Van Douwen's problem on the cellularity of compact homogeneous spaces

(Specialty exam)

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All spaces we consider are Hausdorff.
A space is homogeneous if all points "look alike":

Definition

A topological space X is *homogeneous* if for every $x, y \in X$ there exists an autohomeomorphism f of X such that $f(x) = y$.

Examples:

- $\mathbb{R}$. (Actually, any topological group: just translate!)
- The open interval $(0, 1)$.
- Any discrete space.
- Any product of homogeneous spaces. (Just take the product of the autohomeomorphisms.)
- 2^κ for any cardinal κ.

Non-examples:

- The closed interval $[0, 1]$. (Proof: removing $\frac{1}{2}$ makes it disconnected, while removing 0 doesn't.)
- $[0, 1]^n$ for any finite n. (Idea: still, the points on the boundary are different from the ones in the interior.)
- The one-point compactification of an infinite discrete space. (The point at ∞ is not isolated!)
- The Stone-Čech compactification of the natural numbers $\beta\mathbb{N}$. (Trivial: the points of $\mathbb{N}$ are isolated!)
- The Stone-Čech remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. (Hard!)

Two surprising results

Theorem (Keller, 1931)

The Hilbert cube $[0,1]^\omega$ *is homogeneous.*

In the same spirit, based on work of Lawrence and Motorov:

Theorem (Dow-Pearl, 1997)

If X *is first countable and zero-dimensional then* X^ω *is homogeneous.*

Sometimes we need to take large product before getting homogeneity: Ridderbos observed that $(2^\kappa \bigoplus 2)^\lambda$ is homogeneous if and only if $\lambda \geq \kappa$.

Some cardinal functions

- Character:

$$\chi(X) = \sup_{x \in X} \chi(x, X), \text{ where}$$

$$\chi(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a local base for } X \text{ at } x\}.$$

X is *first-countable* means $\chi(X) \leq \omega$.

- Weight:

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\}.$$

X is *second-countable* means $w(X) \leq \omega$.

- Density:

$$d(X) = \min\{|D| : D \text{ is a dense subset of } X\}.$$

  X is *separable* means $d(X) \leq \omega$.

- Cellularity:

$$c(X) = \sup\{|\mathcal{C}| : \mathcal{C} \text{ is a cellular family on } X\},$$

  where a *cellular family* on X is a collection of disjoint non-empty open subsets of X.
  X has the *countable chain condition* (ccc) means $c(X) \leq \omega$.

Results that bound the cellularity

The product of a large number of separable spaces need not be separable (it is not, provided there are more than $2^{\aleph_0}$ non-trivial factors). However, the following holds:

Theorem

Any product of separable spaces is ccc.

More generally:

Theorem

If $d(X_i) \leq \lambda$ for every $i \in I$ then

$$c\left(\prod_{i \in I} X_i\right) \leq \lambda.$$

Theorem

$$c\left(\prod_{i\in I} X_i\right) = \sup\left\{c\left(\prod_{i\in F} X_i\right) : F \text{ finite subset of } I\right\}$$

Observe that one cannot just look at single factors: a Suslin line has the ccc but its square doesn't.

Theorem (Arhangel'skiĭ,1969)

Any compact space X satisfies $|X| \leq 2^{\chi(X)}$. In particular, any first-countable compact space has size (hence cellularity) at most $2^{\aleph_0}$.

The above theorems show that the compact homogeneous spaces produced by the Dow-Pearl theorem have cellularity at most $2^{\aleph_0}$.

Observe that, just by looking at $X = 2^\kappa$, we can get compact homogeneous spaces of arbitrarily large size, weight, character and density (to get $d(X) > \lambda$, choose $\kappa > 2^\lambda$).
Since 2 is separable, $c(X) \leq \aleph_0$.

Theorem (Maurice, 1964)

The compact LOTS 2^γ_{lex} is homogeneous if and only if γ is a countable indecomposable ordinal or γ is finite.

On the other hand, it's easy to get $2^{\aleph_0}$ disjoint open sets in 2^γ_{lex} whenever $\gamma \geq \omega + 2$.

- So $2^{\omega^2}_{\mathrm{lex}}$ is an example of compact homogeneous space of cellularity $2^{\aleph_0}$.
- Another example is given by X^ω, where $X = 2^{\omega+2}_{\mathrm{lex}}$ is the double of the double of the Cantor set.

Can we do better?

Problem (Van Douwen, 1970s)

Is there a compact homogeneous space of cellularity bigger than $2^{\aleph_0}$?

Van Douwen's problem has been open, under any set-theoretical assumption, for more than 30 years.

> *It would already be a fantastic achievement if it could be shown that every homogeneous compactum has cellularity at most, say, $\beth_{\omega_1}$.*
>
> JAN VAN MILL

We can easily rule out some classes of spaces:

- First-countable spaces. (By Arhangel'skiĭ's theorem.)
- Topological groups. (Because of the Haar measure.)

P-points

Definition

Let $x \in X$.

- x is a *P-point* if the intersection of countably many neighborhoods of P is still a neighborhood of x.
- x is a *weak P-point* if no countable subset of X has x as limit point.

Every P-point is a weak P-point.

Lemma

In an infinite compact space, not every point is a weak-point.

Proof: otherwise, consider a countably infinite discrete subset.

LOTS won't work

Recall that a linearly ordered topological space (LOTS) is compact iff every subset has a least upper bound and a greatest lower bound. (Proof: just like $[0, 1]$.)

Theorem

Every homogeneous compact LOTS X is first countable.

Proof: by homogeneity, it suffices to prove that X has a countable local base at its biggest element b.
Assume that

$$\mathrm{cof}(X \setminus \{b\}) = \chi(b, X) > \aleph_0.$$

Then b is a P-point.
Hence every point is a P-point by homogeneity.
Hence X is finite by compactness. ☕

The non-homogeneity of $\mathbb{N}^*$

Rudin showed that $\mathbb{N}^*$ is not homogeneous under CH:

Theorem (W. Rudin,1954)

CH implies that there is a P-point in $\mathbb{N}^*$.

In 1967, Frolík proved that $\mathbb{N}^*$ is not homogeneous in ZFC. However, he didn't exhibit points such that there is a quotable topological property distinguishing them. ☹

Theorem (Kunen, 1978)

There is a weak P-point in $\mathbb{N}^*$. ☺

Actually, one can find weak P-points $p, q \in \mathbb{N}^*$ that are *Rudin-Keisler incomparable*: $p \not\preccurlyeq q$ and $q \not\preccurlyeq p$.
By $p \preccurlyeq q$ we mean that for some $f : \mathbb{N} \longrightarrow \mathbb{N}$

$$q = \{f^{-1}[S] : S \in p\}.$$

F-spaces

Definition

A normal space is an *F-space* if disjoint F_σ open sets have disjoint closures.

Examples:

- Any discrete space.
- The Stone-Čech compactification of any F-space.
- The Stone space of any complete boolean algebra.
- Any closed subset of a normal F-space.

Basic properties of p-limits

Definition

Let $x \in X$ and $p \in \mathbb{N}^*$. We say that x is the *p-limit* of the sequence $\langle d_n : n \in \omega \rangle$ if for every neighborhood N of x the set $\{n \in \omega : d_n \in N\}$ belongs to p.

- The p-limit is unique. (Proof: there are no disjoint sets in an ultrafilter.)
- The p-limit of a discrete sequence $\langle d_n : n \in \omega \rangle$ in a compact space exists. (Proof: by the maximality of the Stone-Čech compactification, there is a surjective continuous function $\phi : \beta\mathbb{N} \longrightarrow \mathrm{cl}(\{d_n : n \in \omega\})$ such that $\phi(n) = d_n$. Then $\phi(p) = \lim_p \langle d_n : n \in \omega \rangle$.)

Nicely separating neighborhoods

Let U_n be open and $d_n \in U_n$ for $n \in \omega$. We say that the U_n *nicely separate* the d_n if for every $A \subseteq \omega$

$$\text{cl}\left(\bigcup_{n\in A} U_n\right) \cap \text{cl}\left(\bigcup_{n\notin A} U_n\right) = \varnothing.$$

- In an F-space, every discrete sequence $\langle d_n : n \in \omega \rangle$ can be nicely separated. (Proof: choose every U_n to be F_σ.)
- If the d_n are nicely separated in a compact space, then $\text{cl}(\{d_n : n \in \omega\})$ is homeomorphic to $\beta\mathbb{N}$. (Proof: get the continuous surjection, then prove that it is also injective.)

The Rudin-Frolík method

Theorem

No infinite compact F-space X is homogeneous.

Proof: fix p and q Rudin-Keisler incomparable weak-P-points in $\mathbb{N}^*$ and a discrete sequence $\langle d_n : n \in \omega \rangle$ in X.
We will show that the p-limit of $\langle d_n : n \in \omega \rangle$ cannot be the q-limit of any non-trivial sequence.
More precisely we will show that if

$$x = \lim_p \langle d_n : n \in \omega \rangle = \lim_q \langle e_n : n \in \omega \rangle$$

then $\{n : e_n = x\} \in q$.

We identify each d_n with n, so that $\mathrm{cl}(\{d_n : n \in \omega\})$ can be identified with $\beta\mathbb{N}$, and x with p.
Now fix F_σ sets U_n that nicely separate the d_n and are disjoint from $\mathbb{N}^*$. [*Draws a nice picture on the board.*]
Now define the following partition of ω:

- $A = \{n : e_n \in \mathbb{N}^*\}$
- $B = \{n : e_n \in \bigcup_{n\in\omega} U_n\}$
- $C = \omega \setminus (A \cup B)$

We will show that $B \in q$ and $C \in q$ lead to contradictions.
Case 1: $A \in q$. Then $\{n : e_n = x\} \in q$, otherwise

$$\{n : e_n \neq x\} \cap A \in q,$$

contradicting the fact that p is a weak-P-point.

Case 2: $B \in q$. Define

$$\begin{aligned} f \quad : \quad & B \longrightarrow \omega \\ & n \longmapsto m \text{ such that } e_n \in U_m. \end{aligned}$$

Claim: $q = \{f^{-1}[S] : S \in p\}$, contradicting RK-incomparability.
It suffices to show that $S \in q$ implies $f[S] \in p$.
So let $S \in q$. We want

$$p \in \mathrm{cl}(f[S]) = \mathrm{cl}\left(\bigcup_{n \in f[S]} U_n\right) \cap \beta\mathbb{N}.$$

(The $=$ follows from X being an F-space.)
But that follows from

$$p \in \mathrm{cl}(\{e_n : n \in S \cap B\}) \subseteq \mathrm{cl}\left(\bigcup_{m \in f[S \cap B]} U_m\right).$$

Case 3: $C \in q$. One can inductively construct F_σ sets V_n for every $n \in \omega$ and W_n for every $n \in C$ such that

- $d_n \in V_n$ for every $n \in \omega$,
- $e_n \in W_n$ for every $n \in C$,
- $\{V_n : n \in \omega\} \cup \{W_n : n \in C\}$ is a disjoint family.

But then

$$x \in \mathrm{cl}\left(\bigcup_{n \in \omega} V_n\right) \cap \mathrm{cl}\left(\bigcup_{n \in C} W_n\right)$$

contradicting the definition of F-space. ☕

Sequentially small spaces

Definition

A topological space X is *sequentially small* if for every infinite subset A of X there exists an infinite $B \subseteq A$, such that $\beta\mathbb{N}$ does not embed in $\text{cl}(B)$.

Examples:

- Any compact metric (= second countable) space.
- Actually, any sequentially compact space, such as a compact LOTS. ($\beta\mathbb{N}$ contains no non-trivial convergent sequences.)
- Any space of weight less than $2^{\aleph_0}$. (Since $w(\beta\mathbb{N}) = 2^{\aleph_0}$.)
- Actually, any space of character less than $2^{\aleph_0}$. (Since by a theorem of Pospíšil, there is a point of character $2^{\aleph_0}$ in $\beta\mathbb{N}$.)

Non-homogeneity of products

One might hope to construct a large homogeneous compact space by taking products. Using Frolík's method, Kunen showed that the factors should be chosen very carefully.

Theorem (Kunen, 1990)

Assume $X = \prod_{i \in I} X_i$ is a product of infinite F-spaces, spaces containing a weak P-point, and spaces containing a sequentially small open subset. Also assume that X_i is an infinite F-space for at least one index $i \in I$. Then X is not homogeneous.

☢ F-spaces are *intoxicatingly* non-homogeneous... ☢

Continuous images of homogeneous spaces

Since every compact metric space is a continuous image of the Cantor set, the following is a natural question:

Problem

Is every compact space the continuous image of a compact homogeneous space?

If the answer is 'yes' (at least for a compact space of cellularity bigger than $2^{\aleph_0}$) then the answer is 'yes' to Van Douwen's question as well.

Theorem (Motorov,1985)

Not every compact space is a retract of a compact homogeneous space.

Retracts of homogeneous spaces

The *representable cellularity induced by q* (we will not care about the nature of q) on X is a function

$$F : X \longrightarrow \{\text{closed subsets of } X\}$$

satisfying certain properties, among which the following.

- $x \in F(x)$ for every $x \in X$.
- If $y \in F(x)$ then $F(y) \subseteq F(x)$.

The closed sets $F(x)$ are called the *terms* of the cellularity F. An example is given by

$$F(x) = \text{ smallest closed superset of } \{x\} \text{ closed under paths.}$$

Theorem (Arhangel'skiĭ,1985)

A compact space X is homogeneous iff every representable cellularity on X has disjoint terms.

The following lemma shows that representable cellularities behave well with respect to retractions.

Lemma (Arhangel'skiĭ,1985)

Assume $Y \subseteq X$ and let $r : X \longrightarrow Y$ be a retraction. If F and G are the representable cellularities induced by the same q on X and Y respectively, then $G(x) = F(x) \cap Y$ for every $x \in Y$.

In particular, representable cellularities with disjoint terms remain so after retractions.
Hence the following spaces are not retracts of any compact homogeneous space.

- Motorov's original example: the closure of the graph of $\sin(1/x)$ for $x \in (0, 1]$.
- The one-point compactification of the long line: the ordered space $\omega_1 \times_{\text{lex}} [0, 1) + 1$.

Van Douwen's inequality

Definition

A collection $\mathcal{B}$ of non-empty open subsets of a space X is a *π-base* if for every open set U there exists $V \in \mathcal{B}$ such that $V \subseteq U$. The *π-weight* of X is defined by

$$\pi(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi\text{-base for } X\}.$$

The π-weight can be strictly less than the weight: for example $\aleph_0 = \pi(\beta\mathbb{N}) < w(\beta\mathbb{N}) = 2^{\aleph_0}$.

Theorem (Van Douwen, 1978)

If some power of X is homogeneous then $|X| \leq 2^{\pi(X)}$.

De la Vega's inequality

Definition

The *tightness* $t(x, X)$ of x in X is the least cardinal κ such that for every $Y \subseteq X$

$$\text{if } x \in \text{cl}(Y) \text{ then there exists } Z \in [Y]^{\leq\kappa} \text{ such that } x \in \text{cl}(Z).$$

As usual, we define $t(X) = \sup_{x \in X} t(x, X)$.

Theorem (De la Vega, 2005)

If X is compact and homogeneous then $|X| \leq 2^{t(X)}$.

Arhangel'skiĭ, Van Mill and Ridderbos (2007) showed that we can substitute 'homogeneous' with 'some power of X is homogeneous'.

Theorem (Čech-Pospíšil)

If X is compact and $\chi(x, X) \geq \kappa$ for every $x \in X$ then $|X| \geq 2^\kappa$.

Kunen and Hart observed that the above theorem plus Arhangel'skiĭ's theorem imply $|X| = 2^{\chi(X)}$ whenever X is a compact homogeneous space.

Problem (Arhangel'skiĭ,1987)

Is every homogeneous compact space of countable tightness first-countable?

De la Vega's inequality implies that the answer is consistently 'yes': under CH,

$$2^{\chi(X)} = |X| \leq 2^{t(X)} = 2^{\aleph_0} = \aleph_1.$$

Could it be independent?

Sure, why not?

By carefully glueing together many copies of the ω_1-torus along a Cantor set, Van Mill obtained a space whose homogeneity is independent of ZFC.

Theorem (Van Mill, 2003)

There is a compact space X such that X is homogeneous under $MA + \neg CH$ but not homogeneous under CH.

The space X satisfies $\pi(X) = \aleph_0$ and $\chi(X) = \aleph_1$.

Therefore, under CH, non-homogeneity follows from Van Douwen's inequality: otherwise we would have

$$2^{\aleph_1} = 2^{\chi(X)} = |X| \leq 2^{\pi(X)} = 2^{\aleph_0} = \aleph_1.$$