http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 67, 2026

Pages 73–79

On the infinite powers of large zero-dimensional metrizable spaces

by

Andrea Medini

Electronically published on June 30, 2025

Topology Proceedings

Web:	http://topology.nipissingu.ca/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.nipissingu.ca/tp/
TOPOLOGY
PROCEEDINGS
Volume 67 (2026)
Pages 73-79

E-Published on June 30, 2025

ON THE INFINITE POWERS OF LARGE ZERO-DIMENSIONAL METRIZABLE SPACES

ANDREA MEDINI

Dedicated to the memory of Gary Gruenhage

ABSTRACT. We show that X^λ is strongly homogeneous whenever X is a non-separable zero-dimensional metrizable space and λ is an infinite cardinal. This partially answers a question of Terada, and improves a previous result of the author. Along the way, we show that every non-compact weight-homogeneous metrizable space with a π-base consisting of clopen sets can be partitioned into κ many clopen sets, where κ is the weight of X. This improves a result of van Engelen.

CONTENTS

2020 *Mathematics Subject Classification.* 54E35, 54B10, 54F45, 54F99.
Key words and phrases. Zero-dimensional, metrizable, infinite power, strongly homogeneous, h-homogeneous, clopen set, partition, π-base.
This research was funded in whole by the Austrian Science Fund (FWF) DOI 10.55776/P35588.

1. Introduction

By *space* we always mean topological space. Recall that a space X is *homogeneous* if for every $(x, y) \in X \times X$ there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$. This is a classical, well-studied notion (see the survey [1]). Also recall that a space X is *strongly homogeneous* (or *h-homogeneous*) if every non-empty clopen subspace of X is homeomorphic to X. The modifier "strongly" is motivated by the well-known fact that every zero-dimensional first-countable strongly homogeneous space is homogeneous (see [11, Proposition 3.32] for a picture-proof).

It is an interesting theme in general topology that taking infinite powers tends to improve the homogeneity properties of a space. The first instance of this phenomenon is of course the classical theorem of Keller [8] that $[0, 1]^\omega$ is homogeneous. But the situation is particularly pleasant in the zero-dimensional realm, as Lawrence [9] showed that X^ω is homogeneous for every zero-dimensional separable metrizable space X (answering the first part of Problem 387 from the book "Open Problems in Topology," which is due to Fitzpatrick and Zhou [6, Problem 4]).

In fact, in the aptly named article [2], Gruenhage asked whether X^ω is homogeneous for every zero-dimensional first-countable space X, and he obtained several partial answers in (unpublished) collaboration with Zhou (see the last paragraph of [4]). Other related results were obtained by van Engelen [4] and Medvedev [13]. The answer was finally shown to be affirmative by Dow and Pearl [3], who combined Lawrence's method with the technique of elementary submodels.

However, while the issue of homogeneity was resolved in the spectacular fashion described above, the following question [16] remains open (even for separable metrizable spaces).

Question 1.1 (Terada)**.** Is X^ω strongly homogeneous for every zero-dimensional first-countable space X?

Several partial answers to the above question are available (see [12, Section 5] for a mini-survey). In particular, the author [10, Corollary 29] proved that X^ω is strongly homogeneous for every strongly zero-dimensional non-separable metrizable space X. The aim of this article is to show that "strongly zero-dimensional" can be weakened to "zero-dimensional" (see Theorem 3.1).

We conclude this section by clarifying some terminology and notation. Our reference for general topology is [5], and our reference for set theory is [7]. A space is *zero-dimensional* if it is non-empty, T_1, and it has a base consisting of clopen sets. So a space X is zero-dimensional iff X is T_1 and $\mathsf{ind}(X) = 0$. It is easy to see that every zero-dimensional space is Tychonoff. A space X is *strongly zero-dimensional* if X is a

Tychonoff space and $\mathsf{dim}(X) = 0$. By [5, Theorem 6.2.6], every strongly zero-dimensional space is zero-dimensional. Recall that the *weight* of a space X, which we will denote by $\mathsf{w}(X)$, is the maximum between ω and the minimal cardinality of a base for X. Given a metric space X with distance d, a point $x \in X$ and a real number $\varepsilon > 0$, we will denote by $\mathsf{B}(x, \varepsilon) = \{z \in X : \mathsf{d}(z, x) < \varepsilon\}$ the *open ball* around x of radius ε.

2. Partitions into clopen sets

The aim of this section is to show that every non-compact weight-homogeneous zero-dimensional metrizable space X can be partitioned into $\mathsf{w}(X)$ many clopen sets. This result was first obtained by van Engelen [4, Lemma 2.1] under the additional assumption that X is strongly zero-dimensional.[1]

In fact, the weaker assumption that X has a π-base consisting of clopen sets will be sufficient (see Theorem 2.4). We remark that this level of generality will not be needed in the proof of Theorem 3.1. However, this assumption has proven to be a useful one (see [10] and [16]), and the amount of extra work required is rather moderate. So we decided to state our results this way.

Given a metric space X with distance d and a real number $\varepsilon > 0$, recall that $D \subseteq X$ is *ε-dispersed* if $\mathsf{d}(d, e) \geqslant \varepsilon$ whenever $d, e \in D$ and $d \neq e$. Given a space X, recall that $\mathcal{B}$ is a *π-base* for X if $\mathcal{B}$ consists of non-empty open subsets of X and for every non-empty open subset U of X there exists $V \in \mathcal{B}$ such that $V \subseteq U$.

Lemma 2.1. *Let X be a metric space. Assume that X has a π-base consisting of clopen sets. If X has an infinite ε-dispersed subset D for some $\varepsilon > 0$ then X can be partitioned into $|D|$ many clopen sets.*

Proof. Let d denote the metric on X. Fix an infinite ε-dispersed subset D of X, where $\varepsilon > 0$. We will use cl to denote closure in X. For every $d \in D$, fix a non-empty clopen subset U_d of X such that $U_d \subseteq \mathsf{B}(d, \varepsilon/4)$. It is clear that $U_d \cap U_e = \varnothing$ whenever $d, e \in D$ and $d \neq e$. Therefore, to conclude the proof, it will be enough to show that U is closed, where $U = \bigcup_{d \in D} U_d$.

Assume, in order to get a contradiction, that $x_n \in U$ for $n \in \omega$ and $x_n \to x$, but $x \notin U$. Pick $N \in \omega$ such that $\mathsf{d}(x_n, x) < \varepsilon/4$ whenever $n \geqslant N$. If there existed $d \in D$ such that $x_n \in U_d$ for every $n \geqslant N$, then we would have $x \in \mathsf{cl}(U_d) = U_d$, contradicting the assumption that $x \notin U$.

[1] At the very beginning of [4, Section 2], van Engelen assumes that all spaces are metrizable and strongly zero-dimensional.

So we can fix distinct $d, e \in D$ and $m, n \geqslant N$ such that $x_m \in U_d$ and $x_n \in U_e$. Then

$$\mathsf{d}(d,e) \leqslant \mathsf{d}(d,x_m) + \mathsf{d}(x_m,x) + \mathsf{d}(x,x_n) + \mathsf{d}(x_n,e) < 4(\varepsilon/4) = \varepsilon,$$

which contradicts the fact that D is ε-dispersed. □

Lemma 2.2. *Let X be a metrizable space, and let κ be a cardinal of uncountable cofinality. Assume that X has a π-base consisting of clopen sets. If $\kappa \leqslant \mathsf{w}(X)$ then X can be partitioned into κ many clopen sets.*

Proof. Assume that $\kappa \leqslant \mathsf{w}(X)$. Let d be a metric on X. By Zorn's Lemma, for every $n \in \omega$ we can fix a maximal 2^{-n}-dispersed subset D_n of X. It is straightforward to check that

$$\mathcal{B} = \bigcup_{n\in\omega} \{\mathsf{B}(d, 2^{-n}) : d \in D_n\}$$

is a base for X. Assume, in order to get a contradiction, that $|D_n| < \kappa$ for each n. Since κ has uncountable cofinality, it follows that

$$|\mathcal{B}| \leqslant \sum_{n\in\omega} |D_n| = \mathsf{sup}\{|D_n| : n \in \omega\} < \kappa \leqslant \mathsf{w}(X),$$

where the equality holds by [7, Lemma 5.8] and the fact that at least one D_n is infinite (otherwise X would be separable). This is clearly a contradiction, hence $|D_n| \geqslant \kappa$ for some n. An application of Lemma 2.1 concludes the proof. □

The following lemma first appeared (without proof) as [10, Lemma 3]. The proof given here is taken almost verbatim from [11, Lemma 3.3]. According to [5], a space X is *pseudocompact* if it is Tychonoff and every continuous function $f : X \longrightarrow \mathbb{R}$ is bounded. However, being Tychonoff is irrelevant to Lemma 2.3, so we state it more directly as follows. Also recall that a metrizable space is pseudocompact iff it is compact (see [5, Theorem 4.1.17] and the subsequent remark).

Lemma 2.3. *Let X be a space. Assume that X has a π-base $\mathcal{B}$ consisting of clopen sets, and that there exists an unbounded continuous function $f : X \longrightarrow \mathbb{R}$. Then X can be partitioned into infinitely many clopen sets.*

Proof. Fix a metric d on $\mathbb{R}$. Throughout this proof, we will use cl to denote closure in $\mathbb{R}$. It is a simple exercise to construct $D = \{d_n : n \in \omega\} \subseteq f[X]$ and open subsets U_n of $\mathbb{R}$ for $n \in \omega$ such that the following conditions are satisfied:

- D is a closed subset of $\mathbb{R}$,
- $d_n \in U_n$ for each n,
- $U_m \cap U_n = \varnothing$ whenever $m \neq n$.

Then set $V_n = \mathsf{B}(d_n, \varepsilon_n)$ for $n \in \omega$, where the ε_n are such that $0 < \varepsilon_n \leqslant 2^{-n}$ and $\mathsf{cl}(V_n) \subseteq U_n$.

Next, we will show that $V = \bigcup_{n\in\omega} \mathsf{cl}(V_n)$ is closed in $\mathbb{R}$. Pick $x \notin V$. Choose $N \in \omega$ such that $2^{-N} < \mathsf{d}(x, D)$, then set $W = \mathsf{B}(x, 2^{-(N+1)})$. We claim that $W \cap V_n = \varnothing$ for every $n \geqslant N + 1$. Otherwise, for an element z of such an intersection, we would have

$$\mathsf{d}(x, d_n) \leqslant \mathsf{d}(x, z) + \mathsf{d}(z, d_n) \leqslant 2^{-(N+1)} + 2^{-(N+1)} = 2^{-N} < \mathsf{d}(x, D),$$

which is a contradiction. So $W\backslash(\mathsf{cl}(V_0) \cup \cdots \cup \mathsf{cl}(V_N))$ is an open neighborhood of x that is disjoint from V.

Finally, fix $B_n \in \mathcal{B}$ for $n \in \omega$ so that each $B_n \subseteq f^{-1}[V_n]$. To conclude the proof, we will show that $B = \bigcup_{n\in\omega} B_n$ is closed. Pick $x \notin B$. If $x \in f^{-1}[U_n]$ for some $n \in \omega$, then $f^{-1}[U_n]\backslash B_n$ is an open neighborhood of x that is disjoint from B. Now assume that $x \notin \bigcup_{n\in\omega} f^{-1}[U_n]$. Then $y = f(x) \notin V$, so we can find an open neighborhood W of y that is disjoint from V. It is clear that $f^{-1}[W]$ is an open neighborhood of x that is disjoint from B. □

Recall that a space X is *weight-homogeneous* if $\mathsf{w}(U) = \mathsf{w}(X)$ for every non-empty open subspace U of X. Naturally, in the context of this article, the only relevant examples of weight-homogeneous spaces are the infinite powers.

Theorem 2.4. *Let X be a metrizable space. Assume that X is non-compact, weight-homogeneous, and has a π-base consisting of clopen sets. Then X can be partitioned into $\mathsf{w}(X)$ many clopen sets.*

Proof. Set $\kappa = \mathsf{w}(X)$. If $\kappa = \omega$, the desired conclusion follows from Lemma 2.3. On the other hand, if κ has uncountable cofinality, the desired conclusion follows from Lemma 2.2. So assume that κ is uncountable but has countable cofinality, and let κ_n for $n \in \omega$ be cardinals of uncountable cofinality such that $\mathsf{sup}\{\kappa_n : n \in \omega\} = \kappa$.

Since X is non-compact, by Lemma 2.3 we can fix non-empty clopen subsets X_n of X for $n \in \omega$ such that $\bigcup_{n\in\omega} X_n = X$ and $X_m \cap X_n = \varnothing$ whenever $m \neq n$. Notice that each $\mathsf{w}(X_n) = \kappa \geqslant \kappa_n$ by weight-homogeneity. Hence each X_n can be partitioned into κ_n many clopen sets by Lemma 2.2. To conclude the proof, simply consider the union of these partitions. □

It is clear from the above proof that the assumption of weight-homogeneity is only used in the case when $\mathsf{w}(X)$ is uncountable of countable cofinality. Of course, it would be nice to eliminate it altogether.

Question 2.5. Is it possible to drop the the assumption of weight-homogeneity in Theorem 2.4?

3. The main result

As we mentioned in the introduction, the following result shows that the assumption of strong zero-dimensionality in [10, Corollary 29] can be weakened to mere zero-dimensionality. Recall that a space X is *strongly divisible* by 2 if $X \times 2$ is homeomorphic to X, where 2 is the discrete space with two elements.

Theorem 3.1. *Let X be a non-separable zero-dimensional metrizable space, and let λ be an infinite cardinal. Then X^λ is strongly homogeneous.*

Proof. Since strong homogeneity is productive in the zero-dimensional realm (see [10, Corollary 14]) and X^λ is homeomorphic to $(X^\omega)^\lambda$, it will be enough to show that X^ω is strongly homogeneous. By Theorem 2.4, we can fix an uncountable cardinal κ and non-empty clopen subsets X_α of X^ω for $\alpha \in \kappa$ such that $\bigcup_{\alpha\in\kappa} X_\alpha = X^\omega$ and $X_\alpha \cap X_\beta = \varnothing$ whenever $\alpha \neq \beta$. Pick $x \in X^\omega$ and a local base $\{U_n : n \in \omega\}$ for X^ω at x consisting of clopen sets. Since X^ω is homogeneous by [3], for every $\alpha \in \kappa$ there exist $n(\alpha) \in \omega$ and a clopen subspace V_α of X^ω such that $V_\alpha \subseteq X_\alpha$ and V_α is homeomorphic to $U_{n(\alpha)}$. Since κ is uncountable, there must be an uncountable $I \subseteq \kappa$ and $n \in \omega$ such that $n(\alpha) = n$ for all $\alpha \in I$. Set $V = \bigcup_{\alpha\in I} V_\alpha$, and observe that V is a non-empty clopen subspace of X^ω that is strongly divisible by 2. The desired conclusion then follows from [10, Proposition 24]. □

We conclude by observing that there might be a more systematic way of proving Theorem 3.1. The following result is [14, Theorem 5] (see also [15, Theorem 6]).[2]

Theorem 3.2 (Medvedev)**.** *Let X be a strongly zero-dimensional metrizable space. Assume that $\mathsf{w}(X)$ has uncountable cofinality and that X is weight-homogeneous. If X is homogeneous then X is strongly homogeneous.*

Notice that the assumption of weight-homogeneity in the above result cannot be dropped. To see this, simply consider $\kappa \times X$, where X is a strongly zero-dimensional homogeneous metrizable space and $\kappa > \mathsf{w}(X)$ is a cardinal of uncountable cofinality with the discrete topology. Furthermore, as $\omega \times 2^\omega$ shows, the assumption that $\mathsf{w}(X)$ has uncountable cofinality cannot be altogether dropped. However, we do not know the

[2] At the very beginning of [14], Medvedev assumes that all spaces are metrizable. Furthermore, it is well-known that $\mathsf{Ind}(X) = \mathsf{dim}(X)$ for every metrizable space X (see [5, Theorem 7.3.2]). Regarding [15, Theorem 6], although the assumption $\mathsf{ind}(X) = 0$ appears in its statement, we remark that the stronger assumption $\mathsf{dim}(X) = 0$ is in fact used.

answers to the following questions. As we hinted at above, affirmative answers to both questions would yield a better proof of Theorem 3.1 as a by-product.

Question 3.3. Is it possible to weaken "has uncountable cofinality" to "is uncountable" in Theorem 3.2?

Question 3.4. Is it possible to weaken "strongly zero-dimensional" to "zero-dimensional" in Theorem 3.2?

References

[1] A. V. Arhangel'skii, J. van Mill. *Topological homogeneity.* Recent Progress in General Topology III. Atlantis Press, 2014. 1–68.

[2] Z. Balogh, S. W. Davis, A. Dow, G. Gruenhage, P. J. Nyikos, M. E. Rudin, F. D. Tall, S. Watson. *New classic problems.* Topology Proc. **15** (1990), 201–220.

[3] A. Dow, E. Pearl. *Homogeneity in powers of zero-dimensional first-countable spaces.* Proc. Amer. Math. Soc. **125** (1997), 2503–2510.

[4] F. van Engelen. *On the homogeneity of infinite products.* Topology Proc. **17** (1992), 303–315.

[5] R. Engelking. *General topology.* Revised and completed edition. Sigma Series in Pure Mathematics, vol. 6. Heldermann Verlag, Berlin, 1989.

[6] B. Fitzpatrick Jr., H. X. Zhou. *Some open problems in densely homogeneous spaces.* Open problems in topology. North-Holland, Amsterdam, 1990. 251–259.

[7] T. Jech. *Set theory.* The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[8] O. H. Keller. *Die Homoiomorphie der kompakten konvexen Mengen in Hilbertschen Raum.* Math. Ann. **105** (1931), 748–758.

[9] L. B. Lawrence. *Homogeneity in powers of subspaces of the real line.* Trans. Amer. Math. Soc. **350:8** (1998), 3055–3064.

[10] A. Medini. *Products and h-homogeneity.* Topology Appl. **158:18** (2011), 2520–2527.

[11] ______. *The topology of ultrafilters as subspaces of the Cantor set and other topics.* Ph.D. Thesis. University of Wisconsin - Madison. ProQuest LLC, Ann Arbor, MI, 2013.

[12] A. Medini, J. van Mill, L. Zdomskyy. *Infinite powers and Cohen reals.* Canad. Math. Bull. **61:4** (2018), 812–821.

[13] S. V. Medvedev. *On properties of h-homogeneous spaces with the Baire property.* Topology Appl. **159:3** (2012), 679–694.

[14] ______. About closed subsets of spaces of first category. Topology Appl. **159:8** (2012), 2187–2192.

[15] ______. Homogeneity and h-homogeneity. Topology Appl. **160:18** (2013), 2523–2530.

[16] T. Terada. *Spaces whose all nonempty clopen subsets are homeomorphic.* Yokohama Math. Jour. **40** (1993), 87–93.

Institut für Diskrete Mathematik und Geometrie
Technische Universität Wien
Wiedner Hauptstrasse 8–10/104
1040 Vienna, Austria
Email address: andrea.medini@tuwien.ac.at
URL: http://www.dmg.tuwien.ac.at/medini/