Countable dense homogeneity in large products of Polish spaces

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Der Wissenschaftsfonds.

Preliminaries

Denote by $\mathcal{H}(X)$ the group of homeomorphisms of a space X.

- A space X is *countable dense homogeneous* (CDH) if for every pair (D, E) of countable dense subsets of X there exists $h \in \mathcal{H}(X)$ such that $h[D] = E$.
- A space is *Polish* if it is separable and completely metrizable.
- A space X is *strongly locally homogeneous* (SLH) if it has a base $\mathcal{B}$ such that for every $U \in \mathcal{B}$ and $(x, y) \in U \times U$ there exists $h \in \mathcal{H}(X)$ such that $h(x) = y$ and h is supported on U.
- Given $n \in \omega$, a space X is *strongly n-homogeneous* if for every bijection $\sigma : F \longrightarrow G$, where $F, G \subseteq X$ have size n, there exists $h \in \mathcal{H}(X)$ such that $\sigma \subseteq h$.
- A *manifold with boundary* is a separable metrizable space that is locally homeomorphic to either $\mathbb{R}^n$ or $\{x \in \mathbb{R}^n : x(0) \geq 0\}$. The simplest non-trivial example is the unit interval $[0, 1]$.

The fundamental theorem of CDH spaces©

Theorem (Anderson, Curtis and van Mill, 1982)

Every SLH *Polish space is* CDH*.*

Using this result, one sees that many familiar spaces are CDH:

- The reals $\mathbb{R}$ (Cantor, 1895), $\mathbb{R}^n$ (Brouwer, 1913),
- Every manifold (with empty boundary),
- The Cantor set 2^ω, the Baire space ω^ω.

Another example is given by the Hilbert cube $[0,1]^\omega$ (Fort, 1962).

Theorem (Steprāns and Zhou, 1988)

Let $\kappa < \mathfrak{p}$ *be an infinite cardinal. Then* 2^κ, $\mathbb{R}^\kappa$ *and* $[0,1]^\kappa$ *are* CDH*.*

However, they did not give a unified result from which all these particular cases can be deduced. We obtained such a result, not only for powers but for products as well. As a new application, we determined which products of manifolds with boundary are CDH.


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Proof of the fundamental theorem

Let D and E be countable dense subsets of X, then fix injective enumerations $D = \{d_n : n \in \omega\}$ and $E = \{e_n : n \in \omega\}$. The plan is to construct $h_n \in \mathcal{H}(X)$ for $n \in \omega$. If each h_n is sufficiently close to the identity, the Inductive Convergence Criterion will guarantee that

$$h = \lim_{n \to \infty} (h_n \circ \cdots \circ h_0)$$

exists and belongs to $\mathcal{H}(X)$. (This is where "Polish" is used.)

Start with $h_0 = \mathrm{id}$. Given $h_0, \ldots, h_{2n}$, set $x = (h_{2n} \circ \cdots \circ h_0)(d_n)$, and ensure that h_{2n+1} maps x into E (and that this will be its "final destination"). Plus, make sure not to "disturb" the finitely many points whose final destination has already been decided. This can be achieved by choosing h_{2n+1} supported on a sufficiently small neighborhood of x. (This is where "SLH" is used.)
Given $h_0, \ldots, h_{2n+1}$, do a similar thing for e_n.

A non-CDH product of two very nice spaces

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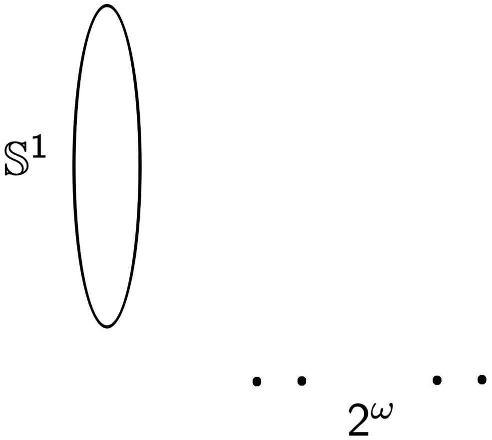
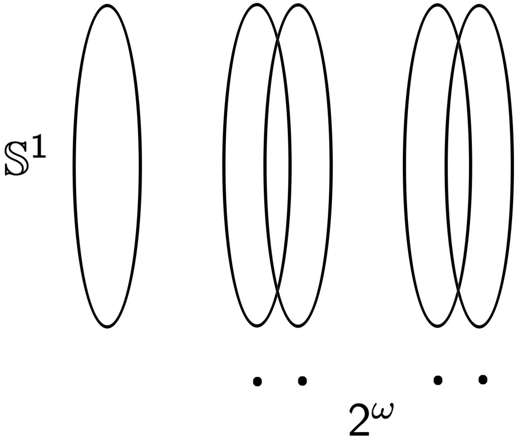


A non-CDH product of two very nice spaces



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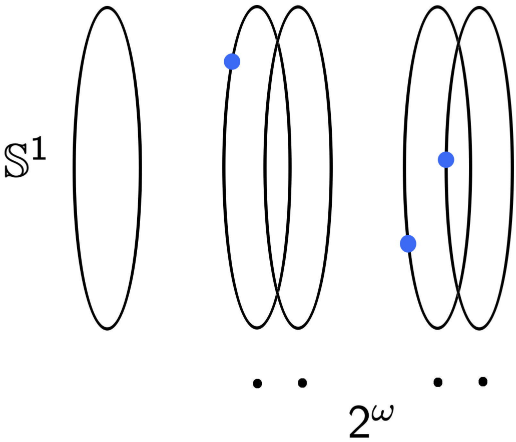
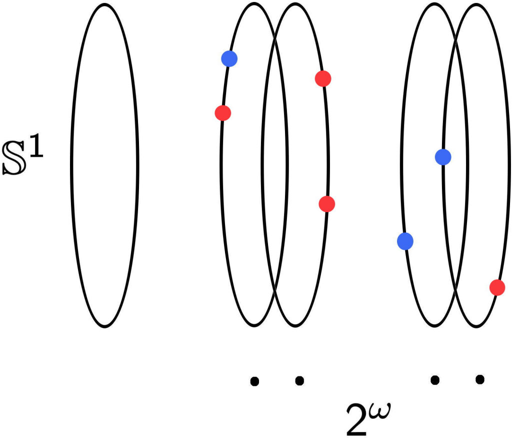


Let D be a countable dense subset of $2^\omega \times \mathbb{S}^1$ such that every vertical section contains at most one point of D.

A non-CDH product of two very nice spaces



Now let E denote a countable dense subset of $2^\omega \times \mathbb{S}^1$ that does not have this property. Since homeomorphisms permute connected components, there is no $h \in \mathcal{H}(2^\omega \times \mathbb{S}^1)$ such that $h[D] = E$.

What are we missing?

The example that we have just given shows that the most natural attempt to generalize the fundamental theorem fails.

Not-yet-a-theorem

Let $X = \prod_{\alpha \in \kappa} X_\alpha$ be a product of Polish spaces, where $\kappa < \mathfrak{p}$. Assume that the following conditions hold:

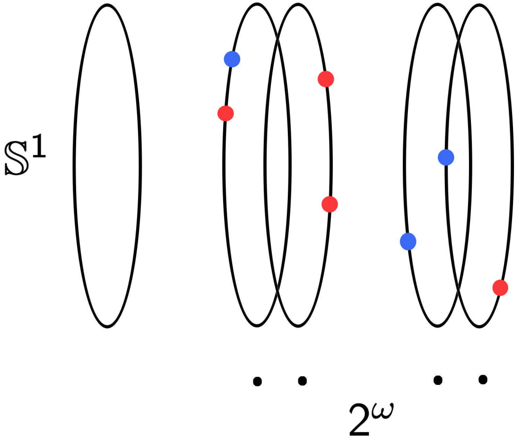
- *Each X_α is strongly locally homogeneous.*
-
-

Then X is CDH*.*

How does one construct a homeomorphism of a product?
The easiest way to do it is coordinate-wise. (Construct a homeomorphism of each factor, then take the product of these homeomorphisms.) For this strategy to be feasible, we need to talk about general position.

General position

We will say that $D \subseteq \prod_{\alpha\in\kappa} X_\alpha$ is in *general position* if all projections are injective when restricted to D. In other words, if $x, y \in D$ are distinct then $x(\alpha) \neq y(\alpha)$ for every $\alpha \in \kappa$.



For example, the set D is in general position, while E is not.

The general position property

We will say that $X = \prod_{\alpha \in \kappa} X_\alpha$ has the *general position property* if every countable set can be brought in general position. (That is, for every countable $D \subseteq X$ there exists $h \in \mathcal{H}(X)$ such that $h[D]$ is in general position.)

It is easy to realize that a product homeomorphism preserves general position. So our strategy (originally due to Steprāns and Zhou, 1988) will necessarily consist of two steps:

- Make sure that if D and E are countable dense sets in general position then there exists $h \in \mathcal{H}(X)$ such that $h[D] = E$,
- Make sure that X has the general position property.

In fact, a large portion of the paper is devoted to showing that suitable products of manifolds with boundary, as well as suitable products of topological groups, have the general position property. (These results hold for arbitrarily large products.)

What are we still missing?

Not-yet-a-theorem

Let $X = \prod_{\alpha\in\kappa} X_\alpha$ *be a product of Polish spaces, where* $\kappa < \mathfrak{p}$*. Assume that the following conditions hold:*

- *Each* X_α *is strongly locally homogeneous,*
-
- X *has the general position property.*

Then X *is* CDH.

Our proof aims to construct, for each coordinate α, a sequence $h_{\alpha,0}, h_{\alpha,1}, \ldots$ of homeomorphisms of X_α as in the proof of the 1-variable version. In the end, the desired homeomorphism will act on this coordinate as

$$h_\alpha = \lim_{n\to\infty}(h_{\alpha,n} \circ \cdots \circ h_{\alpha,0}).$$

Assume that, in the middle of our construction, we have decided:

- d_0 must be mapped to e_{23},
- d_{55} must be mapped to e_0,
- d_1 must be mapped to e_{666}.

Also assume that α is a "new" coordinate, in the sense that we have not started to construct the sequence of homeomorphisms corresponding to this coordinate. Then, the homeomorphism $h_{\alpha,0}$ will need to satisfy the following conditions:

- $d_0(\alpha)$ must be mapped to $e_{23}(\alpha)$,
- $d_{55}(\alpha)$ must be mapped to $e_0(\alpha)$,
- $d_1(\alpha)$ must be mapped to $e_{666}(\alpha)$.

In other words, it would be very convenient if X_α were strongly 3-homogeneous. (Also notice the general position of D and E will ensure that these points are distinct.)

We have all the ingredients...

It's time to cook!

We have all the ingredients...

It's time to cook!

Our main results

Theorem

Let $X = \prod_{\alpha\in\kappa} X_\alpha$ *be a product of Polish spaces, where* $\kappa < \mathfrak{p}$*. Assume that the following conditions hold:*

- *Each* X_α *is strongly locally homogeneous,*
- *Each* X_α *is strongly* n*-homogeneous for every* $n \in \omega$*,*
- X *has the general position property.*

Then X *is* CDH*.*

The above result seems new/interesting even for finite κ.

Corollary

Let $\kappa < \mathfrak{p}$ *be a cardinal, and let* X_α *be connected manifolds with boundary for* $\alpha \in \kappa$*. Assume that either all of the boundaries are empty, or infinitely many are non-empty. Then* $\prod_{\alpha\in\kappa} X_\alpha$ *is CDH.*

The countable case of the above corollary is due to Yang (1990).

(Sketch of) the proof of the main theorem

Fix countable dense subsets D and E of X, and assume without loss of generality that $D \cup E$ is in general position.
Given $\alpha \in \kappa$, denote by $\pi_\alpha : X \longrightarrow X_\alpha$ the canonical projection.
Fix countable subgroups $\mathcal{H}_\alpha$ of $\mathcal{H}(X_\alpha)$ for $\alpha \in \kappa$ that witness the homogeneity properties of X_α with respect to points of $\pi_\alpha[D \cup E]$.
Denote by $\mathbb{P}$ the set of all triples of the form $p = (F, \zeta, \sigma)$ such that the following requirements are satisfied:

- $F \in [\kappa]^{<\omega}$,
- ζ is a function such that $\mathrm{dom}(\zeta) = F$ and each $\zeta(\alpha) = (h_0, \ldots, h_n)$ is a finite sequence of elements of $\mathcal{H}_\alpha$ that satisfies the Inductive Convergence Criterion,
- σ is a finite bijection such that $\mathrm{dom}(\sigma) \subseteq D$ and $\mathrm{ran}(\sigma) \subseteq E$,
- $(h_n \circ \cdots \circ h_0)\big(d(\alpha)\big) = \sigma(d)(\alpha)$ for every $\alpha \in F$ and $d \in \mathrm{dom}(\sigma)$, where $\zeta(\alpha) = (h_0, \ldots, h_n)$.

Given $p \in \mathbb{P}$, we will use the obvious notation $p = (F^p, \zeta^p, \sigma^p)$. Order $\mathbb{P}$ by declaring $q \leq p$ if the following conditions are satisfied:

- $F^q \supseteq F^p$,
- $\zeta^q(\alpha) \supseteq \zeta^p(\alpha)$ for every $\alpha \in F^p$,
- $\sigma^q \supseteq \sigma^p$,
- If $\zeta^p(\alpha) = (h_0, \ldots, h_n)$ and $\zeta^q(\alpha) = (h_0, \ldots, h_m)$ for some $\alpha \in F^p$, then $h_i(x) = x$ whenever $n < i \leq m$ and $x = (h_n \circ \cdots \circ h_0)\big(d(\alpha)\big)$ for some $d \in \mathrm{dom}(\sigma^p)$.

Intuitively, the last condition expresses the "promise" made by p that $d(\alpha)$ has reached its final destination. Furthermore, this final destination will be "the correct one" (in the sense that it is the one dictated by σ) by the definition of $\mathbb{P}$.

Given $\alpha \in \kappa$, $d \in D$ and $e \in E$, make the following definitions:

- $D_\alpha^{\text{coord}} = \{p \in \mathbb{P} : \alpha \in F^p\}$,
- $D_d^{\text{dom}} = \{p \in \mathbb{P} : d \in \text{dom}(\sigma^p)\}$,
- $D_e^{\text{ran}} = \{p \in \mathbb{P} : e \in \text{ran}(\sigma^p)\}$.

Claim 1. Each D_α^{coord} is a dense subset of $\mathbb{P}$.

Proof. Use the fact that the $\mathcal{H}_\alpha$ witness strong n-homogeneity, as well as the general position of $D \cup E$.

Claim 2. Each D_d^{dom} is a dense subset of $\mathbb{P}$.

Proof. Use the fact that the $\mathcal{H}_\alpha$ witness strong local homogeneity.

Claim 3. Each D_e^{ran} is a dense subset of $\mathbb{P}$.

Proof. This is similar to the proof of Claim 2.

Claim 4. $\mathbb{P}$ is σ-centered.

Proof. Use the fact that each $\mathcal{H}_\alpha$ is countable, plus a nice argument using independent families.

Observe that the collection

$$\mathcal{D} = \{D^{\mathrm{coord}}_\alpha : \alpha \in \kappa\} \cup \{D^{\mathrm{dom}}_d : d \in D\} \cup \{D^{\mathrm{ran}}_e : e \in E\}$$

consists of dense subsets of $\mathbb{P}$ by Claims 1-3, and that $|\mathcal{D}| < \mathfrak{p}$ because $\kappa < \mathfrak{p}$. Therefore, since $\mathbb{P}$ is σ-centered by Claim 4, it is possible to apply Bell's Theorem, which guarantees the existence of a filter G on $\mathbb{P}$ that intersects every element of $\mathcal{D}$. Given $\alpha \in \kappa$, let

$$(h_{\alpha,0}, h_{\alpha,1}, \ldots) = \bigcup\{\zeta^p(\alpha) : p \in G\},$$

then set

$$h_\alpha = \lim_{n\to\infty}(h_{\alpha,n} \circ \cdots \circ h_{\alpha,0}).$$

Finally, set $h = \prod_{\alpha\in\kappa} h_\alpha$, and observe that $h \in \mathcal{H}(X)$. The verification that $h[D] = E$ is straightforward.

Optimality

In the following table, we provide explicit counterexamples showing that none of the requirements in the statement of our theorem can be dropped. The first row of the table simply lists these requirements; in the second row, below each requirement, we exhibit a product that satisfies all the other requirements but is not CDH. (In some cases, the product consists of a single factor.)

$\kappa < \mathfrak{p}$	Polish	Strongly locally homogeneous	Strongly n-homogeneous for every $n \in \omega$	General position property
$(2^\omega)^\mathfrak{p}$	$\mathbb{Q}$	Van Mill, 2011	$(\mathbb{S}^1 \oplus \mathbb{S}^1)^\omega$	$2^\omega \times \mathbb{S}^2$

The space $2^\mathfrak{p}$ is not CDH because some of its countable dense subspaces are sequentially crowded, while others are not (Hrušák and Zamora Avilés, 2005).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IN BOTTLES

Reasons to look at our paper

- It gives a unified treatment instead of scattered results.
- It applies to products, instead of just powers.
- It fully answers the question "Which products of manifolds with boundary are CDH?"
- It fully answers the question "Which infinite products of zero-dimensional Polish spaces are CDH?"
- The treatment of the general position property is fully detailed, and it seems to be of wide applicability.
- The treatment of the Inductive Convergence Criterion for Polish spaces is completely rigorous and the statement seems to be as simple as possible. (I am not aware of another source that satisfies both of these conditions.)

Thank you for listening

and have a good lunch!