Countable dense homogeneity and set theory

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April 13, 2013

All spaces are separable, metrizable, and with no isolated points. Let $\mathcal{H}(X)$ be the group of homeomorphisms of X.

Definition (Bennett, 1972)

A space X is *countable dense homogeneous* (briefly, CDH) if for every pair (D, E) of countable dense subsets of X there exists $h \in \mathcal{H}(X)$ such that $h[D] = E$.

Examples:

- $\mathbb{R}$ (Cantor, 1895), $\mathbb{R}^n$ (Brouwer, 1913).
- The Hilbert cube $[0, 1]^\omega$ (Fort, 1962).
- Under MA(σ-centered), some Bernstein sets $X \subseteq 2^\omega$.
- Under MA(countable), some ultrafilters $\mathcal{U} \subseteq 2^\omega$.

Non-examples:

- $\mathbb{Q}^\omega$ (Fitzpatrick and Zhou, 1992).
- Under MA(countable), some ultrafilters $\mathcal{U} \subseteq 2^\omega$.

Proof that $\mathbb{R}^3$ is CDH

It is essentially a back-and-forth argument.

Enumerate $D = \{d_n : n \in \omega\}$ and $E = \{e_n : n \in \omega\}$. Construct $f_n \in \mathcal{H}(\mathbb{R}^3)$ for every $n \in \omega$. Let $h_n = f_n \circ \cdots \circ f_1 \circ f_0$. In the end, we would like to set $h = \lim_{n\to\infty} h_n$. Make sure that

- $h_m(d_n) = h_{2n}(d_n) \in E$ for all $m \geq 2n$,
- $h_m^{-1}(e_n) = h_{2n+1}^{-1}(e_n) \in D$ for all $m \geq 2n+1$.

Problem: the h_n might not converge to a homeomorphism! The Inductive Convergence Criterion (using the fact that $\mathbb{R}^3$ is Polish) guarantees exactly that, provided this additional condition is satisfied:

- f_n is sufficiently close to the identity: actually, we can choose f_n to be supported inside an arbitrarily small open set.

The main positive result

The following is the key property used in the above proof.

Definition

A space X is *strongly locally homogeneous* (briefly, SLH) if there exists an open base $\mathcal{B}$ for X such that for every $U \in \mathcal{B}$ and $x, y \in U$ there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$ and $h \upharpoonright (X \setminus U)$ is the identity.

Examples: $\mathbb{R}^n$ (actually, any manifold), 2^ω, $[0, 1]^\omega$, any zero-dimensional homogeneous space.

Theorem (Curtis, Anderson, Van Mill, 1982)

If X is Polish and SLH then X is CDH.

What about non-complete spaces?

Question (Fitzpatrick and Zhou, 1990)

Is there a CDH space that is not Polish?

Recall that $X \subseteq 2^\omega$ is a *Bernstein set* if neither X nor $2^\omega \setminus X$ contain any copy of 2^ω.

Theorem (Baldwin and Beaudoin, 1989)

Under MA(σ-centered), there is a CDH Bernstein $X \subseteq 2^\omega$.

Theorem (Medini and Milovich, 2012)

Under MA(countable), there exists a CDH non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$.

Theorem (Farah, Hrušák and Martínez-Ranero, 2005)

There exists a CDH subspace X of $\mathbb{R}$ of size $\aleph_1$ that is not completely metrizable (actually, X is a λ-set).

Using MA: the basic poset

Fix countable dense subsets $D, E \subseteq 2^\omega$. Let $\mathbb{P}$ be the set of all pairs $p = (g, \pi) = (g_p, \pi_p)$ such that, for some $n = n_p \in \omega$, the following conditions hold.

- g is a bijection between a finite subset of D and a finite subset of E.
- π is a permutation of ${}^n 2$.
- $\pi(d \restriction n) = g(d) \restriction n$ for every $d \in \mathrm{dom}(g)$.

Order $\mathbb{P}$ by declaring $q \leq p$ if the following conditions hold.

- $g_q \supseteq g_p$.
- $\pi_q(t) \restriction n_p = \pi_p(t \restriction n_p)$ for all $t \in {}^{n_q} 2$.

The generic function will be $h \in \mathcal{H}(2^\omega)$ such that $h[D] = E$.

The basic poset by itself is useless! However, appropriate modifications of it yield the following lemmas. Recall that $D \subseteq X$ is λ-*dense* if $|D \cap U| = \lambda$ for every non-empty open $U \subseteq X$.

Lemma (Baldwin and Beaudoin, 1989)

Assume MA(σ-centered). Fix $\kappa < \mathfrak{c}$ and infinite $\lambda_\alpha < \mathfrak{c}$ for every $\alpha < \kappa$. Let $\{D_\alpha : \alpha \in \kappa\}$ be a collection of pairwise disjoint subsets of 2^ω such that each D_α is λ_α-dense in 2^ω. Let $\{E_\alpha : \alpha \in \kappa\}$ be another such collection. Then there exists $h \in \mathcal{H}(2^\omega)$ such that $h[D_\alpha] = E_\alpha$ for each α.

Lemma (Medini and Milovich, 2012)

Assume MA(countable). Assume that $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a $<\mathfrak{c}$-generated proper ideal. Fix countable dense $D, E \subseteq \mathcal{I}$. Then there exist $x \in \mathcal{P}(\omega)$ and $h \in \mathcal{H}(2^\omega)$ such that $\mathcal{I} \cup \{x\}$ still generates a proper ideal, $h[D] = E$ and $\{d \Delta h(d) : d \in D\} \subseteq \mathcal{P}(x)$.

Constructing a CDH Bernstein set $X \subseteq 2^\omega$

Enumerate all perfect subsets of 2^ω as $\{P_\alpha : \alpha \in \mathfrak{c}\}$, then build increasing sequences $\langle X_\alpha : \alpha \in \mathfrak{c}\rangle$ and $\langle Y_\alpha : \alpha \in \mathfrak{c}\rangle$ such that

- X_α is $|\alpha|$-dense in 2^ω (in particular $|X_\alpha| = |\alpha|$),
- Y_α is $|\alpha|$-dense in 2^ω (in particular $|Y_\alpha| = |\alpha|$),
- $X_\alpha \cap Y_\alpha = \varnothing$.

In the end, we will set $X = \bigcup_{\alpha \in \mathfrak{c}} X_\alpha$.
At stage $\alpha + 1$, make sure that both $X_{\alpha+1}$ and $Y_{\alpha+1}$ contain at least one point from P_α.
New requirements: enumerate as $\{(D_\alpha, E_\alpha) : \alpha \in \mathfrak{c}\}$ all pairs of countable dense subsets of 2^ω, making sure each pair is listed cofinally often.

Also build an increasing sequence $\langle \mathcal{H}_\alpha : \alpha \in \mathfrak{c} \rangle$ of subgroups of $\mathcal{H}(2^\omega)$ of size $< \mathfrak{c}$ such that

- $h[X_\alpha] = X_\alpha$ for every $h \in \mathcal{H}_\alpha$,
- $h[Y_\alpha] = Y_\alpha$ for every $h \in \mathcal{H}_\alpha$.

In the end, let $\mathcal{H} = \bigcup_{\alpha \in \mathfrak{c}} \mathcal{H}_\alpha$. This will guarantee that each $h \in \mathcal{H}$ restricts to a homeomorphism of X.
At stage α, assume $D_\alpha \cup E_\alpha \subseteq X_\alpha$ and $X_\alpha \setminus (D_\alpha \cup E_\alpha)$ is $|\alpha|$-dense (otherwise, skip this part).
Use the first lemma (with $\kappa = 3$) to get $h \in \mathcal{H}(2^\omega)$ such that

- $h[D_\alpha] = E_\alpha$,
- $h[X_\alpha \setminus D_\alpha] = X_\alpha \setminus E_\alpha$,
- $h[Y_\alpha] = Y_\alpha$.

Constructing a CDH ultrafilter $\mathcal{U} \subseteq 2^\omega$

All filters and ideals are proper and non-principal.
Any ultrafilter $\mathcal{U}$ is homeomorphic to its dual maximal ideal $\mathcal{J}$.
So, for notational convenience, we will construct an increasing sequence of $<\mathfrak{c}$-generated ideals $\langle \mathcal{I}_\alpha : \alpha \in \mathfrak{c} \rangle$. In the end, let $\mathcal{J}$ be any maximal ideal extending $\bigcup_{\alpha\in\mathfrak{c}} \mathcal{I}_\alpha$.
The idea is to use the following lemma.

Lemma (Medini and Milovich, 2012)

Let $h \in \mathcal{H}(2^\omega)$. Fix a maximal ideal $\mathcal{J} \subseteq 2^\omega$ and a countable dense $D \subseteq \mathcal{J}$. Then h restricts to a homeomorphism of $\mathcal{J}$ if and only if $cl(\{d\Delta h(d) : d \in D\}) \subseteq \mathcal{J}$.

At stage $\alpha + 1$, make sure that either

- $\omega \setminus x \in \mathcal{I}_{\alpha+1}$ for some $x \in D_\alpha \cup E_\alpha$, or
- there exists $h \in \mathcal{H}(2^\omega)$ and $x \in \mathcal{I}_{\alpha+1}$ such that $h[D_\alpha] = E_\alpha$ and $\{d\Delta h(d) : d \in D_\alpha\} \subseteq \mathcal{P}(x)$ (use the second lemma).

How to prove stuff in ZFC

(Based on Dilip Raghavan's "Almost disjoint families and diagonalizations of length continuum", 2010.)
Suppose we are in the middle (stage $\alpha + 1$) of a construction that uses MA(σ-centered):

> *Since we are using only $|\alpha| < \mathfrak{c}$ dense sets and our poset is σ-centered, using MA(σ-centered) we get a generic object x that takes care of the α-th requirement. Now let $X_{\alpha+1} = X_\alpha \cup \{x\}$.*

Key observation: if $|\alpha| < \mathfrak{p}$, the assumption of MA(σ-centered) is not necessary (Bell, 1981).
Therefore, if we could show that only $\mathfrak{p}$-many requirements need to be satisfied, our result would hold in ZFC.

Similarly, for MA(countable), we would need $|\alpha| < \text{cov}(\text{meager})$. More generally, let ☺ be a cardinal invariant. If the following two conditions hold at the same time:

- in the inductive step, all we need is $|\alpha| <$ ☺,
- globally, we only need to satisfy ☺-many requirements,

then we will have a ZFC proof.
For example, the starting point of the ZFC proof of the existence of a Van Douwen MAD family is the observation that only ☺ $=$ non(meager) requirements need to be satisfied.

Question

In the two previous construction of CDH spaces, can we reduce the number of requirements from $\mathfrak{c}$ to $\mathfrak{p}$ or cov(meager)?

What about infinite powers?

Question (Fitzpatrick and Zhou, 1990)

For which $X \subseteq 2^\omega$ is X^ω CDH?

Recall that a space X is *completely Baire* if every closed subspace of X is a Baire space. Consider the following 'addition of theorems'.

Theorem (Hurewicz)

Every co-analytic completely Baire space is Polish.

Theorem (Hrušák and Zamora Avilés, 2005)

Every analytic CDH space is completely Baire.

Theorem (Hrušák and Zamora Avilés, 2005)

Every Borel CDH space is Polish.

We just proved half of the following theorem.

Theorem (Hrušák and Zamora Avilés, 2005)

For a Borel $X \subseteq 2^\omega$, the following conditions are equivalent.

- X^ω *is CDH.*
- X *is a* G_δ *(equivalently, Polish).*

The other half follows from the next result.

Theorem (Dow and Pearl, 1997)

If X is zero-dimensional and first-countable then X^ω is homogeneous.

Question (Hrušák and Zamora Avilés, 2005)

Is there a non-G_δ subset X of 2^ω such that X^ω is CDH?

Given Baldwin and Beaudoin's result, a Bernstein set seems like a natural candidate. But...

Theorem (Hernández-Gutiérrez, 2013)

If X is crowded and X^ω is CDH, then X contains a copy of 2^ω.

Theorem (Medini and Milovich, 2012)

Assume MA(countable). Then there exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $\mathcal{U}^\omega$ is CDH.

Theorem (Hernández-Gutiérrez and Hrušák, 2013)

Let $\mathcal{F} \subseteq 2^\omega$ be a non-meager P-filter. Then $\mathcal{F}$ and $\mathcal{F}^\omega$ are CDH.

This is particularly interesting because the statement 'There are no non-meager P-filters' is known to have large cardinal strength.

Products and countable dense homogeneity

Since homeomorphisms permute connected components, it is easy to show that $2^\omega \times S^1$ is not CDH. But what about *zero-dimensional* spaces? The following theorem follows easily from work of Hrušák and Zamora Avilés.

Theorem

If X_n is Borel, zero-dimensional and CDH for every $n \in \omega$ then $\prod_{n\in\omega} X_n$ is CDH.

It is natural to ask whether the 'Borel' assumption can be dropped...

Theorem (Medini, 2013)

Under MA(σ-centered), there exists a zero-dimensional CDH space X such that X^2 is not CDH (actually, X^2 has $\mathfrak{c}$-many types of countable dense sets).

Raising cardinals
Part I: Raising the density

Definition

Let λ be an infinite cardinal. A space X is λ-*dense homogeneous* (briefly, λ-DH) if whenever $D, E \subseteq X$ are λ-dense there exists $h \in \mathcal{H}(X)$ such that $h[D] = E$.

Notice that CDH $= \aleph_0$-DH. One might expect that under MA every decent (manifold? Polish and SLH?) space is λ-DH for every $\lambda < \mathfrak{c}$. This is true in some cases...

Theorem (Baldwin and Beaudoin, 1989)

Assume MA(σ-centered). Then 2^ω is λ-DH for every $\lambda < \mathfrak{c}$.

Theorem (Steprāns and Watson, 1989)

Assume MA(σ-centered). Then every manifold X of dimension $n \geq 2$ is λ-DH for every $\lambda < \mathfrak{c}$.

Baumgartner's result

...but the assumption $n \geq 2$ is crucial!

Theorem (Baumgartner, 1973 + 1984)

It is consistent with MA $+ \mathfrak{c} = \aleph_2$ (it actually follows from PFA) that $\mathbb{R}$ is $\aleph_1$-DH.

Theorem (Abraham and Shelah, 1981)

It is consistent with MA $+ \mathfrak{c} = \aleph_2$ that $\mathbb{R}$ is not $\aleph_1$-DH.

Furthermore, the following is still open.

Question (Baumgartner, 1973)

Is it consistent that $\mathbb{R}$ is $\aleph_2$-DH?

Why is dimension $n = 1$ so tough? For example, a bijection $F \longrightarrow G$ between finite subsets $F, G \subseteq \mathbb{R}$ does not necessarily extend to a homeomorphism $\mathbb{R} \longrightarrow \mathbb{R}$...

Kunen's improvements

Recall that the *angle* between two vectors v and w is

$$\angle(v, w) = \arccos\left(\frac{v \cdot w}{|v||w|}\right) \in [0, \pi].$$

Recall that $h \in \mathcal{H}(\mathbb{R}^n)$ is *absolutely continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(U) < \delta$ implies $\mu(f[U]) < \varepsilon$ for every open set $U \subseteq \mathbb{R}^n$, where μ is Lebesgue measure.

Theorem (Kunen, 2012)

Assume PFA. Fix $n \geq 1$ and $\aleph_1$-dense subsets D, E of $\mathbb{R}^n$. Let $\theta \in (\pi/2, \pi)$. Then there exists $h \in \mathcal{H}(\mathbb{R}^n)$ such that

- $h[D] = E$,
- *both h and h^{-1} are absolutely continuous,*
- $\sup\{\angle(x - y, f(x) - f(y)) : x, y \in D, x \neq y\} \leq \theta$.

Raising cardinals
Part II: Raising the weight

In this section, drop the assumption of metrizability.

Theorem (Steprāns and Zhou, 1988)

Every manifold of weight $< \mathfrak{b}$ is CDH. Furthermore, there exists a manifold of weight $\mathfrak{c}$ that is not CDH.

Theorem (Watson, 1988)

It is consistent that there exists a manifold of weight $< \mathfrak{c}$ that is not CDH.

The above results suggest the following definition.

$$\mathfrak{cdh}_{\mathfrak{m}} = \min\{\kappa : \text{ there exists a non-CDH manifold of weight } \kappa\}$$

Question (Watson, 1988)

Is $\mathfrak{cdh}_{\mathfrak{m}}$ one of the known cardinal invariants?

Recall that 2^κ is separable if and only if $\kappa \leq \mathfrak{c}$. So it makes sense to define

$$\mathfrak{cdh}_\mathfrak{e} = \min\{\kappa : 2^\kappa \text{ is not CDH}\}.$$

Question

Is $\mathfrak{cdh}_\mathfrak{e}$ one of the known cardinal invariants?

Yes! It is in fact the same as as the pseudo-intersection number.

Theorem (Hrušák and Zamora Avilés, 2005)

$\mathfrak{cdh}_\mathfrak{e} = \mathfrak{p}$.

The proof is based on results of Matveev and Steprāns and Zhou.

The topological Vaught conjecture

Assume that all spaces are separable metrizable again.

Definition

Let $\mathcal{G}$ be a class of topological groups and $\mathcal{X}$ be a class of spaces. Let $V(\mathcal{G}, \mathcal{X})$ be the statement that for all $G \in \mathcal{G}$ and $X \in \mathcal{X}$, if $A : G \times X \longrightarrow X$ is a continuous action of G on X then A has either countably many or $\mathfrak{c}$-many orbits.

The statement V(Polish groups, Polish spaces) is known as the *topological Vaught conjecture*. It is obviously true under CH, and it implies the classical Vaught conjecture. (The classical Vaught conjecture says that every complete first-order theory in a countable language has either countably many of $\mathfrak{c}$-many isomorphism-classes of countable models.)

Theorem (Hrušák and Van Mill, 2012)

The following are equivalent.

- *V(Polish groups, Locally compact spaces)*
- *Every locally compact space has either countably many or $\mathfrak{c}$-many types of countable dense sets.*

Question (Hrušák and Van Mill, 2012)

Can one prove in ZFC the existence of a Polish space with exactly $\aleph_1$ types of countable dense subsets?

Question

Is there an equivalent formulation of the topological Vaught conjecture involving countable dense homogeneity?