we use that (22) holds for $k = 1, 2$. For I_1 fixed, $u = (x, -1)$ and $v = (y, 1)$, set

$$f_k(x, y) = \lim_{s,t\to 0} z_k([I_1, s\,u, t\,v]).$$

These limits exist by (17). Since z_k is a valuation, we have for $r > 0$ suitably small and $e = (1,0)$

$$z_k([I_1, s\,u, t\,v]) + z_k([I_1, -s\,r\,e, t\,r\,e]) = z_k([I_1, s\,u, t\,r\,e]) + z_k([I_1, -s\,r\,e, t\,v]).$$

Taking the limit as $s, t \to 0$ gives

$$f_k(x, y) + f_k(0, 0) = f_k(x, 0) + f_k(0, y). \tag{24}$$

For

$$\phi = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

we have $\phi[I_1, s\,(x, -1), t\,(y, 1)] = [I_1, s\,(0, -1), t\,(x + y, 1)]$ and by (2)

$$z_k(\phi[I_1, s\,u, t\,v]) = \sum_{l=k}^{2} \binom{2-k}{l-k} x^{l-k} z_l([I_1, s\,u, t\,v]).$$

Combined with (22) this implies

$$f_k(0, x + y) = f_k(x, y). \tag{25}$$

Setting $g_k(x) = f_k(0, x) - f_k(0, 0)$, it follows from (24) and (25) that

$$g_k(x + y) = g_k(x) + g_k(y).$$

This is Cauchy's functional equation (6). Since z_k is measurable, so is g_k and by (7) there is a constant $w_k(I_1) \in \mathbb{R}$ such that

$$g_k(x) = f_k(0, x) - f_k(0, 0) = w_k(I_1)\, x.$$

Thus

$$\lim_{s,t \to 0} z_k([I_1, s\,u, t\,v]) = w_k(I_1)(x + y) + f_k(0, 0). \tag{26}$$

Using this we obtain the following. By (2) z_k is homogeneous of degree $2q + 2$. Therefore (26) implies that for $r > 0$

$$w_k(r\,I_1) = r^{2q+2} w_k(I_1). \tag{27}$$

On the other hand, for

$$\phi = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix},$$

we have $z_k(\phi R) = r^{2q+2-k}\, z_k(R)$ and by (26), $w_k(r\,I_1) = r^{2q+1-k}\, w_k(I_1)$. Combined with (27) this shows that $w_k(I_1) = 0$. Since (14) implies that $f_k(0, 0) = 0$ for $k = 1, 2$, and $f_0(0, 0) = 2\,a_0\,(s_1^2 + t_1^2)$ for $q = 0$, (22) and (23) follow from (26).

Equations (20) and (21) combined with (22) and (23) imply that (15) and (16) hold for $s, t > 0$ suitably small. Since every $[I_1, s\,u, t\,v]$ is the union of $Q_1, Q_2 \in \mathcal{Q}_o^2$ with $Q_1 \cap Q_2 = [I_1, -t'\,v, -s'\,u]$ and $s', t' > 0$ suitably small and since Z is a valuation, (14) implies that (15) and (16) hold for general $s, t > 0$. $\square$

Lemma 3. *Let $Z : \mathcal{P}_o^2 \to \mathcal{M}^2$ be a covariant valuation for which (15) holds. If $q = 1$, then $a_0 = 2\,a_1$. If $q > -1$ and $q \neq 0, 1$, then $Z(Q) = 0$ for every $Q \in \mathcal{Q}_o^2$.*

Proof. Let $T_{c,d}$ be the triangle with vertices $(1, 0)$, $(0, 1)$, $(-c, -d)$, $c, d > 0$. Then $T_{c,d} = [I_1, d\,u, v]$ where $I_1 = [-s_1, 1]$ lies on the x_1-axis, $s_1 = c/(1 + d)$, $u = (x, -1)$, $x = -c/d$, $v = (y, 1)$, $y = 0$. By (15) we have

$$\begin{aligned} z_1(T_{c,d}) &= a_1\,((\frac{c}{1+d})^{q+1} - 1)(d^{q+1} - 1) + a_0\,((\frac{c}{1+d})^q + 1)\,c\,d^{q+1}, \\ z_2(T_{c,d}) &= a_0\,((\frac{c}{1+d})^q + 1)(c^{q+2} + 1). \end{aligned} \tag{28}$$

To determine $z_0(T_{c,d})$, note that $T_{c,d} = \phi T_{d,c}$ with

$$\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By (2) this implies that

$$z_0(T_{c,d}) = z_2(T_{d,c}) = a_0\,((\frac{d}{1+c})^q + 1)(c^{q+2} + 1). \tag{29}$$

For $s > 0$ and $0 \le x < 1/2 < y \le 1$, define the triangle $T^s(x, y) \in \mathcal{P}_o^2$ as the convex hull of $(y, 1-y)$, $(x, 1-x)$, $(-s, -s)$. Then we have $T^s(x, y) = \phi T_{c,d}$ with $c = s\,(1 - 2x)/(y - x)$, $d = s\,(2y - 1)/(y - x)$, and

$$\phi = \begin{pmatrix} y & x \\ 1-y & 1-x \end{pmatrix}.$$

By (2) this implies that

$$Z(T^s(x, y)) = (y - x)^q\,\phi Z(T_{c,d})\phi^t. \tag{30}$$

Since $T^s(0, 1) = T^s(0, 1-x) \cup T^s(x, 1)$ and $T^s(x, 1-x) = T^s(0, 1-x) \cap T^s(x, 1)$ and since z_2 is a valuation, we have

$$z_2(T^s(0, 1-x)) + z_2(T^s(x, 1)) = z_2(T^s(0, 1)) + z_2(T^s(x, 1-x)). \tag{31}$$

We compare coefficients in this equation.

First, let $q > 0$. Then taking the limit as $x \to 1/2$ in (31) and using (28), (29), and (30) gives

$$2\,a_0 s^{q+2} - 2\,a_1 s^{q+1} + a_0 s^q + 2^{-(q+2)}(6\,a_0 + 4\,a_1) = a_0((\frac{s}{1+s})^q + 1)(s^{q+2} + 1).$$

Letting $s \to 0$ shows that

$$2\,a_1 = (2^{q+1} - 3)a_0. \tag{32}$$

This implies that for $q = 1$ we have $a_0 = 2\,a_1$ and that

$$a_0(s+1)^q\left(s^2 - (2^{q+1} - 3)s + 1\right) = a_0(s^{q+2} + 1).$$

If $q > 0$ and $q \neq 1$, setting $s = 1$ shows that this only holds for $a_0 = 0$. Combined with (32) and (14) this completes the proof of the lemma for $q > 0$.

Now, let $-1 < q < 0$. Then multiplying (31) by $(1-2\,x)^{-q}$ and taking the limit as $x \to 1/2$ and using (28), (29), and (30) gives

$$a_0(\frac{s}{1+2s})^q((2s)^{q+2}+1) = a_0((\frac{s}{1+s})^q+1)\,(2s^{q+2}+1)+a_1((\frac{s}{1+s})^{q+1}-1)(s^{q+1}-1).$$

Setting $s = 1$ in this equation shows that $a_0 = 0$ and this implies that also $a_1 = 0$. Combined with (14) this completes the proof of the lemma for $-1 < q < 0$. □

If $q > -1$ and $q \neq 0, 1$, then because of (14) and Lemma 2 we can apply Lemma 3 and obtain that $Z(Q) = 0$ for every $Q \in \mathcal{Q}_o^2$. Therefore we can apply Lemma 5 (stated and proved below) and Lemma 1 and obtain that $Z(P) = 0$ for every $P \in \mathcal{P}_o^2$. This proves Theorem 1 for $n = 2$ in this case.

If $q = 1$, then by Lemma 3 we have $a_0 = 2\,a_1$. For the coefficients m_{ij} of the moment matrix M_2, we obtain by an elementary calculation that

$$\begin{aligned} m_{11}([I_1, I_2]) &= \frac{1}{12}(s_1^3 + t_1^3)(s_2 + t_2), \\ m_{12}([I_1, I_2]) &= \frac{1}{24}(s_1^2 - t_1^2)(s_2^2 - t_2^2), \\ m_{22}([I_1, I_2]) &= \frac{1}{12}(s_1 + t_1)(s_2^3 + t_2^3). \end{aligned} \tag{33}$$

Thus there is a constant $c \in \mathbb{R}$ such that

$$Z(Q) = c\,M_2(Q) \tag{34}$$

for $Q \in \mathcal{Q}_o(x_2)$. Set $W = Z - c\,M_2$. Then W vanishes on $\mathcal{Q}_o^2$ and we can apply Lemma 5 and Lemma 1 and obtain that $W(P) = 0$ for every $P \in \mathcal{P}_o^2$. Thus $Z(P) = c\,M_2(P)$ for every $P \in \mathcal{P}_o^2$ and Theorem 1 is proved for $n = 2$ and $q = 1$.

1.2. We consider the case $q = 0$. By (4) we have

$$z_0([I_1, I_2]) = a_0(I_1)\,\log(\frac{t_2}{s_2}) + b_0(I_1)$$

and by (5)

$$z_k([I_1, I_2]) = a_k(I_1)\,s_2^k + b_k(I_1)\,t_2^k \quad \text{for } k = 1, 2$$

with $I_2 = [-s_2, t_2]$. The functionals $a_k, b_k : \mathcal{P}_o^1 \to \mathbb{R}$ are measurable valuations and homogeneous of degree $2 - k$. Thus, by (4) and (5) there are constants $a_k', b_k', c_k', d_k' \in \mathbb{R}$ such that

$$\begin{aligned} z_0([I_1, I_2]) &= (a_0'\,s_1^2 + b_0'\,t_1^2)\,\log(\frac{t_2}{s_2}) + (c_0'\,s_1^2 + d_0'\,t_1^2) \\ z_1([I_1, I_2]) &= (a_1'\,s_1 + b_1'\,t_1)\,s_2 + (c_1'\,s_1 + d_1'\,t_1)\,t_2 \\ z_2([I_1, I_2]) &= (a_2'\,\log(\frac{t_1}{s_1}) + b_2')\,s_2^2 + (c_2'\,\log(\frac{t_1}{s_1}) + d_2')\,t_2^2 \end{aligned}$$

for every $s_1, t_1, s_2, t_2 > 0$ with $I_1 = [-s_1, t_1]$. Using this and comparing coefficients in (12) and (13) shows that

$$\begin{aligned} z_0([I_1, I_2]) &= 2\, a_0\, (s_1^2 + t_1^2), \\ z_1([I_1, I_2]) &= a_1\, (s_1 - t_1)(s_2 - t_2), \\ z_2([I_1, I_2]) &= 2\, a_0\, (s_2^2 + t_2^2) \end{aligned} \tag{35}$$

for every $s_1, t_1, s_2, t_2 > 0$ with $a_0, a_1 \in \mathbb{R}$. This corresponds to (14) for $q = 0$. Therefore we can apply Lemma 2. The following lemma combined with Lemma 1 shows that Theorem 1 holds for $n = 2$ and $q = 0$.

Lemma 4. *Let $Z : \mathcal{P}_o^2 \to \mathcal{M}^2$ be a covariant valuation for which (15) and (16) hold. If $q = 0$, then $Z(R) = 0$ for every $R \in \mathcal{R}_o^2$.*

Proof. Let $T_{c,d}$ be the triangle with vertices $(1,0)$, $(0,1)$, $(-c,-d)$, $c, d > 0$. Then $T_{c,d} = [I_1, d\,u, v]$ where $I_1 = [-s_1, 1]$ lies on the x_1-axis, $s_1 = c/(1+d)$, $u = (x, -1)$, $x = -c/d$, $v = (y, 1)$, $y = 0$. By (15) and (16) we have

$$\begin{aligned} z_0(T_{c,d}) &= 2\, a_0\, \big((\frac{c}{1+d})^2 + 1 + c^2 \big) + 2\, a_1\, (\frac{c}{1+d} - 1)\, c, \\ z_1(T_{c,d}) &= a_1\, (\frac{c}{1+d} - 1)(d-1) + 2\, a_0\, c\, d, \\ z_2(T_{c,d}) &= 2\, a_0\, (d^2 + 1). \end{aligned}$$

We can determine $z_0(T_{c,d})$ also in the following way. Since $T_{c,d} = \phi T_{d,c}$ with

$$\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we have by (2),

$$z_0(T_{c,d}) = z_2(T_{d,c})$$

for $c, d > 0$. Comparing coefficients in this equation shows that $a_0 = a_1 = 0$. This completes the proof of the lemma. □

2. Let $n \geq 3$. We use induction on the dimension n. Suppose that Theorem 1 is true for $q > -1$ in dimension $(n-1)$.

Let $Q = [P', I] \in \mathcal{Q}_o(x_n)$ where $P' \in \mathcal{P}_o^{n-1}$ and $I = [-s, t]$, $s, t > 0$, is an interval on the x_n-axis. For I fixed, define $Z' : \mathcal{P}_o^{n-1} \to \mathcal{M}^{n-1}$ by $z'_{ij}(P') = z_{ij}([P', I])$ for $i, j = 1, \ldots, n-1$, define $z' : \mathcal{P}_o^{n-1} \to \mathbb{R}^{n-1}$ by $z'_i(P') = z_{in}([P', I])$ for $i = 1, \ldots, n-1$, and define $\mu : \mathcal{P}_o^{n-1} \to \mathbb{R}$ by $\mu(P') = z_{nn}([P', I])$. Then Z', z', and μ are measurable valuations on $\mathcal{P}_o^{n-1}$. For every $\phi' \in \mathrm{GL}(n-1)$ we have

$$\begin{aligned} Z'(\phi' P') &= |\det \phi'|^q\, \phi' Z'(P') \phi'^t, \\ z'(\phi' P') &= |\det \phi'|^q\, \phi' z'(P'), \\ \mu(\phi' P') &= |\det \phi'|^q\, \mu(P'). \end{aligned} \tag{36}$$

This can be seen in the following way. Define $\phi \in \mathrm{GL}(n)$ such that $\phi_{ij} = \phi'_{ij}$ for $i, j = 1, \dots, n-1$, $\phi_{ni} = \phi_{in} = 0$ for $i = 1, \dots, n-1$, and $\phi_{nn} = 1$. Then $\det \phi = \det \phi'$, and (2) shows that equations (36) hold.

First, let $q > -1$, $q \neq 1$. Theorem 1 for $q > -1$ in dimension $(n-1)$ implies that $Z'(P') = 0'$. Theorem 4 implies that $z'(P') = o'$ and Theorem 3 implies that $\mu(P') = 0$ for $q \neq 0$ and that $\mu(P') = a$ with $a \in \mathbb{R}$ for $q = 0$. Therefore $z_{nn}([P', I]) = a(I)$ for every $P' \in \mathcal{P}_o^{n-1}$. To determine $a(I)$, let $Q = [I_1, \dots, I_n]$, where $I_j \in \mathcal{P}_o^1$ lies on the x_j-axis, and let $\phi \in \mathrm{SL}(n)$ be the linear transformation that interchanges the first and last coordinates and leaves the other coordinates unchanged. From (2) we obtain that $z_{11}(\phi Q) = z_{nn}(Q)$ and consequently, $a(I) = 0$. Thus for $q > -1$, $q \neq 1$,

$$Z(Q) = 0 \tag{37}$$

for $Q \in \mathcal{Q}_o(x_n)$.

Now, let $q = 1$. Then Theorem 1 in dimension $(n-1)$ implies that there is a constant $c_1 \in \mathbb{R}$ such that $Z'(P') = c_1\, M_2'(P')$. Theorem 4 implies that $z'(P) = c_2\, m'(P')$ and Theorem 3 implies that $\mu(P') = c_3\, V'(P')$ with $c_2, c_3 \in \mathbb{R}$. Therefore there are measurable valuations $c_1, c_2, c_3 : \mathcal{P}_o^1 \to \mathbb{R}$ such that

$$\begin{aligned} z_{ij}([P', I]) &= c_1(I)\, m'_{ij}(P') \quad \text{for} \quad i, j = 1, \dots, n-1, \\ z_{in}([P', I]) &= c_2(I)\, m'_i(P') \quad \text{for} \quad i = 1, \dots, n-1, \\ z_{nn}([P', I]) &= c_3(I)\, V'(P'). \end{aligned}$$

Here c_1 is homogeneous of degree 1, c_2 is homogeneous of degree 2, and c_3 is homogeneous of degree 3. Therefore by (5) there are constants $a_i, b_i \in \mathbb{R}$ for $i = 1, 2, 3$ such that

$$\begin{aligned} z_{ij}([P', I]) &= (a_1\, s + b_1\, t)\, m'_{ij}(P') \quad \text{for} \quad i, j = 1, \dots, n-1, \\ z_{in}([P', I]) &= (a_2\, s^2 + b_2\, t^2)\, m'_i(P') \quad \text{for} \quad i = 1, \dots, n-1, \\ z_{nn}([P', I]) &= (a_3\, s^3 + b_3\, t^3)\, V'(P'). \end{aligned}$$

Let $\phi \in \mathrm{GL}(n)$ be the linear transformation that multiplies the last coordinate with -1 and leaves the other coordinates unchanged. Then $\phi[P', I] = [P', -I]$ and by (2) we get

$$\begin{aligned} z_{ij}([P', I]) &= a_1\, (t + s)\, m'_{ij}(P') \quad \text{for} \quad i, j = 1, \dots, n-1, \\ z_{in}([P', I]) &= a_2\, (t^2 - s^2)\, m'_i(P') \quad \text{for} \quad i = 1, \dots, n-1, \\ z_{nn}([P', I]) &= a_3\, (t^3 + s^3)\, V'(P'). \end{aligned}$$

To determine a_1, a_2, a_3, let $Q = [I_1, \dots, I_n]$ where $I_j \in \mathcal{P}_o^1$ lies on the x_j-axis, $I_1 = I_n = I$, and let ϕ be the linear transformation that interchanges the first and last coordinates and leaves the other coordinates unchanged. Then $\phi Q = Q$ and by (2)

$$z_{11}(Q) = z_{nn}(Q) \quad \text{and} \quad z_{12}(Q) = z_{2n}(Q). \tag{38}$$

We compare coefficients in these equations. Elementary calculations show that

$$\begin{aligned} m_{ij}([P',I]) &= \mathrm{B}(n+2,1)\,(t+s)\,m'_{ij}(P') \quad \text{for} \quad i,j=1,\dots,n-1,\\ m_{in}([P',I]) &= \mathrm{B}(n+1,2)\,(t^2-s^2)\,m'_i(P') \quad \text{for} \quad i=1,\dots,n-1,\\ m_{nn}([P',I]) &= \mathrm{B}(n,3)\,(t^3+s^3)\,V'(P'), \end{aligned}$$

where m_{ij} are the coefficients of the moment matrix M_2 and $\mathrm{B}(\cdot,\cdot)$ is the Beta function, and that

$$m'_i([I_1,\dots,I_{n-1}]) = \frac{1}{n!}(t_{n-1}+s_{n-1})\cdots(t_1+s_1)(t_i-s_i) \quad \text{for} \quad i=1,\dots,n-2,$$

where m'_i are the coefficients of the moment vector m'. Using this and (33), we obtain from (38) that $2\,a_1 = (n+1)\,n\,a_3$ and $a_1 = (n+1)\,a_2$. Thus there is a constant $c \in \mathbb{R}$ such that $a_1 = \mathrm{B}(n{+}2,1)\,c$, $a_2 = \mathrm{B}(n{+}1,2)\,c$, and $a_3 = \mathrm{B}(n,3)\,c$ and this shows that

$$Z(Q) = c\,M_2(Q) \tag{39}$$

for $Q \in \mathcal{Q}_o(x_n)$.

We need the following result.

Lemma 5. *Let $Z : \mathcal{P}_o^n \to \mathcal{M}^n$ be a measurable covariant valuation. If Z vanishes on $\mathcal{Q}_o^n$ and $q > -1$, then $Z = 0$ for every $R \in \mathcal{R}_o^n$.*

Proof. Let $R = [P', s\,u, t\,v] \in \mathcal{R}_o(x_n)$ where $P' \in \mathcal{P}_o^{n-1}$, $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$ and $s, t > 0$. Since Z is a valuation, we have for $0 < t < t'$ and $t'' > 0$ suitably small

$$Z([P', s\,u, t\,v]) + Z([P', -t''\,v, t'\,v]) = Z([P', s\,u, t'\,v]) + Z([P', -t''\,v, t\,v]).$$

Since $[P', -t''\,v, t'\,v], [P', -t''\,v, t\,v] \in \mathcal{Q}_o^n$ and since Z vanishes on $\mathcal{Q}_o^n$, this implies that $Z([P', s\,u, t\,v])$ does not depend on $t > 0$. A similar argument shows that it does not depend on $s > 0$. Thus

$$Z([P', s\,u, t\,v]) = Z([P', u, v]) \tag{40}$$

for $s, t > 0$.

For P' fixed, set $F(u', v') = Z([P', u, v])$. Since Z is a valuation, we have for $r > 0$ suitably small and $e = (o', 1)$

$$Z([P', u, v]) + Z([P', -r\,e, r\,e]) = Z([P', u, r\,e]) + Z([P', -r\,e, v]). \tag{41}$$

Since $[P', -r\,e, r\,e] \in \mathcal{Q}_o^n$ and since Z vanishes on $\mathcal{Q}_o^n$, we have

$$F(o', o') = Z([P', -r\,e, r\,e]) = 0.$$

Combined with (40) and (41) this implies that

$$F(u', v') = F(u', o') + F(o', v'). \tag{42}$$

Let

$$\phi = \begin{pmatrix} 1 & \dots & 0 & u_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & u_{n-1} \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then $\phi(u', -1) = (o', -1)$, $\phi(v', 1) = (u' + v', 1) = w$, and $\phi[P', u, v] = [P', -e, w]$. By (2) this implies that for the coefficients f_{ij} of F we have

$$f_{nn}(o', u' + v') = f_{nn}(u', v'), \tag{43}$$

that

$$f_{in}(o', u' + v') = f_{in}(u', v') + u_i\, f_{nn}(u', v') \tag{44}$$

for $i = 1, \dots, n-1$, and that

$$f_{ij}(o', u' + v') = f_{ij}(u', v') + f_{in}(u', v')\, u_j + u_i\, f_{jn}(u', v') + f_{nn}(u', v')\, u_i\, u_j \tag{45}$$

for $i, j = 1, \dots, n-1$. Set $g_{nn}(u') = f_{nn}(o', u')$. Then we get by (42) and (43) that

$$g_{nn}(u' + v') = g_{nn}(u') + g_{nn}(v').$$

This is Cauchy's functional equation (6). Since Z is measurable, by (7) there is a vector $w'(P') \in \mathbb{R}^{n-1}$ such that

$$g_{nn}(u' + v') = z_{nn}([P', u, v]) = w'(P') \cdot (u' + v') \tag{46}$$

for every $u', v' \in \mathbb{R}^{n-1}$.

Using this we obtain the following. By (2), f_{nn} is homogeneous of degree $n\,q + 2$. Since we know by (40) that $Z([r\,P', r\,u, r\,v]) = Z([r\,P', u, v])$ for $r > 0$, this and (46) imply that

$$w'(r\,P') = r^{n\,q+2} w'(P'). \tag{47}$$

On the other hand, let $\psi \in \mathrm{GL}(n)$ be the map that multiplies the first $(n-1)$ coordinates with r and the last coordinate with 1. Then $z_{nn}(\psi R) = r^{(n-1)q}\, z_{nn}(R)$ and by (46) this implies that

$$w'(r\,P') = r^{(n-1)q-1}\, w'(P').$$

Since $q > -1$, this combined with (47) shows that $w'(P') = o'$. Thus by (46), $z_{nn}(R) = 0$.

Using this and (44), we obtain by the same arguments as for $i = n$ that there are $w'_{(in)}(P') \in \mathbb{R}^{n-1}$ such that

$$z_{in}([P', u, v]) = w'_{(in)}(P') \cdot (u' + v')$$

for $i = 1, \dots, n-1$. As in (47) we have

$$w'_{(in)}(r\,P') = r^{n\,q+2} w'_{(in)}(P')$$

and using ψ implies that

$$w'_{(in)}(r\,P') = r^{(n-1)q}\,w'_{(in)}(P').$$

Since $q > -1$, this shows that $w'_{(in)}(P') = o'$. Thus by (46), $z_{in}(R) = 0$ for $i = 1, \dots, n-1$.

Using this and (45), we obtain by the same arguments as for $j = n$ that there are $w'_{(ij)}(P') \in \mathbb{R}^{n-1}$ such that

$$z_{ij}([P', u, v]) = w'_{(ij)}(P') \cdot (u' + v')$$

for $i, j = 1, \dots, n-1$. As in (47) we have

$$w'_{(ij)}(r\,P') = r^{n\,q+2} w'_{(ij)}(P')$$

and using ψ shows that

$$w'_{(ij)}(r\,P') = r^{(n-1)q+1}\,w'_{(ij)}(P').$$

Since $q > -1$, this shows that $w'_{(ij)}(P') = o'$. Thus by (46), $z_{ij}(R) = 0$ for $i, j = 1, \dots, n-1$. □

If $q > -1$, $q \neq 1$, then by (37) and (2) we have $Z(Q) = 0$ for every $Q \in \mathcal{Q}_o^n$. Lemma 5 and Lemma 1 therefore imply that $Z(P) = 0$ for every $P \in \mathcal{P}_o^n$. Thus Theorem 1 holds in this case.

If $q = 1$, then by (39) and (2) we have $Z(Q) - c\,M_2(Q) = 0$ for every $Q \in \mathcal{Q}_o^n$. Lemma 5 and Lemma 1 therefore imply that $Z(P) = c\,M_2(P)$ for every $P \in \mathcal{P}_o^n$. Thus Theorem 1 holds in this case.

2.2 Proof of Theorem 2 for $q \leq -1$

1.1. We begin by proving Theorem 2 for $q < -1$ and $n = 2$. The rotation by an angle $\pi/2$ is described by the matrix

$$\psi_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Set $W(P) = \psi_{\pi/2}\,Z(P)\,\psi_{\pi/2}^{-1}$. Since Z is contravariant, we obtain that

$$\begin{aligned} W(\phi P) &= \psi_{\pi/2}\,Z(\phi P)\psi_{\pi/2}^{-1} \\ &= |\det\phi|^{-q}\,\psi_{\pi/2}\,\phi^{-t}\psi_{\pi/2}^{-1}\,W(P)(\psi_{\pi/2}\,\phi^{-t}\psi_{\pi/2}^{-1})^t \\ &= |\det\phi|^{-q-2}\,\phi W(P)\phi^t \end{aligned}$$

for every $\phi \in \mathrm{GL}(2)$. Therefore W is a measurable covariant valuation with $p = -q - 2$. Since $q < -1$, we have $p > -1$ and we can apply Theorem 1.

If $q < -1$, $q \neq -3$, Theorem 1 shows that $W(P) = 0$ for every $P \in \mathcal{P}_o^2$. This proves Theorem 2 for $n = 2$ in this case.

If $q = -3$, Theorem 1 shows that there is a constant $c \in \mathbb{R}$ such that $W(P) = c\, M_2(P)$ for every $P \in \mathcal{P}_o^2$. Therefore $Z(P) = c\, \psi_{\pi/2}^{-1} M_2(P)\, \psi_{\pi/2}$ and this proves Theorem 2 for $n = 2$ in this case.

1.2. We consider the case $q = -1$ and $n = 2$. We use notation and results from Section 2.1. For

$$\phi = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & r \end{pmatrix},$$

it follows from (3) that

$$z_k([r^{-1}\, I_1, r\, I_2]) = z_k([I_1, r^2\, I_2]) = z_k([r^{-2}\, I_1, I_2]) = r^{2-2k}\, z_k([I_1, I_2])$$

for $k = 0, 1, 2$. Thus $z_k([I_1, \cdot])$ is homogeneous of degree $(1 - k)$ and $z_k([\cdot, I_2])$ is homogeneous of degree $(k - 1)$. By (4) we have

$$z_1([I_1, I_2]) = a_1(I_1)\, \log(\frac{t_2}{s_2}) + b_1(I_1)$$

and by (5)

$$z_k([I_1, I_2]) = a_k(I_1)\, s_2^{1-k} + b_k(I_1)\, t_2^{1-k} \quad \text{for} \quad k = 0, 2$$

with $I_2 = [-s_2, t_2]$. The functionals $a_k, b_k : \mathcal{P}_o^1 \to \mathbb{R}$ are measurable valuations and homogeneous of degree $(1 - k)$. Thus, by (4) and (5) there are constants $a_k, b_k, c_k, d_k \in \mathbb{R}$ such that

$$\begin{aligned} z_0([I_1, I_2]) &= (a_0\, s_1^{-1} + b_0\, t_1^{-1})\, s_2 + (c_0\, s_1^{-1} + d_0\, t_1^{-1})\, t_2 \\ z_1([I_1, I_2]) &= (a_1\, \log(\frac{t_1}{s_1}) + b_1)\, \log(\frac{t_2}{s_2}) + (c_1\, \log(\frac{t_1}{s_1}) + d_1) \\ z_2([I_1, I_2]) &= (a_2\, s_1 + b_2\, t_1)\, s_2^{-1} + (c_2\, s_1 + d_2\, t_1)\, t_2^{-1} \end{aligned}$$

for every $s_1, t_1, s_2, t_2 > 0$ with $I_1 = [-s_1, t_1]$. For

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

(3) implies that

$$z_k(\phi[I_1, I_2]) = z_k([I_1, -I_2]) = (-1)^k z_k([I_1, I_2]) \tag{48}$$

and

$$z_k(\psi[I_1, I_2]) = z_k([-I_2, I_1]) = (-1)^k z_{2-k}([I_1, I_2]). \tag{49}$$

Comparing coefficients in (48) and (49) shows that

$$\begin{aligned} z_0([I_1, I_2]) &= a_0\, (s_1^{-1} + t_1^{-1})(s_2 + t_2), \\ z_1([I_1, I_2]) &= a_1\, \log \frac{t_1}{s_1} \log \frac{t_2}{s_2}, \\ z_2([I_1, I_2]) &= a_0\, (s_1 + t_1)(s_2^{-1} + t_2^{-1}) \end{aligned} \tag{50}$$

for every $s_1, t_1, s_2, t_2 > 0$.

We need the following result.

Lemma 6. *Let $Z : \mathcal{P}_o^2 \to \mathcal{M}^2$ be a measurable contravariant valuation for which (50) holds. If $a_0 = 0$, then $Z(Q) = 0$ for every $Q \in \mathcal{Q}_o^2$.*

Proof. Let $R = [I_1, s\,u, s\,v]$, where $I_1 = [-s_1, t_1]$ lies on the x_1-axis, $u = (x, -1)$, $v = (y, 1)$ with $x, y \in \mathbb{R}$, $s_1, t_1, s > 0$. First, we show that for I_1 fixed, $s, s' > 0$ suitably small, and $k = 0, 1$, we have

$$z_k([I_1, s\,u, s\,v]) = z_k([I_1, s'\,u, s'\,v]). \tag{51}$$

Since z_k is a valuation, we have for $s > 0$ suitably small, $0 < s' < s$, and $s'' > 0$ suitably large

$$z_k([I_1, s\,u, s\,v]) + z_k([I_1, -s''\,v, s'\,v]) = z_k([I_1, s\,u, s'\,v]) + z_k([I_1, -s''\,v, s\,v]). \tag{52}$$

Since $a_0 = 0$ and $[I_1, -s''\,v, s\,v] = \phi[I_1, I_2]$ with

$$\phi = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

and $I_2 = [-s'', s]$, we obtain from (3) and (50) that

$$z_0([I_1, -s''\,v, s\,v]) = 0 \quad \text{and} \quad z_1([I_1, -s''\,v, s\,v]) = z_1([I_1, I_2]). \tag{53}$$

Thus we get by (50), (52), and (53) that

$$z_1([I_1, s\,u, s\,v]) - z_1([I_1, s\,u, s'\,v]) = a_1 \,\log\frac{t_1}{s_1}\log\frac{s}{s'}. \tag{54}$$

Similarly, we have for $s > 0$ suitably small, $0 < s' < s$, and $s'' > 0$ suitably large

$$z_1([I_1, s\,u, s'\,v]) - z_1([I_1, s'\,u, s'\,v]) = a_1 \,\log\frac{t_1}{s_1}\log\frac{s'}{s}. \tag{55}$$

This implies that (51) holds.

Next, we show that for every $s > 0$ and $k = 0, 1$, we have

$$z_k([I_1, s\,u, s\,v]) = 0. \tag{56}$$

We start by proving (56) for $k = 0$. If $k = 1$, we use that (56) holds for $k = 0$. Set $e = (1, 0)$ and $f_k(x, y) = z_k([I_1, s\,u, s\,v])$ for $s > 0$ suitably small. Since z_k is a valuation, we have for $s, r > 0$ suitably small,

$$z_k([I_1, s\,u, s\,r\,e]) + z_k([I_1, -s\,r\,e, s\,e]) = z_k([I_1, s\,u, s\,e]) + z_k([I_1, -s\,r\,e, s\,r\,e]).$$

This combined with (50) implies that

$$\begin{aligned} z_0([I_1, s\,u, s\,r\,e]) &= f_0(x, 0), \\ z_1([I_1, s\,u, s\,r\,e]) &= f_1(x, 0) + a_1 \,\log\frac{t_1}{s_1}\log r. \end{aligned} \tag{57}$$

Similarly, we get for $s, r > 0$ suitably small, $z_0([I_1, s\,u, s\,r\,e]) = f_0(0, y)$ and

$$z_1([I_1, -s\,r\,e, s\,v]) = f_1(0, y) - a_1 \log \frac{t_1}{s_1} \log r.$$

Since we have for $s, r > 0$ suitably small,

$$z_k([I_1, s\,u, s\,v]) + z_k([I_1, -s\,r\,e, s\,r\,e]) = z_k([I_1, s\,u, s\,r\,e]) + z_k([I_1, -s\,r\,e, s\,v]),$$

these equations shows that

$$f_k(x, y) = f_k(x, 0) + f_k(0, y). \tag{58}$$

For

$$\phi = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

we have $\phi[I_1, s\,(x, -1), s\,(y, 1)] = [I_1, s\,(0, -1), s\,(x + y, 1)]$, and by (3)

$$z_k(\phi[I_1, s\,u, s\,v]) = z_k([I_1, s\,u, s\,v]).$$

This implies that

$$f_k(0, x + y) = f_k(x, y). \tag{59}$$

Setting $g_k(x) = f_k(0, x)$, it follows from (58) and (59) that

$$g_k(x + y) = g_k(x) + g_k(y).$$

This is Cauchy's functional equation (6). Since Z is measurable, so is g and by (7) there is a constant $w_k(I_1) \in \mathbb{R}$ such that

$$g_k(x) = f_k(0, x) = w_k(I_1)\, x.$$

Thus

$$z_k([I_1, s\,u, s\,v]) = w_k(I_1)(x + y). \tag{60}$$

Using this we obtain the following. By (3) z_k is homogeneous of degree 0. Therefore (60) implies that for $r > 0$

$$w_k(r\, I_1) = w_k(I_1). \tag{61}$$

On the other hand, for

$$\phi = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$$

we have $z_k(\phi R) = r^{k-1}\, z_k(R)$ and by (60),

$$w_k(r\, I_1) = r^{k-2}\, w_k(I_1).$$

Combined with (61) this shows that $w_k(I_1) = 0$. This proves (56) for $s > 0$ suitably small. Since every $[I_1, s\,u, s\,v]$ is the union of $Q_1, Q_2 \in \mathcal{Q}_o^2$ with $Q_1 \cap Q_2 = [I_1, -s'\,v, -s'\,u]$ and $s' > 0$ suitably small and since z_k is a valuation, this and (50) imply that (56) hold for general $s > 0$. Note that it follows from (56), (57), and (50) that $s, r > 0$

$$\begin{aligned} z_0([I_1, s\,u, s\,r\,e]) &= 0, \\ z_1([I_1, s\,u, s\,r\,e]) &= a_1 \,\log \frac{t_1}{s_1} \log r. \end{aligned} \tag{62}$$

Let $T_{c,d}$ be the triangle with vertices $(1,0)$, $(0,1)$, $(-c,-d)$, $c, d > 0$. Then $T_{c,d} = [I_1, d\,u, v]$ where $I_1 = [-s_1, 1]$ lies on the x_1-axis, $s_1 = c/(1+d)$, $u = (x, -1)$, $x = -c/d$, $v = (y, 1)$, $y = 0$. By (62) we have

$$\begin{aligned} z_0(T_{c,d}) &= 0 \\ z_1(T_{c,d}) &= a_1 \,\log \frac{1+d}{c} \,\log \frac{1}{d}. \end{aligned} \tag{63}$$

To determine $z_2(T_{c,d})$, note that $T_{c,d} = \phi T_{d,c}$ with

$$\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By (3) this implies that

$$z_2(T_{c,d}) = z_0(T_{d,c}) = 0. \tag{64}$$

Define the triangle $T^s(x,y)$ as the convex hull of $(y, 1-y)$, $(x, 1-x)$, $(-s,-s)$. Then we have $T^s(x,y) = \phi T_{c,d}$ with $c = s\,(1-2x)/(y-x)$, $d = s\,(2y-1)/(y-x)$, and

$$\phi = \begin{pmatrix} y & x \\ 1-y & 1-x \end{pmatrix}.$$

By (3) this implies that

$$Z(T^s(x,y)) = (y-x)\,\phi^{-t} Z(T_{c,d})\phi^{-1}. \tag{65}$$

Since $T^s(0,1) = T^s(0,1-x) \cup T^s(x,1)$ and $T^s(x,1-x) = T^s(0,1-x) \cap T^s(x,1)$ and since z_2 is a valuation, we have

$$z_2(T^s(0,y)) + z_2(T^s(x,1)) = z_2(T^s(0,1)) + z_2(T^s(x,y)). \tag{66}$$

A simple calculation using (50), (63), (64), and (65) gives that $z_2(T^s(0,y)) = z_2(T^s(0,1)) = 0$ and

$$a_1 \,\frac{x}{1-x} \,\log \frac{1-x+s}{s(1-2x)} \,\log \frac{1-x}{s} = a_1 \,\frac{x\,y}{y-x} \,\log \frac{y-x+s(2y-1)}{s(1-2x)} \,\log \frac{y-x}{s(2y-1)}.$$

Setting $s = 1-x$ shows that this implies that $a_1 = 0$. This completes the proof of the lemma. □

For the coefficients l_{ij} of the matrix M_{-2}, an elementary calculation shows that

$$\begin{aligned} l_{11}([I_1, I_2]) &= (s_1^{-1} + t_1^{-1})(s_2 + t_2), \\ l_{12}([I_1, I_2]) &= 0, \\ l_{22}([I_1, I_2]) &= (s_1 + t_1)(s_2^{-1} + t_2^{-1}). \end{aligned} \tag{67}$$

We apply Lemma 6 to $W(P) = Z(P) - a_0\, M_{-2}(P^*)$ and obtain that

$$Z(Q) = c\, M_{-2}(Q^*)$$

for $Q \in \mathcal{Q}_o(x_n)$. By Lemma 7 and Lemma 1 we obtain that $Z(P) = c\, M_{-2}(P^*)$ for every $P \in \mathcal{P}_o^2$. This proves Theorem 1 for $n = 2$ and $q = -1$.

2. Let $n \geq 3$. We use induction on the dimension n. Suppose that Theorem 2 is true for $q \leq -1$ in dimension $(n-1)$.

Let $Q = [P', I] \in \mathcal{Q}_o(x_n)$ where $P' \in \mathcal{P}_o^{n-1}$ and $I = [-s, t]$, $s, t > 0$, is an interval on the x_n-axis. For I fixed, define $Z' : \mathcal{P}_o^{n-1} \to \mathcal{M}^{n-1}$ by $z'_{ij}(P') = z_{ij}([P', I])$ for $i, j = 1, \dots, n-1$, define $z' : \mathcal{P}_o^{n-1} \to \mathbb{R}^{n-1}$ by $z'_i(P') = z_{in}([P', I])$ for $i = 1, \dots, n-1$, and define $\mu : \mathcal{P}_o^{n-1} \to \mathbb{R}$ by $\mu(P') = z_{nn}([P', I])$. Then Z', z', and μ are measurable valuations on $\mathcal{P}_o^{n-1}$. For every $\phi' \in \mathrm{GL}(n-1)$ we have

$$\begin{aligned} Z'(\phi' P') &= |\det \phi'^{-t}|^q\, \phi'^{-t} Z'(P') \phi'^{-1}, \\ z'(\phi' P') &= |\det \phi'^{-t}|^q\, \phi'^{-t} z'(P'), \\ \mu(\phi' P') &= |\det \phi'^{-t}|^q\, \mu(P'). \end{aligned} \tag{68}$$

This can be seen in the following way. Define $\phi \in \mathrm{GL}(n)$ such that $\phi_{ij} = \phi'_{ij}$ for $i, j = 1, \dots, n-1$, $\phi_{ni} = \phi_{in} = 0$ for $i = 1, \dots, n-1$, and $\phi_{nn} = 1$. Then $\det \phi = \det \phi'$, and (3) shows that equations (68) hold.

First, let $q < -1$, $q \neq -2, -3$. Then Theorem 2 for $q < -1$ in dimension $(n-1)$ implies that $Z'(P') = 0'$. Theorem 5 implies that $z'(P') = o'$ and Theorem 3 implies that $\mu(P') = 0$. Thus we have for $q < -1$, $q \neq -2, -3$

$$Z(Q) = 0 \tag{69}$$

for $Q \in \mathcal{Q}_o(x_n)$.

Let $q = -2$. If $n = 3$, then $Z'(P') = 0'$, $z'(P') = c\, \psi_{\pi/2}^{-1} m(P')$ and $\mu(P') = 0$. To determine c, we take $Q = [I_1, I_2, I_3]$, where $I_j \in \mathcal{P}_o^1$ lies on the x_j-axis, and transformations $\phi, \psi \in \mathrm{SL}(3)$ that interchange the first and last coordinates and the second and third coordinates, respectively. From (3) we obtain that $c = 0$. The same argument as for $q \neq -2, -3$ now implies that (69) holds for $q = -2$, $n \geq 3$.

Let $q = -3$. If $n = 3$, then $Z'(P') = c\, \psi_{\pi/2}^{-1}\, M_2(P)\, \psi_{\pi/2}$, $z'(P') = o'$ and $\mu(P') = 0$. To determine c, let Q, ϕ, and ψ be as in the case $q = -2$. From (3) we obtain that $c = 0$. The same argument as for $q \neq -2, -3$ now implies that (69) holds for $q = -3$, $n \geq 3$.

Now, let $q=-1$. Then Theorem 2 in dimension $(n-1)$ implies that there is a constant $c_1 \in \mathbb{R}$ such that $Z'(P') = c_1 M'_{-2}(P')$. Theorem 5 implies that $z'(P) = o$ and Theorem 3 implies that $\mu(P') = c_2 V'(P')$ with $c_2 \in \mathbb{R}$. Therefore there are measurable valuations $c_1, c_2 : \mathcal{P}_o^1 \to \mathbb{R}$ such that

$$\begin{aligned} z_{ij}([P',I]) &= c_1(I)\, l'_{ij}(P') \quad \text{for} \quad i,j = 1,\dots,n-1, \\ z_{in}([P',I]) &= 0 \quad \text{for} \quad i = 1,\dots,n-1, \\ z_{nn}([P',I]) &= c_2(I)\, V'(P'). \end{aligned}$$

Here c_1 is homogeneous of degree 1 and c_2 is homogeneous of degree -1. By (5) there are constants $a_i, b_i \in \mathbb{R}$ for $i = 1,2$ such that

$$\begin{aligned} z_{ij}([P',I]) &= (a_1\, s + b_1\, t)\, l'_{ij}(P') \quad \text{for} \quad i,j = 1,\dots,n-1, \\ z_{nn}([P',I]) &= (a_2\, s^{-1} + b_2\, t^{-1})\, V'(P'). \end{aligned}$$

Let $\phi \in \mathrm{GL}(n)$ be the linear transformation that multiplies the last coordinate with -1 and leaves the other coordinates unchanged. Then $\phi[P',I] = [P',-I]$ and by (3) we get

$$\begin{aligned} z_{ij}([P',I]) &= a_1\,(t+s)\, l'_{ij}(P') \quad \text{for} \quad i,j = 1,\dots,n-1, \\ z_{nn}([P',I]) &= a_2\,(t^{-1} + s^{-1})\, V'(P'). \end{aligned}$$

To determine a_1, a_2, let $Q = [I_1, \dots, I_n]$ where $I_j \in \mathcal{P}_o^1$ lies on the x_j-axis, $I_1 = I_n = I$, and let ϕ be the linear transformation that interchanges the first and last coordinates and leaves the other coordinates unchanged. Then $\phi Q = Q$ and by (3)

$$z_{11}(Q) = z_{nn}(Q). \tag{70}$$

We compare coefficients in this equation. An elementary calculation shows that

$$\begin{aligned} l_{ij}([P',I]) &= \frac{1}{n-1}(t+s)\, l'_{ij}(P') \quad \text{for} \quad i,j = 1,\dots,n-1, \\ l_{in}([P',I]) &= 0 \quad \text{for} \quad i = 1,\dots,n-1, \\ l_{nn}([P',I]) &= (t^{-1} + s^{-1})\, V'(P'), \end{aligned}$$

where l_{ij} are the coefficients of the matrix M_{-2}. Therefore it follows from (70) that $a_1 = 1/(n-1)\, a_2$, and this shows that

$$Z(Q) = c\, M_{-2}(Q) \tag{71}$$

for $Q \in \mathcal{Q}_o(x_n)$.

We need the following result.

Lemma 7. *Let $Z : \mathcal{P}_o^n \to \mathcal{M}^n$ be a measurable contravariant valuation. If Z vanishes on $\mathcal{Q}_o^n$ and $q \le -1$, then $Z = 0$ for every $R \in \mathcal{R}_o^n$.*

Proof. Let $R = [P', s\,u, t\,v] \in \mathcal{R}_o(x_n)$ where $P' \in \mathcal{P}_o^{n-1}$, $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$ and $s, t > 0$. Since Z is a valuation, we have for $0 < t < t'$ and $t'' > 0$ suitably small

$$Z([P', s\,u, t\,v]) + Z([P', -t''\,v, t'\,v]) = Z([P', s\,u, t'\,v]) + Z([P', -t''\,v, t\,v]).$$

Since $[P', -t''\,v, t'\,v], [P', -t''\,v, t\,v] \in \mathcal{Q}_o^n$ and since Z vanishes on $\mathcal{Q}_o^n$, this implies that $Z([P', s\,u, t\,v])$ does not depend on $t > 0$. A similar argument shows that it does not depend on $s > 0$. Thus

$$Z([P', s\,u, t\,v]) = Z([P', u, v]) \tag{72}$$

for $s, t > 0$.

For P' fixed, set $F(u', v') = Z([P', u, v])$. Since Z is a valuation, we have for $r > 0$ suitably small and $e = (o', 1)$

$$Z([P', u, v]) + Z([P', -r\,e, r\,e]) = Z([P', u, r\,e]) + Z([P', -r\,e, v]). \tag{73}$$

Since $[P', -r\,e, r\,e] \in \mathcal{Q}_o^n$ and since Z vanishes on $\mathcal{Q}_o^n$, we have

$$F(o', o') = Z([P', -r\,e, r\,e]) = 0.$$

This combined with (72) and (73) gives

$$F(u', v') = F(u', o') + F(o', v'). \tag{74}$$

Let

$$\phi = \begin{pmatrix} 1 & \dots & 0 & u_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & u_{n-1} \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then $\phi(u', -1) = (o', -1)$, $\phi(v', 1) = (u' + v', 1) = w$, and $\phi[P', u, v] = [P', -e, w]$. Since (3) holds, this implies that for the coefficients f_{ij} of F we have

$$f_{ij}(o', u' + v') = f_{ij}(u', v') \tag{75}$$

for $i, j = 1, \dots, n-1$, that

$$f_{in}(o', u' + v') = -F'(u', v')\,u' + f_{in}(u', v') \tag{76}$$

for $i = 1, \dots, n-1$, and that

$$f_{nn}(o', u' + v') = u' \cdot F'(u', v')\,u' - 2\,f'(u', v') \cdot u' f_{nn}(u', v'), \tag{77}$$

where F' is the $(n-1) \times (n-1)$ matrix with coefficient f_{ij} for $i, j = 1, \dots, n-1$ and f' is the vector with coefficients f_{in} for $i = 1, \dots, n-1$. Set $g_{ij}(u') = f_{ij}(o', u')$. Then we get by (74) and (75) that

$$g_{ij}(u' + v') = g_{ij}(u') + g_{ij}(v').$$

These are functional equations of Cauchy's type. Since Z is measurable, by (7) there are vectors $w'_{(ij)}(P') \in \mathbb{R}^{n-1}$ such that

$$g_{ij}(u'+v') = z_{ij}([P',u,v]) = w'_{(ij)}(P')\cdot(u'+v') \tag{78}$$

for every $u',v' \in \mathbb{R}^{n-1}$.

Using this we obtain the following. By (3), z_{ij} is homogeneous of degree $-(n\,q+2)$. Since we know by (72) that $Z([r\,P', r\,u, r\,v]) = Z([r\,P', u, v])$ for $r>0$, this and (78) imply that

$$w'_{(ij)}(r\,P') = r^{-(n\,q+2)} w'_{(ij)}(P'). \tag{79}$$

On the other hand, let $\psi \in \mathrm{GL}(n)$ be the map that multiplies the first $(n-1)$ coordinates with r and the last coordinate with 1. Then we have $z_{ij}(\psi R) = r^{-((n-1)q+2)}\, z_{ij}(R)$ and by (78) this implies that

$$w'_{(ij)}(r\,P') = r^{-((n-1)q+3)}\, w'_{(ij)}(P').$$

Since $q \le -1$, this combined with (79) shows that $w'_{(ij)}(P') = o'$. Thus by (78), $z_{ij}(R) = 0$ for $i,j = 1,\dots,n-1$.

Using this and (76), we obtain by the same arguments as for $i,j = 1,\dots,n-1$ that there are $w'_{(in)}(P') \in \mathbb{R}^{n-1}$ such that

$$z_{in}([P',u,v]) = w'_{(in)}(P')\cdot(u'+v')$$

for $i = 1,\dots,n-1$. As in (79) we have

$$w'_{(in)}(r\,P') = r^{-(n\,q+2)} w'_{(in)}(P')$$

and using ψ shows that

$$w'_{(in)}(r\,P') = r^{-((n-1)q+2)}\, w'_{(in)}(P').$$

Since $q < -1$, this shows that $w'_{(in)}(P') = o'$. Thus by (78), $z_{in}(R) = 0$ for $i = 1,\dots,n-1$.

Using this and (77), we obtain by the same arguments as for $j = n$ that there is a $w'_{(nn)}(P') \in \mathbb{R}^{n-1}$ such that

$$z_{nn}([P',u,v]) = w'_{(nn)}(P')\cdot(u'+v').$$

The functional $w'_{(nn)} : \mathcal{P}_o^{n-1} \to \mathbb{R}^{n-1}$ is a measurable valuation. For $\phi' \in \mathrm{GL}(n-1)$, define $\phi \in \mathrm{GL}(n)$ such that $\phi_{ij} = \phi'_{ij}$ for $i,j = 1,\dots,n-1$, $\phi_{ni} = \phi_{in} = 0$ for $i = 1,\dots,n-1$, and $\phi_{nn} = 1$. By (3),

$$z_{nn}([\phi' P', \phi u, \phi v]) = |\det \phi'^{-t}|^q z_{nn}([P',u,v]).$$

Thus

$$w'_{(nn)}(\phi' P') = |\det \phi'^{-t}|^q \phi'^{-t} w'_{(nn)}(P')$$

and for $q \le -1$ we obtain from Theorem 5 that $w'_{(nn)}(P') = o$. Thus $z_{nn}(R) = 0$ and the lemma is proved. □

If $q < -1$, then by (69) and (3) we have $Z(Q) = 0$ for every $Q \in \mathcal{Q}_o^n$. Lemma 7 and Lemma 1 therefore imply that $Z(P) = 0$ for every $P \in \mathcal{P}_o^n$. Thus Theorem 2 holds in this case.

If $q = -1$, then by (71) and (3) we have $Z(Q) - c\,M_{-2}(Q) = 0$ for every $Q \in \mathcal{Q}_o^n$. Lemma 7 and Lemma 1 therefore imply that $Z(P) = c\,M_{-2}(P)$ for every $P \in \mathcal{P}_o^n$. Thus Theorem 2 holds in this case.

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