Homogeneous spaces and Wadge theory

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Everybody loves

homogeneous stuff!

Topological homogeneity

A space is homogeneous if all points "look alike" from a global point of view:

Definition

A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$.

Non-examples:

- $2^\omega \oplus \omega^\omega$ (Because of local compactness)
- $[0,1]^n$ whenever $1 \leq n < \omega$ (Points on the boundary are different from points in the interior)
- The Stone-Čech remainder $\omega^* = \beta\omega \setminus \omega$
(W. Rudin, 1956, under CH, because of P-points)
(Frolík, 1967, using a cardinality argument)
(Kunen, 1978, by proving the existence of weak P-points)

Examples:

- Any topological group (Just translate)
- Any product of homogeneous spaces
  (Just take the product homeomorphism)
- Any open subspace of a zero-dimensional homogeneous space
- The Hilbert cube $[0,1]^\omega$ (Keller, 1931)

Homogeneous spaces are decently understood.
Compact homogeneous spaces are shrouded in mystery:

Question (Van Douwen, 1970s)

Is there a compact homogeneous space with more than $\mathfrak{c}$ pairwise disjoint non-empty open sets?

Question (W. Rudin, 1958)

Is there a compact homogeneous space with no non-trivial convergent ω-sequences?

Strong homogeneity

Definition

A space X is *strongly homogeneous* (or *h-homogeneous*) if every non-empty clopen subspace of X is homeomorphic to X.

Examples:

- Any connected space
- $\mathbb{Q}$, 2^ω, ω^ω (Use their characterizations)
- Any product of zero-dimensional strongly homogeneous spaces (Medini, 2011, building on work of Terada, 1993)
- Erdős space $\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\}$. (Dijkstra and van Mill, 2010)

Non-examples:

- Discrete spaces with at least two elements
- $\omega \times 2^\omega$

Is “strong” a good choice of word?

Not particularly. For example, ω^* is strongly homogeneous but not homogeneous. Things get better under additional assumptions:

Theorem (folklore)

Let X be a first-countable zero-dimensional space. If X is strongly homogeneous then X is homogeneous.

The locally compact case (the trivial case)

From now on, all spaces are separable and metrizable.

Proposition

Let X be a locally compact zero-dimensional space. Then the following conditions are equivalent:

- *X is homogeneous*
- *X is discrete, $X \approx \omega \times 2^\omega$, or $X \approx 2^\omega$*

An example of van Douwen

Theorem (van Douwen, 1984)

There exists a subspace X of $\mathbb{R}$ with the following properties:

- *X is a Bernstein subset of $\mathbb{R}$*
- *X is a subgroup of $(\mathbb{R}, +)$*
- *There exists a measure μ on the Borel subsets of X such that $A \approx B$ implies $\mu(A) = \mu(B)$ whenever $A, B \subseteq X$ are Borel*

Given a Borel subset A of X, the measure of A is defined by:

$$\mu(A) = \text{Lebesgue measure of } \tilde{A}$$

where $\tilde{A}$ is a Borel subset of $\mathbb{R}$ such that $\tilde{A} \cap X = A$.

Corollary

There exists a zero-dimensional homogeneous space that is not locally compact space and not strongly homogeneous.

The main result

In his remarkable Ph.D. thesis, van Engelen obtained a complete classification of the zero-dimensional homogeneous Borel spaces. As a corollary, he proved the following:

Theorem (van Engelen, 1986)

Let X be a zero-dimensional Borel space that is not locally compact. If X is homogeneous then X is strongly homogeneous.

Can the "Borel" assumption be dropped? Certainly not in ZFC, by van Douwen's example. However:

Theorem (Carroy, Medini, Müller)

Work in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$. *Let X be a zero-dimensional space that is not locally compact. If X is homogeneous then X is strongly homogeneous.*

Until now, our ambient theory was ZFC. From now on, we will be working in $\mathsf{ZF} + \mathsf{DC}$.

The Wadge Brigade

Raphaël Carroy

Andrea Medini

Sandra Müller

Wadge theory: basic definitions

Let Z be a set and $\boldsymbol{\Gamma} \subseteq \mathcal{P}(Z)$. Define $\check{\boldsymbol{\Gamma}} = \{Z \setminus A : A \in \boldsymbol{\Gamma}\}$. We say that $\boldsymbol{\Gamma}$ is *selfdual* if $\boldsymbol{\Gamma} = \check{\boldsymbol{\Gamma}}$. Also define $\Delta(\boldsymbol{\Gamma}) = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$.

Definition (Wadge, 1984)

Let Z be a space. Given $A, B \subseteq Z$, we will write $A \leq B$ if there exists a continuous function $f : Z \longrightarrow Z$ such that $A = f^{-1}[B]$. In this case, we will say that A is *Wadge-reducible* to B, and that f *witnesses* the reduction.

Definition (Wadge, 1984)

Let Z be a space. Given $A \subseteq Z$, define

$$[A] = \{B \subseteq Z : B \leq A\}$$

We will say that $\boldsymbol{\Gamma} \subseteq \mathcal{P}(Z)$ is a *Wadge class* if there exists $A \subseteq Z$ such that $\boldsymbol{\Gamma} = [A]$. The set A is *selfdual* if $[A]$ is selfdual.

First examples of Wadge classes

From now on, we will always assume that Z is an uncountable zero-dimensional Polish space.

- $\{\varnothing\}$ and $\{Z\}$ (These are the minimal ones)
- $\mathbf{\Delta}^0_1(Z)$ is their immediate successor
  (Generated by an arbitrary proper clopen set)

Let $1 \leq \xi < \omega_1$. Recall that $\mathbf{\Sigma}^0_\xi(Z)$ has a 2^ω-universal set U. This means that $U \subseteq 2^\omega \times Z$ and

$$\mathbf{\Sigma}^0_\xi(Z) = \{U_x : x \in 2^\omega\}$$

where $U_x = \{y \in Z : (x, y) \in U\}$ denotes the vertical section.

- $\mathbf{\Sigma}^0_\xi(Z)$ and $\mathbf{\Pi}^0_\xi(Z)$ (Generated by a universal set)
- $\mathbf{\Sigma}^1_n(Z)$ and $\mathbf{\Pi}^1_n(Z)$ for $n \geq 1$ (As above)

Why do we need determinacy?

Lemma (Wadge, 1984)

Assume AD. *Let $A, B \subseteq Z$. Then either $A \leq B$ or $B \leq Z \setminus A$.*

The above result is the most fundamental in Wadge theory.
In particular, it shows that antichains have size at most 2 in the poset $\mathbb{W}(Z)$ of all Wadge classes in Z ordered by $\subseteq$.

Theorem (Martin, Monk)

Assume AD. *The poset $\mathbb{W}(Z)$ is well-founded.*

This yields the definition of Wadge rank. By the two result above, $\mathbb{W}(Z)$ becomes a well-order if we identify every Wadge class $\boldsymbol{\Gamma}$ with its dual class $\check{\boldsymbol{\Gamma}}$. The length of this well-order is Θ.

In fact, all the consequences of AD that we need seem to be Wadge's Lemma and the fact that all subsets of a Polish space have the Baire property.

The analysis of selfdual sets

The following fundamental result reduces the study of self-dual Wadge classes to the study of non-selfdual Wadge classes:

Theorem (see Motto-Ros, 2009)

Assume AD*. Let A be a selfdual subset of Z. Then there exists a partition $\{V_n : n \in \omega\} \subseteq \mathbf{\Delta}^0_1(Z)$ of Z and non-selfdual $A_n < A$ for $n \in \omega$ such that*

$$\bigcup_{n\in\omega}(A_n \cap V_n) = A$$

The above result shows that Wadge classes whose rank has uncountable cofinality can never be selfdual.

By the same argument, if Z is compact, this is the case for all Wadge classes whose rank is a limit ordinal.

On the other hand, under AD, it can be shown that the poset of non-selfdual Wadge classes does not depend on the ambient space.

Hausdorff operations

Definition (Hausdorff, 1927)

Given $D \subseteq 2^\omega$, define

$$\mathcal{H}_D(A_0, A_1, \ldots) = \{x \in Z : \{n \in \omega : x \in A_n\} \in D\}$$

whenever $A_0, A_1, \ldots \subseteq Z$. (We identify 2^ω with $\mathcal{P}(\omega)$.)
We will call functions of this form *Hausdorff operations*.

Given $n \in \omega$, define $s_n : \{n\} \longrightarrow 2$ by setting $s_n(n) = 1$. Then:

- $\mathcal{H}_{[s_n]}(A_0, A_1, \ldots) = A_n$
- $\bigcap_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \ldots) = \mathcal{H}_D(A_0, A_1, \ldots)$, where $D = \bigcap_{i \in I} D_i$
- $\bigcup_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \ldots) = \mathcal{H}_D(A_0, A_1, \ldots)$, where $D = \bigcup_{i \in I} D_i$
- $X \setminus \mathcal{H}_D(A_0, A_1, \ldots) = \mathcal{H}_{2^\omega \setminus D}(A_0, A_1, \ldots)$ for all $D \subseteq 2^\omega$

Hence, any operation obtained by combining unions, intersections and complements can be expressed as a Hausdorff operation.

The difference hierarchy

Given $1 \leq \eta < \omega_1$, define the Hausdorff operation D_η as follows:

- $\mathrm{D}_1(A_0) = A_0$
- $\mathrm{D}_2(A_0, A_1) = A_1 \setminus A_0$
- $\mathrm{D}_3(A_0, A_1, A_2) = A_0 \cup (A_2 \setminus A_1)$

$$\vdots$$

- $\mathrm{D}_\omega(A_0, A_1, \ldots) = (A_1 \setminus A_0) \cup (A_3 \setminus A_2) \cup \cdots$
- $\mathrm{D}_{\omega+1}(A_0, A_1, \ldots, A_\omega) = A_0 \cup (A_2 \setminus A_1) \cup \cdots \cup (A_\omega \setminus \bigcup_{n<\omega} A_n)$

$$\vdots$$

Given $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$, define:

$$\mathrm{D}_\eta(\mathbf{\Gamma}) = \{\mathrm{D}_\eta(A_\mu : \mu < \eta) : \text{each } A_\mu \in \mathbf{\Gamma} \text{ and } (A_\mu : \mu < \eta) \text{ is increasing}\}$$

Hausdorff and Kuratowski showed that $\mathbf{\Delta}^0_{\xi+1} = \bigcup_{1\leq\eta<\omega_1} \mathrm{D}_\eta(\mathbf{\Sigma}^0_\xi)$.

More Wadge classes from Hausdorff operations

Definition

Given $D \subseteq 2^\omega$, define

$$\mathbf{\Gamma}_D(Z) = \{\mathcal{H}_D(A_0, A_1, \dots) : A_0, A_1, \dots \in \mathbf{\Sigma}^0_1(Z)\}$$

By fixing a 2^ω-universal set for $\mathbf{\Sigma}^0_1(Z)$ and "applying $\mathcal{H}_D$ to it", one obtains the following:

Theorem (Addison for $Z = \omega^\omega$)

Let $D \subseteq 2^\omega$. Then $\mathbf{\Gamma}_D(Z)$ is a non-selfdual Wadge class.

In particular, each $\mathrm{D}_\eta(\mathbf{\Sigma}^0_1(Z))$ is a non-selfdual Wadge class.
In fact, it can be shown that they exhaust the non-selfdual Wadge classes contained in $\mathbf{\Delta}^0_2(Z)$. (Not easy! What about $\mathbf{\Delta}^0_3(Z)$?)

Why do we need Hausdorff operations?

When one tries to give a systematic exposition of Wadge theory, it soon becomes apparent that it would be very useful to be able to talk about "abstract" Wadge classes, as opposed to Wadge classes in a particular space. More precisely, given a Wadge class $\mathbf{\Gamma}$ in some space Z, one would like to find a way to define what a "$\mathbf{\Gamma}$ subset of W" is, for every other space W, while of course preserving suitable coherence properties.

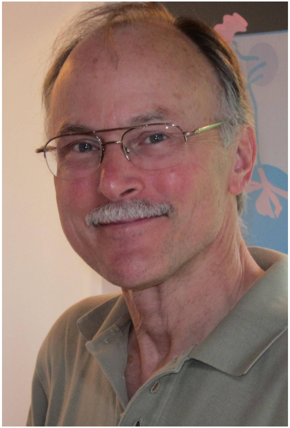
It turns out that Hausdorff operations allow us to do exactly that in a rather elegant way. The first ingredient is the following result, proved by Van Wesep in his Ph.D. thesis:

Theorem (Van Wesep, 1977, for $Z = \omega^\omega$)

Let $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$. Then the following are equivalent:

- $\mathbf{\Gamma}$ *is a non-selfdual Wadge class in Z*
- *There exists $D \subseteq 2^\omega$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(Z)$*

Robert Van Wesep: medical scientist, mathematician, poet



Plus Ultra

The whole world having been
into its ultrapower injected
The latter being founded well,
if all goes as expected
The sets whose images contain
the point of criticality
Return an ultrafilter with
a dividend: normality!

Remember when I said "preserving suitable coherence properties"?

The second ingredient is the following Relativization Lemma. Similar result have appeared in work of van Engelen, and even earlier in work of Louveau and Saint-Raymond.

Lemma

Let Z and W be spaces, and let $D \subseteq 2^\omega$.

- *Assume that $W \subseteq Z$. Then $A \in \mathbf{\Gamma}_D(W)$ iff there exists $\tilde{A} \in \mathbf{\Gamma}_D(Z)$ such that $A = \tilde{A} \cap W$.*
- *If $f : Z \longrightarrow W$ is continuous and $B \in \mathbf{\Gamma}_D(W)$ then $f^{-1}[B] \in \mathbf{\Gamma}_D(Z)$.*
- *If $h : Z \longrightarrow W$ is a homeomorphism then $A \in \mathbf{\Gamma}_D(Z)$ iff $h[A] \in \mathbf{\Gamma}_D(W)$.*

Reasonably closed Wadge classes

Given $i \in 2$, set:

$$Q_i = \{x \in 2^\omega : x(n) = i \text{ for all but finitely many } n \in \omega\}$$

Notice that every element of $2^\omega \setminus (Q_0 \cup Q_1)$ is obtained by alternating finite blocks of zeros and finite blocks of ones.

Define the function $\phi : 2^\omega \setminus (Q_0 \cup Q_1) \longrightarrow 2^\omega$ by setting

$$\phi(x)(n) = \begin{cases} 0 & \text{if the } n^{\text{th}} \text{ block of zeros of } x \text{ has even length} \\ 1 & \text{otherwise} \end{cases}$$

where we start counting with the 0^{th} block of zeros. It is easy to check that ϕ is continuous.

Definition (Steel, 1980)

Let $\mathbf{\Gamma}$ be a Wadge class in 2^ω. We will say that $\mathbf{\Gamma}$ is *reasonably closed* if $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$ for every $A \in \mathbf{\Gamma}$.

A sufficient condition for reasonability

Lemma (Step 3)

Assume AD*. Let* $\mathbf{\Gamma}$ *be a Wadge class in* 2^ω *that is closed under intersections with* $\mathbf{\Pi}^0_2$ *sets and unions with* $\mathbf{\Sigma}^0_2$ *sets. Then* $\mathbf{\Gamma}$ *is reasonably closed.*

Proof.

Pick $A \in \mathbf{\Gamma}$. We need to show that $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$.
By Van Wesep's Theorem, fix $D \subseteq 2^\omega$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(2^\omega)$.

Set $Z = 2^\omega \setminus (Q_0 \cup Q_1)$, and notice that $\phi^{-1}[A] \in \mathbf{\Gamma}_D(Z)$ by the Relativization Lemma. Therefore, again by the Relativization Lemma, there exists $B \in \mathbf{\Gamma}_D(2^\omega) = \mathbf{\Gamma}$ such that $B \cap Z = \phi^{-1}[A]$.

Since $Z \in \mathbf{\Pi}^0_2(2^\omega)$, it follows from our assumptions that $\phi^{-1}[A] \in \mathbf{\Gamma}$, hence $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$. $\square$

Our main tool: Steel's theorem

Given a Wadge class $\boldsymbol{\Gamma}$ in 2^ω and $X \subseteq 2^\omega$, we will say that X is *everywhere properly* $\boldsymbol{\Gamma}$ if $X \cap [s] \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$ for every $s \in 2^{<\omega}$.

Theorem (Steel, 1980)

Assume AD*. Let* $\boldsymbol{\Gamma}$ *be a reasonably closed Wadge class in* 2^ω*. Assume that* X *and* Y *are subsets of* 2^ω *that satisfy the following:*

- X *and* Y *are everywhere properly* $\boldsymbol{\Gamma}$
- X *and* Y *are either both meager or both comeager*

Then there exists a homeomorphism $h : 2^\omega \longrightarrow 2^\omega$ *such that* $h[X] = Y$*.*

The following (due to Harrington) is crucial for the proof: if $\boldsymbol{\Gamma} = [B]$ is a reasonably closed non-selfdual Wadge class in 2^ω, then $A \leq B$ can always be witnessed by an injective function.

From now on, we will always assume that AD holds.

Three steps to reasonability

$\mathbf{\Gamma} = [X]$ for some homogeneous $X \subseteq 2^\omega$ of high enough complexity

$\downarrow$

$\mathbf{\Gamma}$ is a good Wadge class

$\downarrow$

$\mathbf{\Gamma}$ is closed under $\cap \, \mathbf{\Pi}^0_2$ and $\cup \, \mathbf{\Sigma}^0_2$

$\downarrow$

$\mathbf{\Gamma}$ is reasonably closed

Finishing the proof (assuming the three steps)

Let X be a zero-dimensional homogeneous space that is not locally compact. Without loss of generality, assume that X is a dense subspace of 2^ω. If $X \in \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$, then X is strongly homogeneous by van Engelen's results. So assume that $X \notin \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$. (This is the "high enough complexity".)

Fix $s \in 2^{<\omega}$, and let $Y = X \cap [s]$. Using the Relativization Lemma, it is easy to see that Y is everywhere properly $\mathbf{\Gamma}$ in $[s] \approx 2^\omega$.

Since X is homogeneous, either X is meager or it is Baire (hence comeager in 2^ω by AD). The same will be true of Y.

Hence $Y \approx X$ by Steel's theorem. The following result concludes the proof that X is strongly homogeneous.

Theorem (Terada, 1993)

Let X be a space. Assume that X has a base $\mathcal{B} \subseteq \mathbf{\Delta}^0_1(X)$ such that $U \approx X$ for every $U \in \mathcal{B}$. Then X is strongly homogeneous.

Partitioned unions and the notion of level

From now on, $\xi < \omega_1$ and $\mathbf{\Gamma}$, $\mathbf{\Lambda}$ are Wadge classes in Z.

Definition (Louveau, Saint-Raymond, 1988)

Define $\mathrm{PU}_\xi(\mathbf{\Gamma})$ to be the collection of all sets of the form

$$\bigcup_{n\in\omega}(A_n \cap V_n)$$

where $A_n \in \mathbf{\Gamma}$ for $n \in \omega$ and $\{V_n : n \in \omega\} \subseteq \mathbf{\Delta}^0_{1+\xi}(Z)$ is a partition of Z. A set in this form is called a *partitioned union* of sets in $\mathbf{\Gamma}$.

Definition (Louveau, Saint-Raymond, 1988)

- $\ell(\mathbf{\Gamma}) \geq \xi$ if $\mathrm{PU}_\xi(\mathbf{\Gamma}) = \mathbf{\Gamma}$
- $\ell(\mathbf{\Gamma}) = \xi$ if $\ell(\mathbf{\Gamma}) \geq \xi$ and $\ell(\mathbf{\Gamma}) \not\geq \xi + 1$
- $\ell(\mathbf{\Gamma}) = \omega_1$ if $\ell(\mathbf{\Gamma}) \geq \xi$ for every $\xi < \omega_1$

We refer to $\ell(\mathbf{\Gamma})$ as the *level* of $\mathbf{\Gamma}$.

The expansion theorem

Definition (Wadge, 1984)

$$\mathbf{\Gamma}^{(\xi)} = \{f^{-1}[A] : A \in \mathbf{\Gamma} \text{ and } f : Z \longrightarrow Z \text{ is } \mathbf{\Sigma}^0_{1+\xi}\text{-measurable}\}$$

We will refer to $\mathbf{\Gamma}^{(\xi)}$ as an *expansion* of $\mathbf{\Gamma}$. To see what happens with regard to Hausdorff operations, it can be shown that

$$\mathbf{\Gamma}_D(Z)^{(\xi)} = \{\mathcal{H}_D(A_0, A_1, \ldots) : A_0, A_1, \ldots \in \mathbf{\Sigma}^0_{1+\xi}(Z)\}$$

Theorem

Assume that $\mathbf{\Gamma}$ is a non-selfdual Wadge class. Then the following conditions are equivalent:

- $\ell(\mathbf{\Gamma}) \geq \xi$
- *There exists a non-selfdual class $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}^{(\xi)} = \mathbf{\Gamma}$*

Good Wadge classes

Definition

We will say that $\boldsymbol{\Gamma}$ is *good* if the following are satisfied:

- $\boldsymbol{\Gamma}$ is non-selfdual
- $\Delta(\mathrm{D}_\omega(\boldsymbol{\Sigma}^0_2)) \subseteq \boldsymbol{\Gamma}$
- $\ell(\boldsymbol{\Gamma}) \geq 1$

Lemma (Step 2)

If $\boldsymbol{\Gamma}$ is good then $\boldsymbol{\Gamma}$ is closed under $\cap\, \boldsymbol{\Pi}^0_2$ and $\cup\, \boldsymbol{\Sigma}^0_2$.

Proof.

Andretta, Hjorth and Neeman proved that if $\Delta(\mathrm{D}_\omega(\boldsymbol{\Sigma}^0_1)) \subseteq \boldsymbol{\Lambda}$ then $\boldsymbol{\Lambda}$ is closed under $\cap\, \boldsymbol{\Pi}^0_1$ and $\cup\, \boldsymbol{\Sigma}^0_1$. Since $\ell(\boldsymbol{\Gamma}) \geq 1$ there exists $\boldsymbol{\Lambda}$ such that $\boldsymbol{\Lambda}^{(1)} = \boldsymbol{\Gamma}$. Apply the above mentioned result to $\boldsymbol{\Lambda}$, then transfer it to $\boldsymbol{\Gamma}$ using expansions. $\square$

The proof of Step 1

Let $X \subseteq 2^\omega$ be dense and homogeneous, with $X \notin \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$. We need to show that $[X]$ is a good Wadge class.

Fix a minimal $\mathbf{\Gamma}$ such that there exists a non-empty $U \in \mathbf{\Sigma}^0_1(2^\omega)$ such that $X \cap U \in \{\mathbf{\Gamma}, \check{\mathbf{\Gamma}}\}$. Fix such U as well. First we will show that $\mathbf{\Gamma}$ is a good Wadge class, then that $[X] = \mathbf{\Gamma}$.

Assume, in order to get a contradiction, that $X \cap U \in \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$. Notice that $\mathcal{U} = \{h[X \cap U] : h$ is a homeomorphism of $X\}$ is a cover of X because X is homogeneous and dense in 2^ω.

Furthermore, since $\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$ is a good Wadge class, the following lemma shows that each element of $\mathcal{U}$ belongs to it.

Lemma (Good Wadge classes are "topological")

Let $\mathbf{\Gamma}$ is a good Wadge class in Z. If $A \in \mathbf{\Gamma}$ and $B \approx A$ then $B \in \mathbf{\Gamma}$.

Using a countable subcover of $\mathcal{U}$, write X as a partitioned union of sets in $\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$, where the elements of the partition are $\mathbf{\Delta}^0_2$.

Since $\ell(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)) \geq 1$, it follows that $X \in \mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$. A similar argument shows that $X \in \check{\mathrm{D}}_\omega(\mathbf{\Sigma}^0_2)$. This contradicts our assumptions, so $X \cap U \notin \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$.

It remains to show that $\ell(\mathbf{\Gamma}) \geq 1$. Assume, in order to get a contradiction, that $\ell(\mathbf{\Gamma}) = 0$. Then, applying the following with $Z = U$ will contradict the minimality of $\mathbf{\Gamma}$.

Lemma

Assume that $\mathbf{\Gamma}$ is a non-selfdual Wadge class with $\ell(\mathbf{\Gamma}) = 0$, and let $X \in \mathbf{\Gamma}$ be codense in Z. Then there exists a non-empty $V \in \mathbf{\Delta}^0_1(Z)$ and a non-selfdual $\mathbf{\Lambda}$ such that $\mathbf{\Lambda} \subsetneq \mathbf{\Gamma}$ and $X \cap V \in \mathbf{\Lambda}$.

Now that we know that $\mathbf{\Gamma}$ is a good Wadge class, since $X \cap U \in \mathbf{\Gamma}$, we can apply the same homogeneity argument as above to see that $X \in \mathbf{\Gamma}$, so $[X] \subseteq \mathbf{\Gamma}$. It remains to show that $[X] \subsetneq \mathbf{\Gamma}$ is impossible.

If X is non-selfdual, this would directly contradict minimality of $\mathbf{\Gamma}$. Otherwise, minimality would be contradicted after applying the analysis of the selfdual sets.

Open questions

As we have seen, for spaces of complexity higher than $\Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$, Baire category and Wadge class are sufficient to uniquely identify a homogeneous zero-dimensional space. This is the "uniqueness" part of the classification. But the "existence" part is still open:

Question

For exactly which good Wadge classes $\mathbf{\Gamma}$ is there a homogeneous X such that $\mathbf{\Gamma} = [X]$? For which ones is there a meager such X? For which ones is there a Baire such X?

Does the usual pattern of results under AD hold?

Question

Assuming $\mathrm{V} = \mathrm{L}$, is it possible to construct a zero-dimensional $\mathbf{\Pi}^1_1$ or $\mathbf{\Sigma}^1_1$ space that is homogeneous, not locally compact, and not strongly homogeneous?

Thank you for your attention

and have a good afternoon!