The topology of ultrafilters as subspaces of 2^ω

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All ultrafilters are non-principal and on ω.
By identifying a subset of ω with an element of 2^ω in the obvious way, we can view any ultrafilter $\mathcal{U}$ as a subspace of 2^ω.

Proposition (folklore)

There are $2^{\mathfrak{c}}$ non-homeomorphic ultrafilters.

Proof.

Using Lavrentiev's lemma, one sees that the homeomorphism classes have size $\mathfrak{c}$. □

The above proof is a cardinality argument: it is not 'honest' in the sense of Van Douwen. ☹
It would be desirable to get 'quotable' topological properties that distinguish ultrafilters up to homeomorphism.

The distinguishing properties

From now on, all spaces are separable and metrizable.
Recall the following definitions.

Definition

- A space X is *completely Baire* if every closed subspace of X is a Baire space.
- A space X is *countable dense homogeneous* if for every pair (D, E) of countable dense subsets of X there exists a homeomorphism $h : X \longrightarrow X$ such that $h[D] = E$.
- Given a space X, a subset A of X has the *perfect set property* if A is countable or A contains a homeomorphic copy of 2^ω.

Main results

Theorem

Assume MA(countable). Let P be one of the following topological properties.

- *P = being completely Baire.*
- *P = countable dense homogeneity.*
- *P = every closed subset has the perfect set property.*

Then there exist ultrafilters $\mathcal{U}, \mathcal{V} \subseteq 2^\omega$ such that $\mathcal{U}$ has property P and $\mathcal{V}$ does not have property P. ☺

Question

Can the assumption of MA(countable) be dropped?

Kunen's closed embedding trick

Theorem (Kunen, private communication)

Let C be a zero-dimensional space. Then there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ with a closed subspace homeomorphic to C.

By choosing $C = \mathbb{Q}$ or $C =$ a Bernstein set one obtains the following corollaries.

Corollary

There exists an ultrafilter $\mathcal{V} \subseteq 2^\omega$ that is not completely Baire.

Corollary

There exists an ultrafilter $\mathcal{V} \subseteq 2^\omega$ with a closed subset that does not have the perfect set property.

Proof of Kunen's trick

Lemma (folklore)

There exists a perfect set $P \subseteq 2^\omega$ such that P is an independent family: that is, every word

$$x_1 \cap \cdots \cap x_m \cap \omega \setminus y_1 \cap \cdots \cap \omega \setminus y_n \text{ is infinite,}$$

where $x_1, \ldots, x_m, y_1, \ldots, y_n \in P$ are distinct.

Let C be the space you want to embed in $\mathcal{V}$ as a closed subset. Since $P \cong 2^\omega$, assume $C \subseteq P$. Now simply define

$$\mathcal{G} = C \cup \{\omega \setminus x : x \in P \setminus C\}.$$

Notice that $\mathcal{G}$ has the finite intersection property because P is independent. Any ultrafilter $\mathcal{V} \supseteq \mathcal{G}$ will intersect P exactly on C.

An ultrafilter that is not countable dense homogeneous

We will use Sierpiński's technique for killing homeomorphisms.

Lemma

Assume MA(countable). Fix D_1 and D_2 disjoint countable dense subsets of 2^ω such that $\mathcal{D} = D_1 \cup D_2$ is an independent family. Then there exists $\mathcal{A} \supseteq \mathcal{D}$ satisfying the following conditions.

- *$\mathcal{A}$ is an independent family.*
- *If $G \supseteq \mathcal{D}$ is a G_δ subset of 2^ω and $f : G \longrightarrow G$ is a homeomorphism such that $f[D_1] = D_2$, then there exists $x \in G$ such that $\{x, \omega \setminus f(x)\} \subseteq \mathcal{A}$.*

In the end, let $\mathcal{V}$ be any ultrafilter extending $\mathcal{A}$.

Enumerate as $\{f_\eta : \eta \in \mathfrak{c}\}$ all such homeomorphisms.
We will construct an increasing sequence of independent families $\mathcal{A}_\xi$ for $\xi \in \mathfrak{c}$. Set $\mathcal{A}_0 = \mathcal{D}$ and take unions at limit stages.
We will take care of f_η at stage $\xi = \eta + 1$, using $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.
List as $\{w_\alpha : \alpha \in \kappa\}$ all the words in $\mathcal{A}_\eta$.
It is easy to check that, for any fixed $n \in \omega$, $\alpha \in \kappa$ and $\varepsilon_1, \varepsilon_2 \in 2$,

$$W_{\alpha,n,\varepsilon_1,\varepsilon_2} = \{x \in G_\eta : |w_\alpha \cap x^{\varepsilon_1} \cap f_\eta(x)^{\varepsilon_2}| \geq n\}$$

is open dense in G_η, so comeager in 2^ω.
So pick x in the intersection of every $W_{\alpha,n,\varepsilon_1,\varepsilon_2}$.

A countable dense homogeneous ultrafilter

Any ultrafilter $\mathcal{U}$ is homeomorphic to its dual maximal ideal $\mathcal{J}$. So, for notational convenience, we will construct an increasing sequence of ideals $\mathcal{I}_\xi$, for $\xi \in \mathfrak{c}$. In the end, let $\mathcal{J}$ be any maximal ideal extending $\bigcup_{\xi\in\mathfrak{c}} \mathcal{I}_\xi$.
The idea is to use the following lemma.

Lemma

Let $f : 2^\omega \longrightarrow 2^\omega$ be a homeomorphism. Fix a maximal ideal $\mathcal{J} \subseteq 2^\omega$ and a countable dense subset D of $\mathcal{J}$. Then f restricts to a homeomorphism of $\mathcal{J}$ iff $cl(\{d + f(d) : d \in D\}) \subseteq \mathcal{J}$.

Enumerate as $\{(D_\eta, E_\eta) : \eta \in \mathfrak{c}\}$ all pairs of countable dense subsets of 2^ω. At stage $\xi = \eta + 1$, make sure that either

- $\omega \setminus x \in \mathcal{I}_\xi$ for some $x \in D_\eta \cup E_\eta$, or
- there exists an homeomorphism $f : 2^\omega \longrightarrow 2^\omega$ and $x \in \mathcal{I}_\xi$ such that $f[D_\eta] = E_\eta$ and $\{d + f(d) : d \in D_\eta\} \subseteq x\downarrow$.

To construct $f : 2^\omega \longrightarrow 2^\omega$ and x, use MA(countable) on the poset $\mathbb{P}$ consisting of all triples $p = (s, g, \pi) = (s_p, g_p, \pi_p)$ such that, for some $n = n_p \in \omega$, the following conditions hold.

- $s : n \longrightarrow 2$.
- g is a bijection between a finite subset of D and a finite subset of E.
- π is a permutation of ${}^n2$.
- $(t + \pi(t))(i) = 1$ implies $s(i) = 1$ for every $t \in {}^n2$ and $i \in n$.
- $\pi(d \upharpoonright n) = g(d) \upharpoonright n$ for every $d \in \mathrm{dom}(g)$.

Order $\mathbb{P}$ by declaring $q \leq p$ if the following conditions hold.

- $s_q \supseteq s_p$.
- $g_q \supseteq g_p$.
- $\pi_q(t) \upharpoonright n_p = \pi_p(t \upharpoonright n_p)$ for all $t \in {}^{n_q}2$.

A completely Baire ultrafilter

We will construct an increasing sequence of filters $\mathcal{F}_\xi$, for $\xi \in \mathfrak{c}$.
In the end, let $\mathcal{U}$ be any ultrafilter extending $\bigcup_{\xi\in\mathfrak{c}} \mathcal{F}_\xi$.
The idea is to use the following lemma.

Lemma (Hurewicz)

A space is completely Baire iff it does not contain any closed copies of $\mathbb{Q}$.

Enumerate as $\{Q_\eta : \eta \in \mathfrak{c}\}$ all copies of $\mathbb{Q}$ in 2^ω.
At stage $\xi = \eta + 1$, make sure that either

- $\omega \setminus x \in \mathcal{F}_\xi$ for some $x \in Q_\eta$, or
- there exists $x \in \mathcal{F}_\xi$ such that $x \in \mathrm{cl}(Q_\eta) \setminus Q_\eta$.

To construct x, use MA(countable) on
$\mathbb{P} = \{q \restriction n : q \in Q_\eta, n \in \omega\}$, ordered by reverse inclusion.

An ultrafilter $\mathcal{U}$ such that $A \cap \mathcal{U}$ has the perfect set property whenever A is analytic

Recall that a play of the *strong Choquet game* on a topological space $(X, \mathcal{T})$ is of the form

$$\begin{array}{c|cccccc} \text{I} & (q_0, U_0) & & (q_1, U_1) & & \cdots \\ \hline \text{II} & & V_0 & & V_1 & \cdots, \end{array}$$

where $U_n, V_n \in \mathcal{T}$ are such that $q_n \in V_n \subseteq U_n$ and $U_{n+1} \subseteq V_n$ for every $n \in \omega$.

Player II wins if $\bigcap_{n \in \omega} U_n \neq \varnothing$.

The topological space $(X, \mathcal{T})$ is *strong Choquet* if II has a winning strategy in the above game.

Define an A-*triple* to be a triple of the form $(\mathcal{T}, A, Q)$ such that the following conditions are satisfied.

- $\mathcal{T}$ is a strong Choquet, second-countable topology on 2^ω that is finer than the standard topology.
- $A \in \mathcal{T}$.
- Q is a non-empty countable subset of A with no isolated points in the subspace topology it inherits from $\mathcal{T}$.

For every analytic A there exists a topology $\mathcal{T}$ as above. Also, such a topology $\mathcal{T}$ necessarily consists only of analytic sets. In particular, we can enumerate all A-triples as $\{(\mathcal{T}_\eta, A_\eta, Q_\eta) : \eta \in \mathfrak{c}\}$, making sure that each A-triple appears cofinally often.

We will construct an increasing sequence of filters $\mathcal{F}_\xi$, for $\xi \in \mathfrak{c}$. Enumerate as $\{z_\eta : \eta \in \mathfrak{c}\}$ all subsets of ω.
At stage $\xi = \eta + 1$, make sure that the following conditions hold.

- Either $z_\eta \in \mathcal{F}_\xi$ or $\omega \setminus z_\eta \in \mathcal{F}_\xi$.
- If $Q_\eta \subseteq \mathcal{F}_\eta$ then there exists $x \in \mathcal{F}_\xi$ such that $x\!\uparrow \cap A_\eta$ contains a perfect subset.

Let $\mathcal{U} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$. If $A \cap \mathcal{U}$ is uncountable for some analytic A then it must have an uncountable subset S with no isolated points. Hence there exists some $Q \subseteq S$ and $\mathcal{T}$ such that $(\mathcal{T}, A, Q)$ is an A-triple. So we took care of it.

Given an A-triple $(\mathcal{T}, A, Q) = (\mathcal{T}_\eta, A_\eta, Q_\eta)$, construct x by applying MA(countable) to the following poset.
Fix a winning strategy Σ for player II in the strong Choquet game in $(2^\omega, \mathcal{T})$. Also, fix a countable base $\mathcal{B}$ for $(2^\omega, \mathcal{T})$.
Let $\mathbb{P}$ be the countable poset consisting of all functions p such that for some $n = n_p \in \omega$ the following conditions hold.

- $p : {}^{\leq n}2 \longrightarrow Q \times \mathcal{B}$. We will use the notation $p(s) = (q_s^p, U_s^p)$.
- $U_\varnothing^p = A$.
- For every $s, t \in {}^{\leq n}2$, if s and t are incompatible (that is, $s \nsubseteq t$ and $t \nsubseteq s$) then $U_s^p \cap U_t^p = \varnothing$.

- For every $s \in {}^n 2$,

$$\begin{array}{c|cccccc} \text{I} & (q^p_{s\restriction 0}, U^p_{s\restriction 0}) & & \cdots & (q^p_{s\restriction n}, U^p_{s\restriction n}) & \\ \hline \text{II} & & V^p_{s\restriction 0} & \cdots & & V^p_{s\restriction n} \end{array}$$

  is a partial play of the strong Choquet game in $(2^\omega, \mathcal{T})$, where the open sets $V^p_{s\restriction i}$ played by II are the ones dictated by the strategy Σ.

Order $\mathbb{P}$ by setting $p \leq p'$ whenever $p \supseteq p'$.

The generic tree will naturally yield a perfect set P such that $\mathcal{F}_\eta \cup \{\bigcap P\}$ has the finite intersection property.
So set $x = \bigcap P$.

A question of Hrušák and Zamora Avilés

Hrušák and Zamora Avilés showed that, for a Borel $X \subseteq 2^\omega$, the following conditions are equivalent.

- X^ω is countable dense homogeneous.
- X is a G_δ.

Then they asked whether there exists a non-G_δ subset X of 2^ω such that X^ω is countable dense homogeneous.
The following theorem consistently answers their question.

Theorem

Assume MA(countable). Then there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $\mathcal{U}^\omega$ is countable dense homogeneous.

Extending the perfect set property

Under V=L, there exists a co-analytic subset of 2^ω without the perfect set property. So MA(countable) is not enough to extend the perfect set property to $\mathcal{U} \cap A$ for all co-analytic A.

Theorem

Assume the consistency of a Mahlo cardinal. Then it is consistent that there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such $A \cap \mathcal{U}$ has the perfect set property for all $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$.

At least an inaccessible is needed for the above theorem.

Question

Does the Levy collapse of an inaccessible κ to ω_1 force such an ultrafilter?

P-points and completely Baire ultrafilters

We constructed the following examples.

	P-point	non-P-point
cB	✓	?
non-cB	?	✓

Question

For a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$, is being a P-point equivalent to being completely Baire?

P-points and the perfect set property

We constructed the following examples.

	P-point	non-P-point
psp	✓	?
non-psp	?	✓

Question

For an ultrafilter $\mathcal{U} \subseteq 2^\omega$, is being a P-point equivalent to $\mathcal{U} \cap A$ having the perfect set property whenever $A \subseteq 2^\omega$ is analytic?

Theorem

Let $\mathcal{U}$ be a P_{ω_2}-point. Then $A \cap \mathcal{U}$ has the perfect set property whenever $A \subseteq 2^\omega$ is such that every closed subset of A has the perfect set property. (For example, whenever A is analytic).

P-points and countable dense homogeneity

We constructed the following examples.

	P-point	non-P-point
cdh	✓	✓
non-cdh	?	✓

The following is the only question left open.

Question

Is a P-point necessarily countable dense homogeneous?