Zero-dimensional σ-homogeneous spaces

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Der Wissenschaftsfonds.

Preliminaries

All spaces are assumed to be separable and metrizable. Given a space X, denote by $\mathcal{H}(X)$ the group of homeomorphisms of X.

- A space X is *homogeneous* if for every $(x, y) \in X \times X$ there exists $h \in \mathcal{H}(X)$ such that $h(x) = y$.
- A zero-dimensional space X is *strongly homogeneous* if all its non-empty clopen subspaces are homeomorphic.
- A space X is *rigid* if $|X| \geq 2$ and $\mathcal{H}(X) = \{\text{id}\}$.
- A space is *σ-homogeneous* if it is the union of countably many of its homogeneous subspaces.
- A space is *Borel* if it can be embedded into some Polish space as a Borel set. Similarly define *analytic* and *coanalytic*.
- A space X is *$\mathfrak{c}$-crowded* if it is non-empty and every non-empty open subset of X has size $\mathfrak{c}$.

Exercise: every zero-dimensional strongly homogeneous space is homogeneous.

An established pattern in set theory

Many properties $\mathcal{P}$ behave as follows:

- Every Borel set of reals satisfies $\mathcal{P}$,
- Under AD, all sets of reals satisfy $\mathcal{P}$,
- Under AC, there exist counterexamples to $\mathcal{P}$,
- Under $\mathsf{V} = \mathsf{L}$, there exist definable (usually coanalytic) counterexamples to $\mathcal{P}$.

The classical regularity properties ($\mathcal{P} =$ "perfect set property", $\mathcal{P} =$ "Lebesgue measurable" and $\mathcal{P} =$ "Baire property") are the most famous instances of this pattern. More entertaining examples include $\mathcal{P} =$ "not a Hamel basis" and $\mathcal{P} =$ "not an ultrafilter". A recent example is $\mathcal{P} =$ "Effros group". This talk is about

$$\mathcal{P} = \text{“}\sigma\text{-homogeneity”},$$

in the context of zero-dimensional spaces.

A theorem of Steel

Recall that a *Wadge class* in 2^ω is a collection of the form

$$\mathbf{\Gamma} = \{f^{-1}[A] | f : 2^\omega \longrightarrow 2^\omega \text{ is continuous}\}$$

for some $A \subseteq 2^\omega$. Given $\mathbf{\Gamma} \subseteq \mathcal{P}(2^\omega)$, set $\check{\mathbf{\Gamma}} = \{2^\omega \setminus A : A \in \mathbf{\Gamma}\}$. We will say that $\mathbf{\Gamma}$ is *reasonably closed* if [redacted] for every [redacted].

Theorem (Steel, 1980)

Assume AD. *Let* $\mathbf{\Gamma}$ *be a reasonably closed Wadge class in* 2^ω*, and let* $X, Y \subseteq 2^\omega$ *be such that the following conditions hold:*

- X *and* Y *are either both comeager or both meager,*
- *For every basic clopen subset* U *of* 2^ω*, both* $X \cap U$ *and* $Y \cap U$ *have complexity exactly* $\mathbf{\Gamma}$ *(i.e. they belong to* $\mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$*).*

Then there exists $h \in \mathcal{H}(2^\omega)$ *such that* $h[X] = Y$*.*

Exercise: show that $\mathbb{Q}^\omega \approx \{x \in \omega^\omega : \lim_{n\to\infty} x_n = \infty\}$.

The positive results

Theorem (Ostrovsky, 2011)

Every zero-dimensional Borel space is σ-homogeneous.

Ostrovsky used the techniques of van Engelen's remarkable Ph.D. thesis, where he employed Louveau's 1983 article to classify all zero-dimensional homogeneous Borel spaces. Using instead material from Louveau's unpublished book, it is possible to extend these techniques beyond the Borel realm.

Theorem

Assume AD*. Then every zero-dimensional space is σ-homogeneous.*

Lemma

Assume AD*. Then it is possible to associate to every non-empty $X \subseteq 2^\omega$ a non-empty homogeneous clopen subspace* $\mathrm{HC}(X)$ *of X.*

Corollary (van Engelen, Miller and Steel, 1987)

Assume AD*. Then there are no zero-dimensional rigid spaces.*

Proof of the theorem, using the lemma

Given $X \subseteq 2^\omega$, define X_α for every ordinal α as follows:

- $X_0 = X$,
- $X_{\alpha+1} = X_\alpha \setminus \mathrm{HC}(X_\alpha)$,
- $X_\gamma = \bigcap_{\alpha<\gamma} X_\alpha$ if γ is a limit ordinal.

Since $X_0 \supseteq X_1 \supseteq \cdots$ are closed in X, the sequence must stabilize at some countable ordinal δ, and clearly $X_\delta = \varnothing$.

☕

"Proof" of the lemma

Take a non-empty clopen subspace U of X of "minimal complexity" (in the sense of Wadge theory). This is possible because, under AD, the Wadge hierarchy is well-founded (by the Martin-Monk theorem). It can be shown that the Wadge class generated by U in 2^ω will be reasonably closed. Using Steel's theorem, one sees that U is (strongly) homogeneous.

☕

A counterexample in ZFC

The naive definition of "hereditarily rigid" would be silly. But:

Definition

A space X is $\mathfrak{c}$-*hereditarily rigid* if X is $\mathfrak{c}$-crowded and every $\mathfrak{c}$-crowded subspace of X is rigid.

Theorem

There exists a ZFC *example of a zero-dimensional* $\mathfrak{c}$*-hereditarily rigid space.*

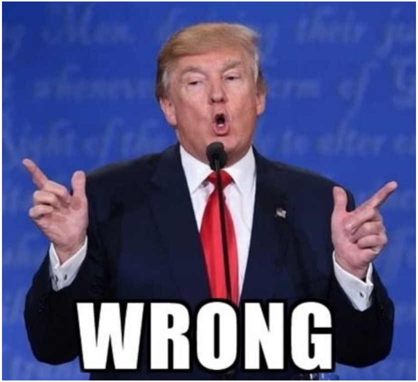
Corollary

There exists a ZFC *example of a zero-dimensional space that is not* σ*-homogeneous.*

Question

Is there a ZFC *example of a zero-dimensional space that is rigid and* σ*-homogeneous? (Yes, by van Engelen and van Mill, 1983.)*

Obviously, if you're a topologist, studying computability theory is a complete waste of time...



Definable counterexamples under $V = L$

In his 1989 paper, Miller sketched a method for constructing coanalytic versions of certain pathological sets of reals (in the spirit of Gödel's coanalytic set without the perfect set property).
In 2014, Vidnyánszky gave a "black box" version of Miller's method. Using this, it's not hard to prove the following:

Lemma

Assume $V = L$. Then there exists $X \subseteq \omega^\omega$ such that:

- *X is coanalytic,*
- *X is dense in ω^ω and $\mathfrak{c}$-crowded,*
- *Every element of X is self-constructible,*
- *If $x, y \in X$ and $x \neq y$ then $\omega_1^x \neq \omega_1^y$.*

Given $x \in \omega^\omega$, we denote by ω_1^x the smallest ordinal not computable from x. We say that x is *self-constructible* if $x \in L_{\omega_1^x}$.

Lemma

Assume V = L*. Let* $X \subseteq \omega^\omega$ *be as in the previous lemma and set* $Y = \omega^\omega \setminus X$*. Then:*

- X *and* Y *are* $\mathfrak{c}$*-crowded,*
- X *is* $\mathfrak{c}$*-hereditarily rigid,*
- X *is not* σ*-homogeneous,*
- Y *is rigid but not* $\mathfrak{c}$*-hereditarily rigid,*
- Y *is not* σ*-homogeneous with Borel witnesses.*

Theorem

Assume V = L*. Then there exists a zero-dimensional coanalytic space that is not* σ*-homogeneous.*

Theorem (van Engelen, Miller, Steel, 1987)

Assume V = L*. Then there there exist both analytic and coanalytic examples of zero-dimensional rigid spaces.*

Proof that X is $\mathfrak{c}$-hereditarily rigid

Pick a $\mathfrak{c}$-crowded subspace S of X, and let $h : S \longrightarrow S$ be a homeomorphism. By Lavrentieff's Lemma, we can fix a homeomorphism $\widetilde{h} : G \longrightarrow G$ that extends h, where $G \in \mathbf{\Pi}^0_2(\omega^\omega)$. Pick a countable ordinal δ such that $\widetilde{h}$ is coded in L_δ.

Pick $x \in S$ such that $\omega_1^x \geq \delta$. (Notice that, by the injectivity condition, all but countably many elements of S have this property.) Observe that $x \in \mathsf{L}_{\omega_1^x}$ by self-constructibility.
Set $y = h(x) = \widetilde{h}(x)$, and observe that $y \in \mathsf{L}_{\omega_1^x}$.
Since $\omega_1^x \notin \mathsf{L}_{\omega_1^x}$, it follows that ω_1^x is not computable from y.
In conclusion, we see that $\omega_1^y \leq \omega_1^x$.

A similar argument, applied to $\widetilde{h}^{-1}$, shows that $\omega_1^x \leq \omega_1^y$.
Therefore $\omega_1^x = \omega_1^y$, hence $x = y$ by the injectivity condition.
Since S is $\mathfrak{c}$-crowded, this shows that h is the identity on S.

Two more open questions

Question

Is every analytic zero-dimensional space σ-homogeneous?

Theorem (Medini, van Mill, Zdomskyy, 2016)

There exists a ZFC *example of a subspace X of 2^ω with the following properties, where $Y = 2^\omega \setminus X$:*

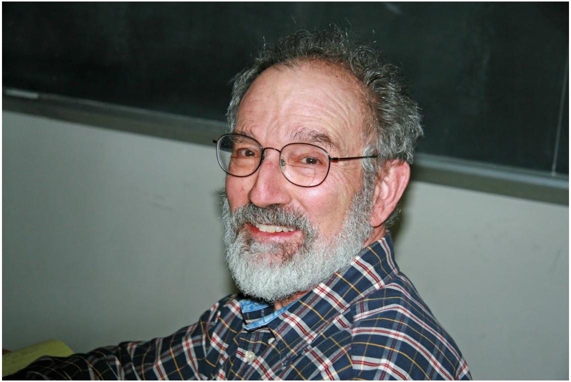
- *X is Bernstein,*
- *X is rigid,*
- *Y is homogeneous.*

It turns out that such an X cannot be $\mathfrak{c}$-hereditarily rigid. But:

Question

Under $\mathsf{V} = \mathsf{L}$*, is there a coanalytic zero-dimensional rigid space that is not $\mathfrak{c}$-hereditarily rigid?*

Kenneth Kunen (1943-2020)



"Thanks to my advisor Ken Kunen for all the wonderful lectures, several useful contributions to my research, and his overall no-nonsense approach."