Almost all homogeneous Borel spaces are semifilters

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All spaces are assumed to be separable and metrizable.

What do you mean by "homogeneous"?

Definition

A space X is *homogeneous* if for every pair (x, y) of points of X there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$.

Every topological group is homogeneous. The spaces $\mathbb{S}$ and $\mathbb{T}$ (which I might never define) are homogeneous spaces that cannot be given a topological group structure.

What do you mean by "Almost"?

Our results will not work for locally compact spaces.
However, this case is trivial. In fact, it is easy to see that the following conditions are equivalent.

- X is a zero-dimensional locally compact homogeneous space.
- Either X is discrete, $X \approx 2^\omega$, or $X \approx \omega \times 2^\omega$.

What do you mean by “Borel”?

Definition

A space is an *absolute Borel set* (or simply *Borel*) if it is homeomorphic to a Borel subspace of a Polish space.

The following reassuring proposition can be easily proved using Lavrentiev’s Lemma.

Proposition

Let X be a space. Then the following are equivalent.

- *X is Borel.*
- *For every Polish space Z, every homeomorphic copy of X inside Z is a Borel subset of Z.*

What do you mean by "semifilter"?

Definition

A *semifilter* is a collection $\mathcal{S}$ of subsets of ω that satisfies the following conditions.

1. $\varnothing \notin \mathcal{S}$ and $\omega \in \mathcal{S}$.
2. If $X \in \mathcal{S}$ and $X =^* Y \subseteq \omega$ then $Y \in \mathcal{S}$.
3. If $X \in \mathcal{S}$ and $X \subseteq Y \subseteq \omega$ then $Y \in \mathcal{S}$.

Notice that $\mathrm{Fin} \cap \mathcal{S} = \varnothing$ and $\mathrm{Cof} \subseteq \mathcal{S}$ for every semifilter $\mathcal{S}$. In particular, no semifilter is locally compact.

Definition

A *filter* is a semifilter $\mathcal{S}$ that satisfies the following additional property.

4. If $X, Y \in \mathcal{S}$ then $X \cap Y \in \mathcal{S}$.

A characterization of Borel filters

Notice that every collection $\mathcal{X} \subseteq \mathcal{P}(\omega)$ can be identified with the subspace X of 2^ω consisting of the characteristic functions of elements of $\mathcal{X}$.

In particular, this applies to filters. Given a space, is it possible to recognize whether it is homeomorphic to a filter? This problem has a very elegant solution for definable spaces.

Theorem (van Engelen, 1994)

Let X be a zero-dimensional Borel space. Then the following conditions are equivalent.

- *X is homeomorphic to a filter.*
- *X is homogeneous, meager, homeomorphic to its square, and not locally compact.*

The above characterization inspired the following result.

Theorem (Medini and Zdomskyy, 2016)

Every filter is homeomorphic to its square.

What about semifilters?

Theorem

Let X be a zero-dimensional Borel space. Then the following conditions are equivalent.

- *X is homeomorphic to a semifilter.*
- *X is homogeneous and not locally compact.*

Proposition

There exists a homogeneous, zero-dimensional and not locally compact space X that is not homeomorphic to a semifilter. Furthermore, if $\mathsf{MA} + \neg\mathsf{CH} + \omega_1 = \omega_1^{\mathsf{L}}$ holds then X is coanalytic.

Proof: the "easy" direction

Filters are homogeneous because they are topological groups.
For semifilters, we need to work harder...

Theorem (van Mill, 1982)

Let X be a zero-dimensional metric space. Fix $x, y \in X$. Assume that, for every $\varepsilon > 0$, there exist clopen neighborhoods U, V of x, y respectively that satisfy the following conditions.

- $\mathrm{diam}(U) < \varepsilon$ *and* $\mathrm{diam}(V) < \varepsilon$.
- $U \approx V$.

Then there is a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$.

Corollary

Let X be a subspace of 2^ω. If X is closed under finite modifications then X is homogeneous.

Corollary

Every semifilter is homogeneous.

Louveau's description of Borel Wadge classes

Given $A, B \subseteq 2^\omega$, recall that $A \leq_W B$ if $A = f^{-1}[B]$ for some continuous function $f : 2^\omega \longrightarrow 2^\omega$.

The *Borel Wadge classes* are those of the form

$$[B] = \{A \subseteq 2^\omega : A \leq_W B\},$$

where B is a Borel subset of 2^ω.

Given $\mathbf{\Gamma} \subseteq \mathcal{P}(2^\omega)$, define $\check{\mathbf{\Gamma}} = \{2^\omega \setminus A : A \in \mathbf{\Gamma}\}$ and $\Delta(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$.

Recall that $\mathbf{\Gamma}$ is *self-dual* if $\mathbf{\Gamma} = \check{\mathbf{\Gamma}}$.

Louveau was able to explicitly describe these classes by defining a suitable $D \subseteq {\omega_1}^\omega$, together with a Borel Wadge class $\mathbf{\Gamma}_u$ for $u \in D$.

Theorem (Louveau, 1983, for ω^ω; van Engelen, 1986, for 2^ω)

The set of non-self dual Borel Wadge classes is

$$\{\mathbf{\Gamma}_u : u \in D\} \cup \{\check{\mathbf{\Gamma}}_u : u \in D\}.$$

Van Engelen's classification: the case above Δ

In his remarkable Ph.D. thesis, Fons van Engelen classified the homogeneous zero-dimensional Borel sets. There is a crucial dividing line: the class $\Delta = \Delta(\mathrm{D}_\omega(\boldsymbol{\Sigma}^0_2))$.
For spaces of complexity higher than Δ, **Baire category and Wadge class are sufficient to uniquely identify a homogeneous zero-dimensional Borel space.** The main ingredients here are a theorem of Steel and Borel Determinacy.

Theorem (van Engelen, 1986)

Let X and Y be homogeneous Borel subspaces of 2^ω whose complexity is higher than Δ. Assume that $[X] = [Y]$.

- *If X and Y are both meager then $X \approx Y$.*
- *If X and Y are both Baire then $X \approx Y$.*

(He also showed that, for these spaces, $[X]$ does not depend on the embedding of X into 2^ω.)

From homogeneous space to semifilter: the case above Δ

Let X be a homogeneous Borel subspace of 2^ω complexity higher than Δ. First assume that X is meager. It will be enough to find a meager semifilter $\mathcal{S}$ such that $[\mathcal{S}] = [X]$.

Instead of a semifilter, we will construct a semiideal. Instead of ω, the base set of this semiideal will be $2^{<\omega}$. Given $z \in 2^\omega$, let $z^* = \{z \upharpoonright n : n \in \omega\} \in \mathcal{P}(2^{<\omega})$. Define

$$\mathcal{S}(X) = \{y \cup e : y \subseteq z^* \text{ for some } z \in X,\ e \subseteq 2^{<\omega} \text{ is finite}\}.$$

It is easy to see that $\mathcal{S}(X)$ is a semiideal. Furthermore, $\mathcal{S}(X)$ has the finite union property, hence it is meager. The proof that $[\mathcal{S}(X)] = [X]$ relies heavily on the closure properties of the classes $\mathbf{\Gamma}_u$ obtained by Louveau.

Now assume that X is Baire. Then it is possible to prove that $X \approx \mathcal{P}(2^{<\omega}) \setminus \mathcal{S}(2^\omega \setminus X)$.

Van Engelen's classification: the case below Δ

$$
\begin{tikzcd}
\mathbb{Q} \arrow[dr] & & \mathbb{Q} \times 2^\omega \arrow[dl] \\
& \omega^\omega \arrow[d] & \\
& \mathbb{Q} \times \omega^\omega \arrow[dl] \arrow[dr] & \\
\mathbb{T} \arrow[d] & & \mathbb{S} \arrow[d] \\
\mathbb{Q} \times \mathbb{T} \arrow[dr] & & \mathbb{Q} \times \mathbb{S} \arrow[dl] \\
& \dots &
\end{tikzcd}
$$

From homogeneous space to semifilter: the case below Δ

This proof is by induction (of length ω).

Base case: $\mathbb{Q}$ and $\mathbb{Q} \times 2^\omega$ are homeomorphic to a semifilter (actually, to a filter).

- $\mathbb{Q} \approx \mathsf{Cof}$.
- Fix any infinite co-infinite $\Omega \subseteq \omega$. Then $\mathbb{Q} \times 2^\omega \approx \{x \subseteq \omega : \Omega \subseteq^* x\}$.

Inductive step: show that if one applies certain operations to a semifilter (the ones that van Engelen uses), then one obtains a space that is still homeomorphic to a semifilter.

- $\mathcal{S} \longmapsto \mathbb{Q} \times \mathcal{S}$.
- $\mathcal{S} \longmapsto 2^\omega \setminus \mathcal{S}$. (Notice that this wouldn't work for filters!)
- $\mathcal{S} \longmapsto (2^\omega \times 2^\omega) \setminus (\mathsf{Fin} \times (2^\omega \setminus \mathcal{S}))$.

Open problems

Question

In ZFC, is it possible to substitute "Borel" with "analytic" in the main theorem?

Question

Assuming Projective Determinacy, is it possible to substitute "Borel" with "projective" in the main theorem?

If one wants to use the same techniques, then it will be necessary to extend van Engelen's classification of homogeneous spaces. For this, it will be necessary to extend Louveau's description of Wadge classes. This seems to be hard... (It has been done by Fournier for differences of coanalytic sets.)
However, it might be possible to circumvent this obstacle altogether by using a more direct approach.
The problem is that I have no idea how...

Two concrete non-trivial examples: $\mathbb{S}$ and $\mathbb{T}$

Theorem (van Mill, 1983; van Douwen)

Let X be a zero-dimensional space.

- *$X \approx \mathbb{S}$ if and only if X is the union of a complete subspace and a σ-compact subspace, X is nowhere σ-compact, and X is nowhere the union of a complete and a countable subspace.*
- *$X \approx \mathbb{T}$ if and only if X is the union of a complete subspace and a countable subspace, X is nowhere σ-compact, and X is nowhere complete.*

Fix infinite sets Ω_1 and Ω_2 such that $\Omega_1 \cup \Omega_2 = \omega$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Define

$$\mathcal{T} = \{x_1 \cup x_2 : x_1 \subseteq \Omega_1,\ x_2 \subseteq \Omega_2, \text{ and}$$

$$(x_1 \notin \mathrm{Fin}(\Omega_1) \text{ or } x_2 \in \mathrm{Cof}(\Omega_2))\}.$$

It is clear that $\mathcal{T}$ is a semifilter. Furthermore, $\mathcal{T}$ is the union of the following spaces.

- $\{x \subseteq \omega : x \cap \Omega_1 \notin \mathrm{Fin}(\Omega_1)\} \approx \omega^\omega \times 2^\omega \approx \omega^\omega$.
- $\{x_1 \cup x_2 : x_1 \in \mathrm{Fin}(\Omega_1) \text{ and } x_2 \in \mathrm{Cof}(\Omega_2)\} \approx \mathbb{Q}$.

Using the fact that $\mathcal{T}$ is homogeneous, one can easily see that $\mathcal{T}$ is nowhere σ-compact and nowhere complete. Hence $\mathcal{T} \approx \mathbb{T}$.

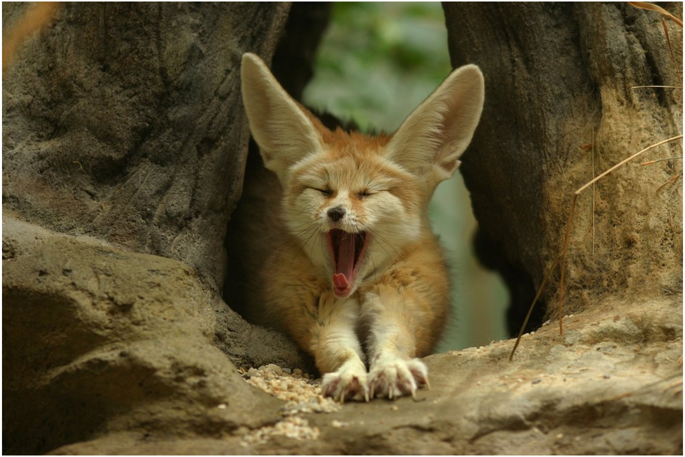
To describe $\mathbb{S}$, also fix an infinite $\Omega \subseteq \Omega_2$ such that $\Omega_2 \setminus \Omega$ is infinite. Define

$$\mathcal{S} = \{x_1 \cup x_2 : x_1 \subseteq \Omega_1,\ x_2 \subseteq \Omega_2, \text{ and}$$

$$(x_1 \notin \mathrm{Fin}(\Omega_1) \text{ or } \Omega \subseteq^* x_2)\}.$$

Using an argument similar to the one that works for $\mathcal{T}$, one can show that $\mathcal{S} \approx \mathbb{S}$.

Thank you for your attention



and good night!