Every finite-dimensional analytic space is σ-homogeneous

Andrea Medini
(joint work with Claudio Agostini)

Institut für Diskrete Mathematik und Geometrie
Technische Universität Wien

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Der Wissenschaftsfonds.

Preliminaries

All spaces are assumed to be separable and metrizable.
Since certain results involve determinacy, we will work in $\mathsf{ZF} + \mathsf{DC}$.

- A space X is *homogeneous* if for every $(x, y) \in X \times X$ there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$.
- A zero-dimensional space X is *strongly homogeneous* if all its non-empty clopen subspaces are homeomorphic.
- A space is *σ-homogeneous* if it is the union of countably many of its homogeneous subspaces.
- A space is *Borel* if it can be embedded into some Polish space as a Borel set. (Same for *analytic* and *coanalytic* spaces.)
- Declare $A \in \mathbf{\Sigma}^1_1(X)$ if there exists a Polish space $Z \supseteq X$ and $\widetilde{A} \in \mathbf{\Sigma}^1_1(Z)$ such that $A = \widetilde{A} \cap X$. (Same for $\mathbf{\Pi}^1_1(X)$ etc.)
- A space is *nowhere countable* if it is non-empty and all its non-empty open subsets are uncountable.

Exercise: every strongly homogeneous space is homogeneous.

An established pattern in set theory

Many properties $\mathcal{P}$ behave as follows:

- Every Borel set of reals satisfies $\mathcal{P}$,
- Under AD, all sets of reals satisfy $\mathcal{P}$,
- Under AC, there exist counterexamples to $\mathcal{P}$,
- Under $V = L$, there exist definable (usually coanalytic) counterexamples to $\mathcal{P}$.

The classical regularity properties ($\mathcal{P}$ = "perfect set property", $\mathcal{P}$ = "Lebesgue measurable" and $\mathcal{P}$ = "Baire property") are the most famous instances of this pattern. More entertaining examples include $\mathcal{P}$ = "not a Hamel basis" and $\mathcal{P}$ = "not an ultrafilter." See my other talk for $\mathcal{P}$ = "Effros group." This talk is about

$$\mathcal{P} = \text{“}\sigma\text{-homogeneity,”}$$

in the context of finite-dimensional spaces.

What was known about σ-homogeneity

The following is the result that sparked our interest in this subject:

Theorem (Ostrovsky, 2011)

Every zero-dimensional Borel space is σ-homogeneous.

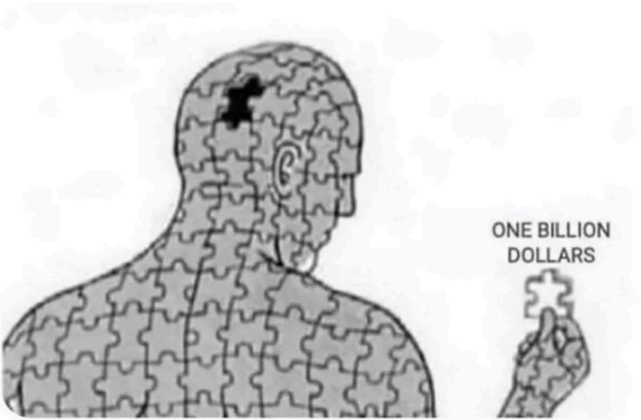
Inspired by the "established pattern," we obtained the following:

Theorem (Medini and Vidnyánszky, 2024)

- *Under* AD, *every zero-dimensional space is σ-homogeneous,*
- *Under* AC, *there exists a zero-dimensional space that is not σ-homogeneous,*
- *Under* $\mathsf{V} = \mathsf{L}$, *there exists a coanalytic zero-dimensional space that is not σ-homogeneous.*

Moreover, the positive results yield witnesses to σ-homogeneity that are closed, strongly homogeneous, and pairwise disjoint.

Sometimes all a person needs is that one missing piece



The missing piece

The following seemed to be the most pressing open question:

Question (Medini and Vidnyánszky, 2024)

Is every zero-dimensional analytic space σ-homogeneous?

The following result shows that the answer is a rather strong "yes:"

Theorem

- *Every zero-dimensional analytic space is σ-homogeneous with analytic, strongly homogeneous witnesses.*
- *Every zero-dimensional analytic space is σ-homogeneous with pairwise disjoint, $\mathbf{\Delta}^1_2$, strongly homogeneous witnesses.*

In both cases, the complexity of the witnesses is optimal.
But first, let's actually say something about the proof!

Our two fundamental tools: tool #1

The following is part of a family of results: earlier versions are due to Ostrovsky and Medvedev (also Steel, if you like Wadge theory):

Theorem (van Engelen, 1992)

Let X be a zero-dimensional space that satisfies the following conditions:

- *Every non-empty clopen subspace of X contains a closed subspace homeomorphic to X,*
- *X is either a meager space or it has a Polish dense subspace.*

Then X is strongly homogeneous.

Regarding the second assumption, the following will be relevant:

Lemma (folklore)

Let X be a Baire space. Assume that X is analytic or coanalytic. Then X has a Polish dense subspace.

Our two fundamental tools: tool #2

Recall that $A \in \mathbf{\Sigma}^1_1(2^\omega)$ is $\mathbf{\Sigma}^1_1$-*complete* if for every $B \in \mathbf{\Sigma}^1_1(2^\omega)$ there exists a continuous $f : 2^\omega \longrightarrow 2^\omega$ such that $f^{-1}[A] = B$. (In other words, A is a generator for the Wadge class $\mathbf{\Sigma}^1_1(2^\omega)$.)

Lemma (Harrington, 1980)

Let $A \in \mathbf{\Sigma}^1_1(2^\omega)$ be $\mathbf{\Sigma}^1_1$-complete. If $B \in \mathbf{\Sigma}^1_1(2^\omega)$ then there exists a continuous **injection** *$f : 2^\omega \longrightarrow 2^\omega$ such that $f^{-1}[A] = B$.*

The above lemma also holds for $\mathbf{\Pi}^1_1(2^\omega)$. In fact, it is originally stated for *reasonably closed* Wadge classes.
Its original statement also makes determinacy assumptions, but it is clear from the proof that the above determinacy-free version holds. This observation was first applied to prove the following:

Theorem (Michalewski, 2000)

$\mathcal{K}(\mathbb{Q})$ is a topological group.

Proof that every zero-dimensional analytic space is σ-homogeneous: preliminaries

By considering $X \setminus \bigcup \mathcal{U}$, where

$$\mathcal{U} = \{U : U \text{ is a countable open subset of } X\},$$

we can assume without loss of generality that X is nowhere countable. (Singletons are strongly homogeneous!)

Now consider

$$\mathcal{V} = \{V : V \text{ is an open meager subset of } X\},$$

and observe that $M = \bigcup \mathcal{V}$ is an open meager subset of X. It follows that M is a meager space, and that M is either empty or nowhere countable. Set $B = X \setminus M$. It is easy to check that B is a Baire space, and that B is either empty or nowhere countable.

In conclusion, we can assume that either X is meager or X is Baire.

Proof that every zero-dimensional analytic space is σ-homogeneous: the construction

Let $\{U_n : n \in \omega\}$ be a clopen base for X. Obtain $K_{n,i}$ for $(n, i) \in \omega \times 2$ satisfying the following conditions:

- Each $K_{n,i} \approx 2^\omega$,
- Each $K_{n,i}$ is nowhere dense in X,
- Each $K_{n,i} \subseteq U_n$,
- $K_{n,i} \cap K_{m,j} = \varnothing$ whenever $(n, i) \neq (m, j)$.

Fix a $\mathbf{\Sigma}^1_1$-complete $A_{n,i} \subseteq K_{n,i}$ for each (n, i). Define

$$X_i = \left(X \setminus \bigcup_{n \in \omega} K_{n,i} \right) \cup \left(\bigcup_{n \in \omega} A_{n,i} \right)$$

for $i \in 2$. It is clear that each X_i is analytic, and that $X = X_0 \cup X_1$. Therefore, it remains to show that each X_i is homogeneous.

Proof that every zero-dimensional analytic space is σ-homogeneous: the verification

So fix $i \in 2$. First observe that X_i is dense in X. In particular, if X is a meager space, then X_i is also a meager space. On the other hand, if X is a Baire space then X has a dense Polish subspace by the folklore lemma, hence the same is true of X_i, because the $K_{n,i}$ are closed nowhere dense in X.

Therefore, by van Engelen's Theorem, it will be enough to show that each $U_n \cap X_i$ contains a closed copy of X_i. Let X_i' be a subspace of $K_{n,i}$ that is homeomorphic to X_i. By Harrington's Lemma, we can fix a continuous injection $f : K_{n,i} \longrightarrow K_{n,i}$ such that $f^{-1}[A_{n,i}] = X_i'$.

Notice that $f : K_{n,i} \longrightarrow f[K_{n,i}]$ is a homeomorphism by compactness. Therefore $f[X_i'] = f[K_{n,i}] \cap A_{n,i}$ is a copy of X_i that is closed in $A_{n,i}$. Since $A_{n,i}$ is closed in X_i, this concludes the proof.

Pairwise disjoint witnesses

Begin by observing that

$$X \setminus X_0 = \bigcup_{n \in \omega} (K_{n,0} \setminus A_{n,0}),$$

and that $X \setminus X_0$ is a meager space. The rest of the proof is almost identical, except that Harrington's Lemma must be applied to the reasonably closed Wadge classes $\mathbf{\Pi}^1_1(K_{n,0})$.

Optimality of the complexity

In both cases (mere σ-homogeneity and σ-homogeneity with pairwise disjoint witnesses) this follows from:

Theorem (Medini and Vidnyánszky, 2024)

Under $\mathsf{V} = \mathsf{L}$, *there exists a zero-dimensionaly analytic space that is not* σ*-homogeneous with* ~~*Borel*~~ *coanalytic witnesses.*

Higher dimensions

It is a fundamental result of dimension theory that every finite-dimensional space X is homeomorphic to a subspace of $\mathbb{R}^n$ for some $n \in \omega$. Since $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both zero-dimensional and analytic, it follows that X can be written as $X = \bigcup_{k \in n} X_k$ for some $n \in \omega$, where the X_k are zero-dimensional, analytic, and pairwise disjoint.

Theorem

- *Every ~~zero-dimensional~~ finite-dimensional analytic space is σ-homogeneous with analytic, strongly homogeneous witnesses.*
- *Every ~~zero-dimensional~~ finite-dimensional analytic space is σ-homogeneous with pairwise disjoint, $\mathbf{\Delta}^1_2$, strongly homogeneous witnesses.*

The argument that we just gave was first employed by Ostrovsky to obtain the following (notice that "closed" became "G_δ"):

Theorem (Ostrovsky, 2011)

Every finite-dimensional Borel space is σ-homogeneous with pairwise disjoint G_δ witnesses.

Similarly, one obtains the following from the result under AD:

Theorem

Under AD, *every finite-dimensional space is σ-homogeneous with pairwise disjoint G_δ witnesses.*

Recall that a space is *countable-dimensional* if it is a countable union of zero-dimensional subspaces.

Question

Do the above two results hold for countable-dimensional spaces?

(They do if one drops "pairwise disjoint," by G_δ-Enlargement.)

...aaaaand now, it's propeller time!

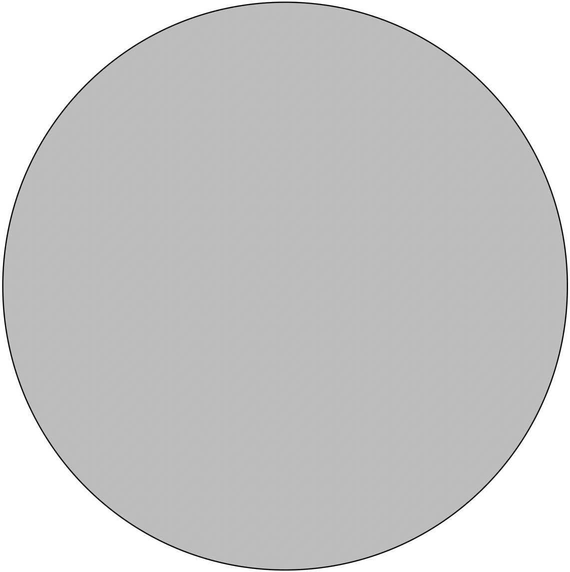
The Propeller Space

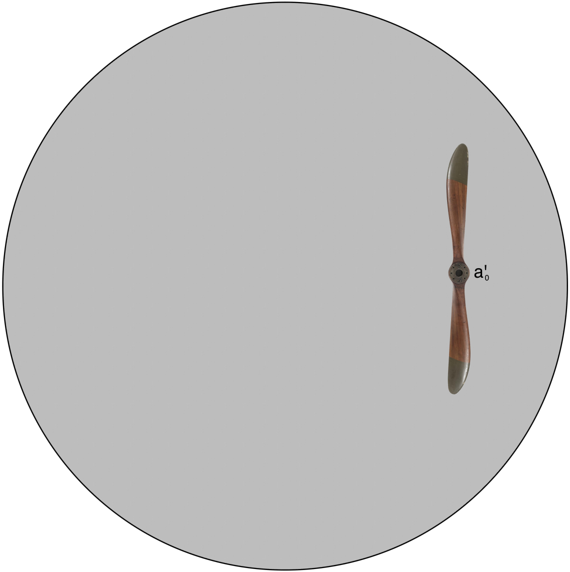
It is natural to wonder whether G_δ is the optimal complexity in the above two result. It turns out that a charming old example (the "Propeller Space" invented by de Groot and Wille in 1958) already gives us the required counterexample.

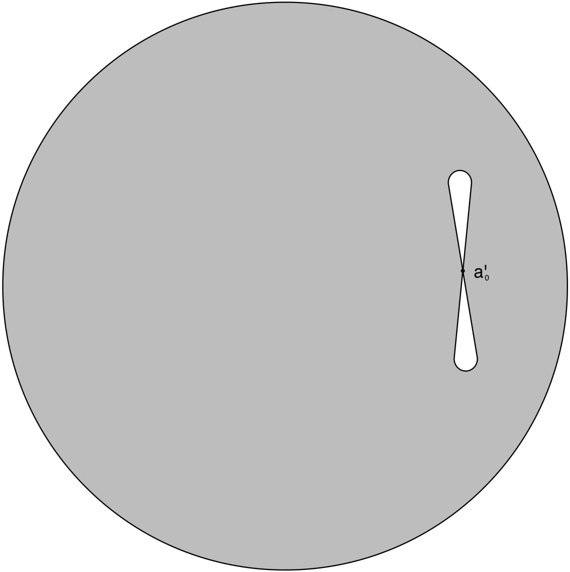
Theorem

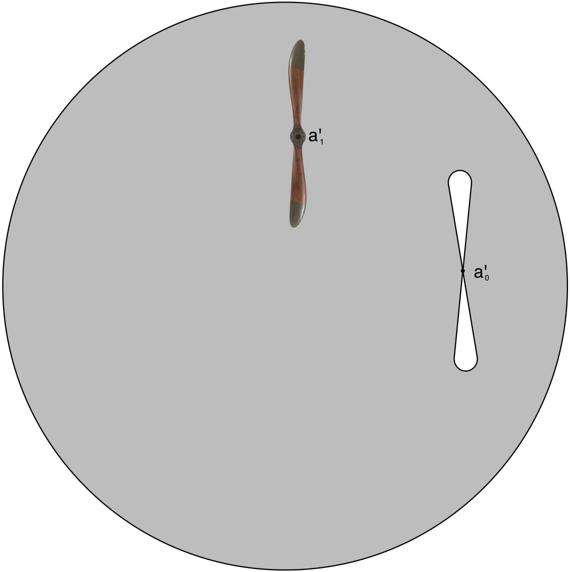
There exists a one-dimensional compact space that is not σ-homogeneous with F_σ witnesses.

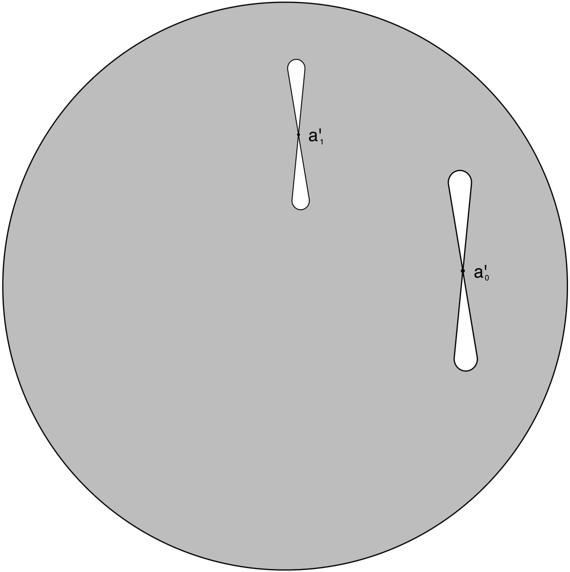
Start by considering the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.
Fix a countable dense subset $A = \{a_i : i \in \omega\} \subseteq \text{int}(D)$ of D.
At each stage, we will remove the interior of a "2-bladed propeller."
At stage i, let a'_i be the first element of A that is not in any of the propellers considered so far (so $a'_0 = a_0$). Put down the next propeller so that it is centered on a'_i, and has sufficiently small diameter (say less than 2^{-i}). Then remove its interior.
Keep going like this for ω many steps, then take the intersection.



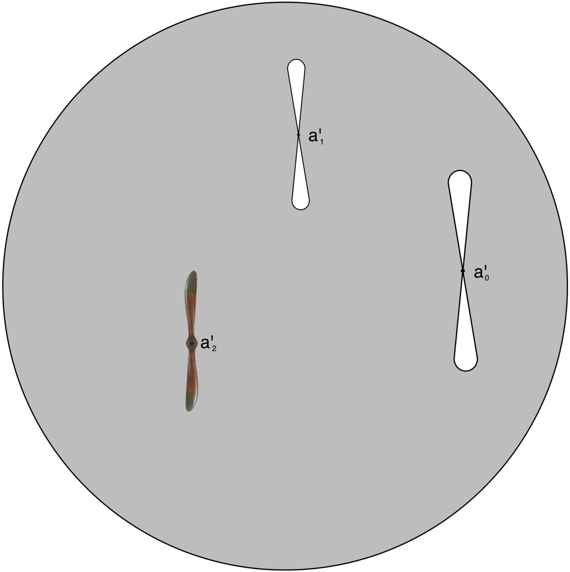

a'_0


a'_0

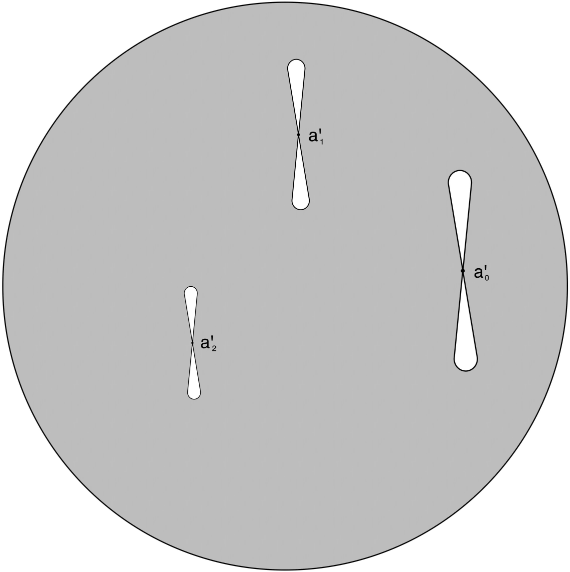

a'_1
a'_0



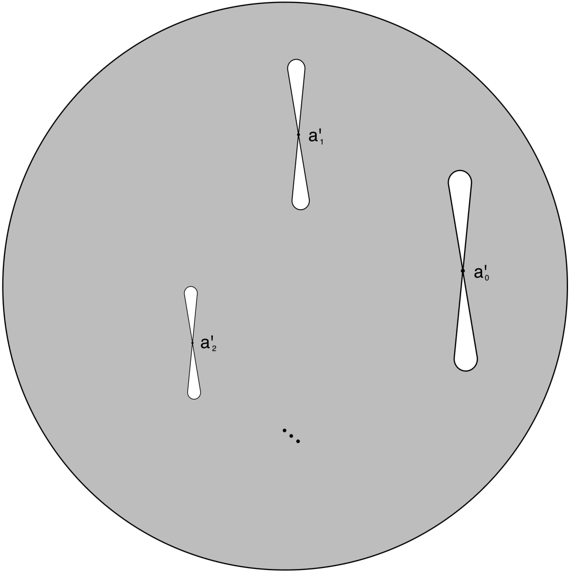
a'_1
a'_0



a'_1
a'_0
a'_2



a'_1
a'_0
a'_2



a'₁
a'₀
a'₂

The resulting space P is a continuum (compact connected) because it is the intersection of a countable decreasing sequence of continua. So P is not zero-dimensional. Since P has empty interior in $\mathbb{R}^2$, it is not two-dimensional. Hence P is one-dimensional.

The key property of P is that if $x \in P \setminus \{a'_i : i \in \omega\}$, then for every neighborhood V of x in P there exists a neighborhood $V' \subseteq V$ of x in P such that $V' \setminus \{x\}$ is connected. On the other hand, no a'_i has this property.

Assume that $P = \bigcup_{n\in\omega} P_n$, where each $P_n \in \mathbf{\Sigma}^0_2(P)$. Pick closed subsets $P_{n,k}$ of P for $(n, k) \in \omega \times \omega$ with each $P_n = \bigcup_{k\in\omega} P_{n,k}$. Since P is a Baire space, we can fix $(n, k) \in \omega \times \omega$ and a non-empty open subset U of P with $U \subseteq P_{n,k}$.

Now fix $x \in U \setminus \{a'_i : 1 \leq i < \omega\}$ and $a'_j \in U$. It is clear that there can be no homeomorphism $h : P_n \longrightarrow P_n$ such that $h(x) = a'_j$.

More open questions

We find the following question very intriguing. We would not know the answer even if we substituted "analytic" with "compact."

Question

Is every analytic space σ-homogeneous?

Question (Medini and Vidnyánszky, 2024)

In ZFC, *is there a zero-dimensional σ-homogeneous space that is not σ-homogeneous with pairwise disjoint witnesses? At least under additional set-theoretic assumptions?*

Question

In ZFC, *is there a zero-dimensional σ-homogeneous (or even homogeneous) space that is not σ-homogeneous with strongly homogeneous witnesses? At least under additional set-theoretic assumptions?*

Thank you for listening!