Countable dense homogeneity and topological groups

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Der Wissenschaftsfonds.

Preliminaries

All spaces are assumed to be separable and metrizable. Given a space X, denote by $\mathcal{H}(X)$ the set of homeomorphisms of X.

- A space X is *homogeneous* if for every $(x, y) \in X \times X$ there exists $h \in \mathcal{H}(X)$ such that $h(x) = y$.
- A space X is *countable dense homogeneous* (CDH) if for every pair (D, E) of countable dense subsets of X there exists $h \in \mathcal{H}(X)$ such that $h[D] = E$.
- A space is *h-homogeneous* if it is zero-dimensional and all its non-empty clopen subspaces are homeomorphic to each other.
- A space is *Polish* if it is (separable and) completely metrizable.
- A subset X of 2^ω is *Bernstein* if both X and $2^\omega \setminus X$ intersect every copy of 2^ω in 2^ω.

Exercise 1: every h-homogeneous space is homogeneous.
Exercise 2: every Bernstein set is a Baire space.

The fundamental theorem of CDH spaces©

Recall that a space X is *strongly locally homogeneous* if it has a base $\mathcal{B}$ such that for every $U \in \mathcal{B}$ and $(x, y) \in U \times U$ there exists $h \in \mathcal{H}(X)$ such that $h(x) = y$ and $h(z) = z$ for all $z \in X \setminus U$.

Theorem (Anderson, Curtis, van Mill, 1982)

Every strongly locally homogeneous Polish space is CDH.

The proof involves the classical back-and-forth argument, plus the Inductive Convergence Criterion for homeomorphisms.
Using this result, one sees that many familiar spaces are CDH:

- The reals $\mathbb{R}$ (Cantor, 1895), $\mathbb{R}^n$ (Brouwer, 1913),
- Every (separable metrizable) topological manifold,
- The Cantor set 2^ω, the Baire space ω^ω.

A non-interesting non-example of CDH space is given by $\mathbb{Q}$.
An interesting non-example is given by $\mathbb{Q}^\omega$. (Just wait...)

An introduction to λ-sets

Definition (Kuratowski, 1933)

A space X is a *λ-set* if every countable subset of X is G_δ.

Every countable space is a trivial example of λ-set. No λ-set can contain copies of 2^ω, hence no uncountable λ-set is Polish. Furthermore, every crowded λ-set is meager.

A classical result (that we will not need) is that there exists a λ-set (in fact, a particularly nice one) of size ω_1 in ZFC. The example in question is the Hausdorff gap (viewed as a subspace of 2^ω).

Rothberger established a strong connection between λ-sets and $\mathfrak{b}$:

Lemma (Rothberger, 1939)

Let X be a space. If $|X| < \mathfrak{b}$ then X is a λ-set.

Theorem (Rothberger, 1939)

There exists a λ-set of size $\mathfrak{b}$.

Proof of Rothberger's Lemma

Let X be a space, and assume that $|X| < \mathfrak{b}$. In particular, X is zero-dimensional, so we can assume that X is a subspace of ω^ω.

Now fix a compactification Z of ω^ω such that Z is homeomorphic to 2^ω and $Q = Z \setminus \omega^\omega$ is a countable dense subset of Z.

Pick a countable $C \subseteq X$. Let $D \supseteq C$ be a countable dense subset of Z. Since Z is CDH, we can fix $h \in \mathcal{H}(Z)$ such that $h[D] = Q$. Clearly, it will be enough to show that $h[C]$ is G_δ in $h[X]$.

Notice that $|h[X] \setminus Q| < \mathfrak{b}$, hence there exists $f \in \omega^\omega$ such that

$$h[X] \setminus Q \subseteq \bigcup_{g =^* f} \prod_{n \in \omega} g(n).$$

Since the right-hand side is σ-compact, Q is G_δ in $h[X] \cup Q$. But $Q = h[D] \supseteq h[C]$, so $h[C]$ is G_δ in $h[X] \cup Q$, so in $h[X]$.

Proof of Rothberger's Theorem (part 1/2)

Fix a cardinal $\kappa \leq \mathfrak{b}$. It is an easy exercise to find $f_\alpha \in \omega^\omega$ for $\alpha \in \kappa$ such that

$$f_\alpha <^* f_\beta \text{ whenever } \alpha < \beta.$$

Set $X = \{f_\alpha : \alpha \in \kappa\}$ and $X_\delta = \{f_\alpha : \alpha < \delta\}$ for $\delta \in \kappa$.
We will show that X is a λ-set.

In particular, for $\kappa = \mathfrak{b}$, this will yield Rothberger's Theorem.
For the sake of clarity, we will assume that $\kappa = \mathfrak{b}$.

So pick a countable $C \subseteq X$. Since $\mathfrak{b}$ has uncountable cofinality, there exists $\delta < \mathfrak{b}$ such that $C \subseteq X_\delta$. Since X_δ is a λ-set by Rothberger's Lemma, it will be enough to show that X_δ is a G_δ-subset of X.

Proof of Rothberger's Theorem (part 2/2)

Set

$$G = \{z \in \omega^\omega : z(n) < f_\delta(n) \text{ for infinitely many } n\},$$

and observe that G is a G_δ subset of ω^ω. We will conclude the proof by showing that $X_\delta = X \cap G$.

Proof of $\subseteq$. This is clear, because if $\alpha < \delta$ then

$$f_\alpha(n) < f_\delta(n) \text{ for all but finitely many } n.$$

Proof of $\supseteq$. Pick $z \in X \cap G$. In particular, $z = f_\alpha$ for some $\alpha \in \mathfrak{b}$. But if we had $\alpha \geq \delta$, then we would have $f_\alpha \geq^* f_\delta$, or in other words

$$f_\alpha(n) \not< f_\delta(n) \text{ for all but finitely many } n,$$

which contradicts $f_\alpha = z \in G$.

Why do we care about λ-sets?

Question (Fitzpatrick, Zhou, 1990)

Is there a non-Polish CDH *space in* ZFC*?*

(Examples under CH or MA were constructed by them, by Baldwin and Beaudoin, and by Gruenhage.)

Theorem (Fitzpatrick, Zhou, 1992)

Let X be a meager CDH *space. Then X is a λ-set.*

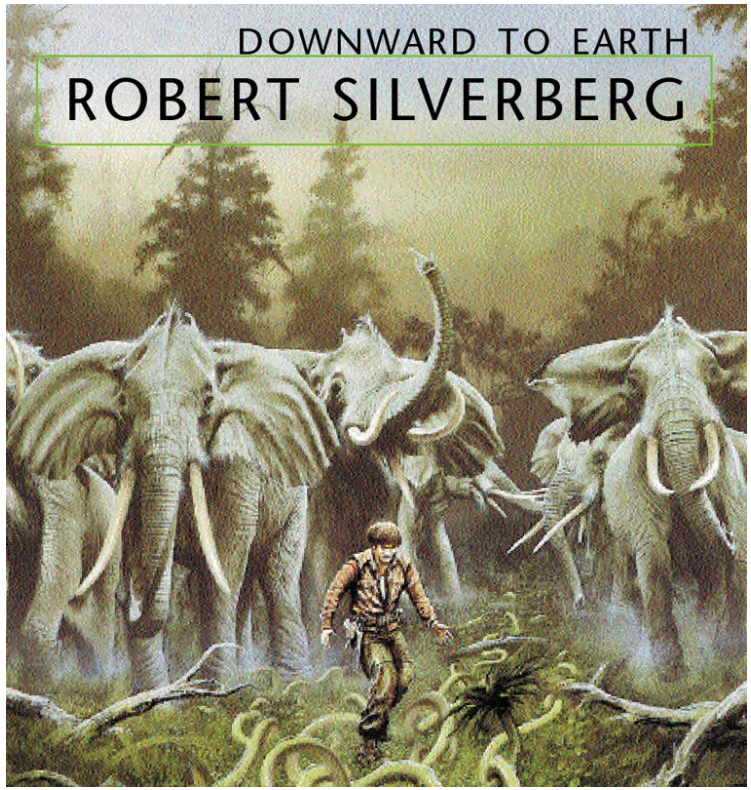
Now we finally understand why $\mathbb{Q}^\omega$ is not CDH!

The first answer to the above question employed very fancy methods (namely, an absoluteness argument involving Keisler's completeness theorem for the $L_{\omega_1\omega}(Q)$ logic).

Theorem (Farah, Hrušák, Martínez Ranero, 2005)

There exists a CDH *λ-set of size ω_1.*

Sure, being fancy is nice and all,



but is it really necessary?

Downward to Earth

In 2014, Hernández-Gutiérrez, Hrušák and van Mill gave a more "down-to-earth" ZFC construction, using the technique of Knaster-Reichbach covers. This inspired further results:

Theorem (Medini, 2015)

There exists a non-Polish X such that X^ω is CDH*.*

(In fact, the complement of the Hausdorff gap in 2^ω works.)

Theorem (Medvedev, 2015)

Every uncountable h-homogeneous λ-set is CDH*.*

It would be interesting to know if "h-homogeneous" can be weakened to "zero-dimensional and homogeneous". In fact, the following question is still open:

Question (Medvedev, 2012)

Is every zero-dimensional meager homogeneous space h-homogeneous?

Three Polish topologists

looking for a non-Polish example

Rekindling the non-Polish flame

The search for non-Polish examples has continued. The following result was motivated by a question of Tkachuk:

Theorem (Hernández-Gutiérrez, 2020)

Consistently, there exists a C_p*-space that is non-Polish and* CDH.

Question (Dobrowolski, Krupski, Marciszewski, 2021)

Is there a non-Polish CDH *topological vector space in* ZFC*?*

In one of my FWF research projects, I noticed that the following (easier) question was also open:

Question (Medini, 2021)

Is there a non-Polish CDH *topological group in* ZFC*?*

For a consistent example it is not necessary to resort to C_p-spaces. The first example was constructed in 2012 by Medini and Milovich. In fact, every non-meager P-filter (viewed as a subspace of 2^ω) will work by a 2013 result of Hernández-Gutiérrez and Hrušák.

The missing piece

Recall the result of Medvedev:

Theorem (Medvedev, 2015)

Every uncountable h-homogeneous λ-set is CDH*.*

Would it be too much to ask for a criterion for the h-homogeneity of topological groups? Actually, no.

Theorem (Medvedev, 2012)

Let X be a zero-dimensional topological group. If X is not locally precompact then X is h-homogeneous.

Even not knowing any of the definitions, it is clear that any λ-group that is not locally precompact would answer my question. (Of course, a *λ-group* is a topological group that is a λ-set.) Recall that a group is *precompact* if its completion is compact. Similarly, a group is *locally precompact* if its completion is locally compact.

The ~~missing~~ piece

Theorem (Agostini, Medini, Zdomskyy)

There exists a dense subgroup of $\mathbb{Z}^\omega$ of size $\mathfrak{b}$ that is a λ-set.

Since $\mathbb{Z}^\omega$ is not locally precompact, neither is any of its dense subgroups. Hence the theorem implies the following:

Corollary (Agostini, Medini, Zdomskyy)

There exists a non-Polish CDH *topological group.*

Proof of the theorem. Construct $f_\alpha \in \mathbb{Z}^\omega$ for $\alpha \in \mathfrak{b}$ such that:

1. $f_\alpha(n) > 0$ for each $\alpha \in \mathfrak{b}$ and $n \in \omega$,
2. $f_\alpha >^* k_0 \cdot f_{\beta_0} + \cdots + k_n \cdot f_{\beta_n}$ whenever $n \in \omega$, $k_0, \ldots, k_n \in \omega$, and $\beta_0, \ldots, \beta_n < \alpha < \mathfrak{b}$.

Observe that $f_\alpha \neq f_\beta$ whenever $\alpha \neq \beta$. Set $Q = \{z \in \mathbb{Z}^\omega : z =^* \vec{0}\}$ and

$$X = \langle \{f_\alpha : \alpha \in \mathfrak{b}\} \cup Q \rangle.$$

It is clear that X is dense in $\mathbb{Z}^\omega$ and that $|X| = \mathfrak{b}$.

Proof of the theorem: the verification

It remains to show that X is a λ-set. For any given $z \in X \setminus Q$, fix $k(z) \in \mathbb{Z} \setminus \{0\}$ and $\alpha(z) \in \mathfrak{b}$ such that

$$z =^* k(z) \cdot f_{\alpha(z)} + z'$$

for some $z' \in \langle \{f_\beta : \beta < \alpha(z)\} \rangle$. Now pick a countable $C \subseteq X$. Since $\mathfrak{b}$ has uncountable cofinality, it is possible to fix $\delta \in \mathfrak{b}$ such that $\delta > \sup\{\alpha(z) : z \in C\}$.

Claim. Let $x \in C$, and let $y \in X \setminus Q$. Assume that $\alpha(y) > \delta$. Then

$$|x| \leq^* f_\delta <^* |y|.$$

Proof of the claim. The first inequality follows easily from our choice of δ, conditions (1) and (2), and the triangle inequality.

Claim. Let $x \in C$, and let $y \in X \setminus Q$. Assume that $\alpha(y) > \delta$. Then

$$|x| \leq^* f_\delta <^* |y|.$$

Continued proof of the claim. It remains to prove the second inequality. So pick $n \in \omega$, $k_0, \ldots, k_n = k(y) \in \mathbb{Z}$ and $\alpha_0 < \cdots < \alpha_n = \alpha(y) < \mathfrak{b}$ such that

$$y =^* k_0 \cdot f_{\alpha_0} + \cdots + k_n \cdot f_{\alpha_n}.$$

Set

$$y' = f_{\alpha(y)} - \sum_{i<n} |k_i| \cdot f_{\alpha_i},$$

and observe that $y' \geq^* \vec{0}$ by condition (2).

Next, we will prove that $|y| \geq^* y'$. First assume that $k(y) > 0$. In this case, it is easy to realize that $y \geq^* y' \geq^* \vec{0}$, hence $|y| =^* y \geq^* y'$.

Claim. Let $x \in C$, and let $y \in X \setminus Q$. Assume that $\alpha(y) > \delta$. Then

$$|x| \leq^* f_\delta <^* |y|.$$

Continued proof of the claim. Now assume that $k(y) < 0$. In this case, one sees that

$$-y =^* (-k(y)) \cdot f_{\alpha(y)} - \sum_{i<n} k_i \cdot f_{\alpha_i} \geq f_{\alpha(y)} - \sum_{i<n} |k_i| \cdot f_{\alpha_i} = y' \geq^* \vec{0},$$

so $|y| =^* -y \geq^* y'$. In conclusion, we must have

$$|y| \geq^* y' = f_{\alpha(y)} - \sum_{i<n} |k_i| \cdot f_{\alpha_i} >^* f_\delta,$$

where the last inequality follows from condition (2) plus the assumption that $\alpha(y) > \delta$.

Claim. Let $x \in C$, and let $y \in X \setminus Q$. Assume that $\alpha(y) > \delta$. Then

$$|x| \leq^* f_\delta <^* |y|.$$

Conclusion of the proof of the theorem. Finally, define

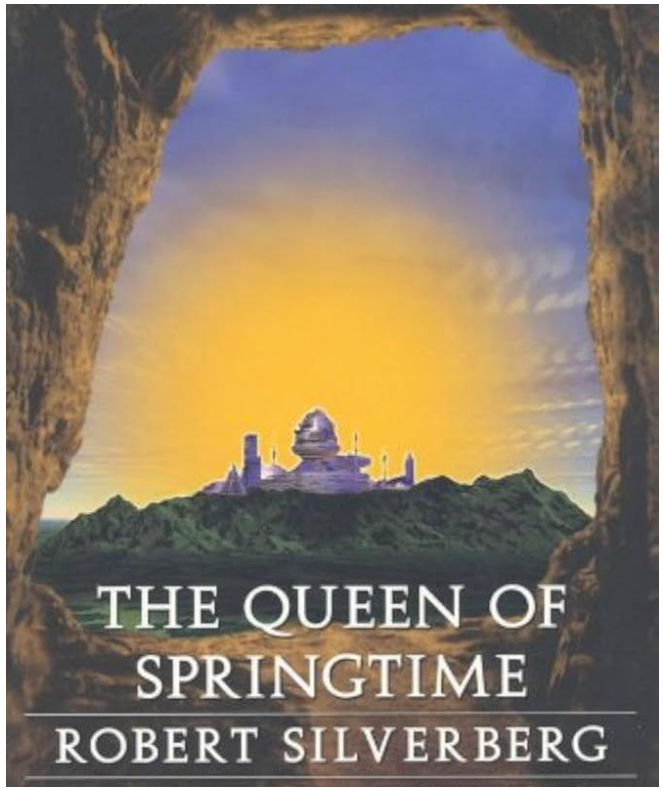
$$G = \{z \in \mathbb{Z}^\omega : |z(n)| \leq f_\delta(n) \text{ for infinitely many } n \in \omega\},$$

and observe that G is a G_δ-subset of $\mathbb{Z}^\omega$. The first inequality of the claim immediately shows that $C \subseteq X \cap G$, so it will be enough to prove that C is G_δ in $X \cap G$.

But the second inequality of the claim shows that $|X \cap G| < \mathfrak{b}$. So $X \cap G$ is a λ-set by Rothberger's Lemma, hence C is G_δ in it.

We remark that the above theorem (to the best of our knowledge) gives the first ZFC example of an uncountable λ-group.
(For a consistent example, assume that $\omega_1 < \mathfrak{b}$, consider an arbitrary group of size ω_1, and apply Rothberger's Lemma to it.)

“Every answer holds the next question,



throbbing impatiently within it”

Open questions (focusing on Baire spaces)

Theorem (Dobrowolski, Krupski, Marciszewski, 2021)

Every countable dense homogeneous topological vector space is a Baire space.

(In fact, they showed that every homogeneous CDH space containing a copy of 2^ω is Baire.) So the following can be viewed as a further "baby-step" towards answering their question:

Question (Agostini, Medini, Zdomskyy)

In ZFC*, is there a non-Polish* CDH *topological group that is Baire?*

(Again, every non-meager P-filter gives a consistent example.)

Theorem (Baldwin, Beaudoin, 1989)

Under MA, there exists a CDH *Bernstein subset of* 2^ω*.*

Question (Baldwin, Beaudoin, 1989)

Is there a CDH *Bernstein subset of* 2^ω *in* ZFC*?*

Question

Is there a CDH *Bernstein subgroup of* 2^ω*? At least under additional set-theoretic assumptions?*

Our failure inspired the following conjecture:

Conjecture (Agostini, Medini, Zdomskyy)

Every crowded Baire CDH *topological group contains a copy of* 2^ω*.*

At least, we were able to give a proof of a very special case:

Theorem (Agostini, Medini, Zdomskyy)

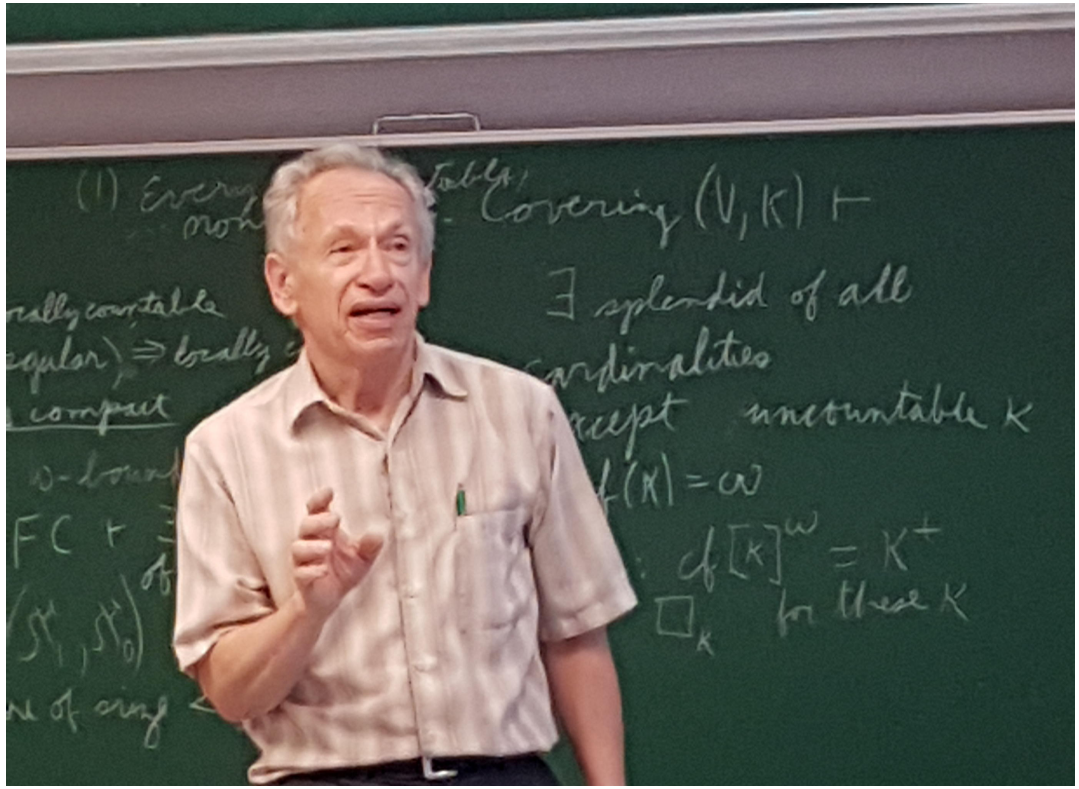
The conjecture holds for groups of index 2 *in their completion.*

Notice that the conjecture would resolve the following open problem in the context of topological groups:

Question (Hernández-Gutiérrez, Hrušák, van Mill, 2014)

Is it consistent that there exists a crowded Baire CDH *space* X *such that* $|X| < \mathfrak{c}$*?*

Dedicated to the memory of



Peter Nyikos (1946-2024)