Lattice Path Conference
21-25 June 2021
Presentation times for this poster:

Monday	1:30-2:30 pm
Tuesday	1:30-2:30 pm

Latticepathology and symmetric functions

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Lattice paths

- **Step set:** $\mathcal{S} = \{s_1, s_2, \dots, s_m\} \subset \mathbb{Z}$
- largest left and right steps: $-c := \min \mathcal{S}$ and $d := \max \mathcal{S}$
- ***n*-step lattice path:** sequence of steps $(v_1, \dots, v_n) \in \mathcal{S}^n$
  $\rightsquigarrow$ can be seen as a *directed lattice path* in $\mathbb{N} \times \mathbb{Z}$

New tool for lattice path surgery: prime walks

- Set $\mathcal{A}_k$ of arches $=$ walks starting at 0, ending at altitude k, and staying always strictly above altitude k except for its first and final position.
- The set $\mathcal{P}$ of *prime walks* is defined as the following sets of arches

$$\mathcal{P} = \bigcup_{k=0}^{d} \mathcal{A}_k.$$

Theorem (Universal context-free grammar)

Meanders and excursions are generated by the following grammar:

$$\mathcal{M} \to \varepsilon + \mathcal{P}\,\mathcal{M} \qquad \text{(meanders)},$$
$$\mathcal{E} \to \varepsilon + \mathcal{A}_0\mathcal{E} \qquad \text{(excursions)},$$

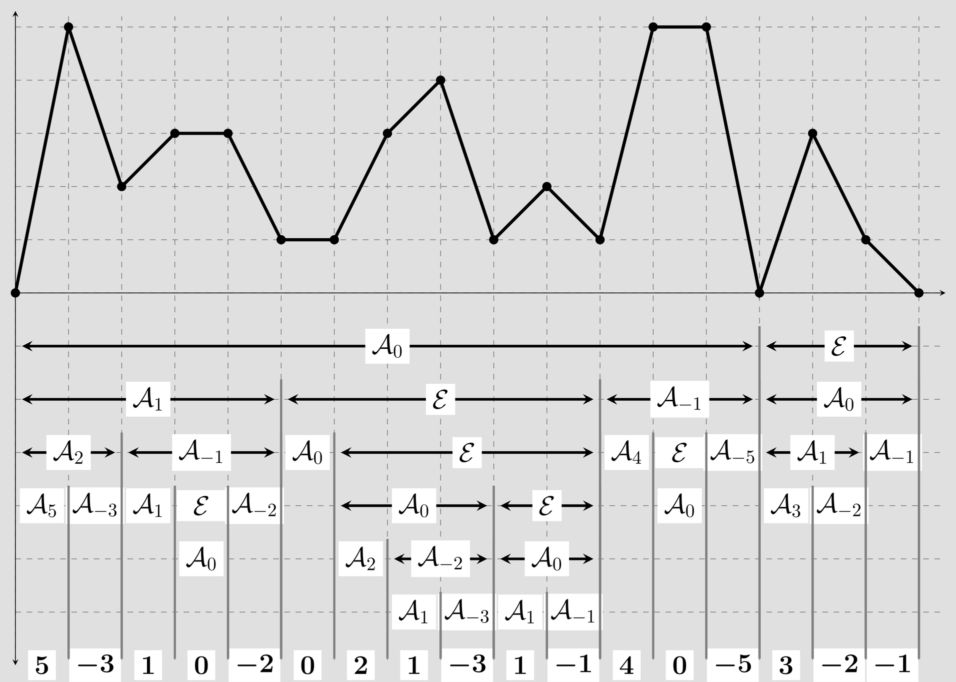
i.e., "meanders are sequences of prime walks": $\mathcal{M} = \mathrm{Seq}\left(\sum_{k=0}^{d} \mathcal{A}_k\right)$
and "excursions are sequences of arches": $\mathcal{E} = \mathrm{Seq}(\mathcal{A}_0)$,
where the arches $\mathcal{A}_k$ from 0 to k are generated by

$$\mathcal{A}_k \to k + \sum_{j=k+1}^{d} \mathcal{A}_j\,\mathcal{E}\,\mathcal{A}_{k-j} \qquad \text{(arches for } k \geq 0),$$

$$\mathcal{A}_k \to k + \sum_{j=-c}^{k-1} \mathcal{A}_{k-j}\,\mathcal{E}\,\mathcal{A}_j \qquad \text{(arches for } k < 0),$$

with the convention that the part $\mathcal{A}_k \to k$ is omitted whenever $k \notin \mathcal{S}$.

Prime walk decomposition



Theorem (Bivariate Spitzer/Sparre Andersen's identities)

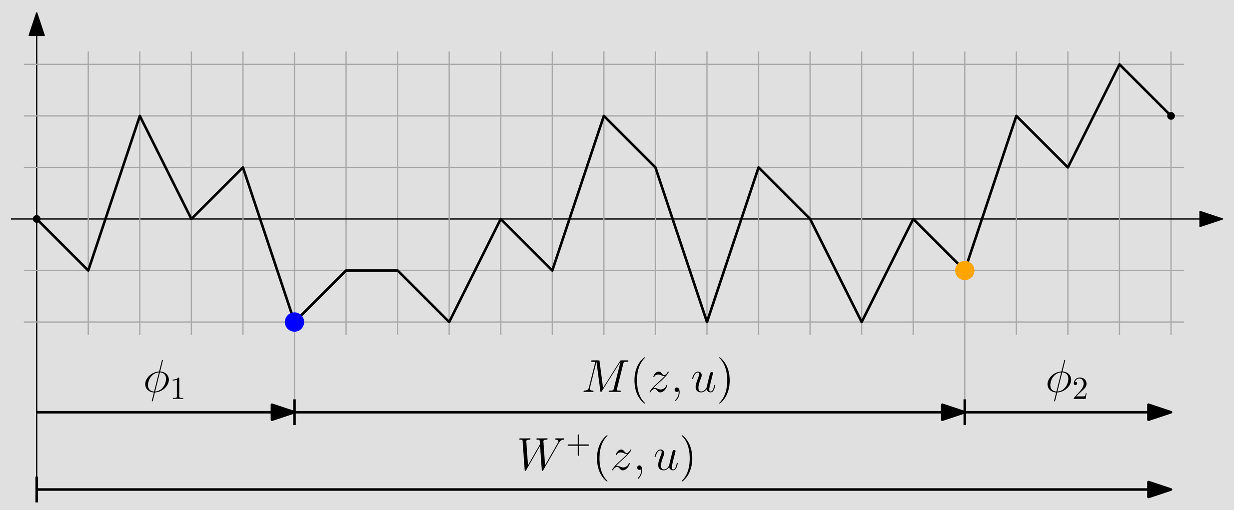
The GF $W^+(z,u) = \sum_n w_n^+(u) z^n$ of walks ending at a positive altitude and the GF $M(z,u) = \sum_n m_n(u) z^n$ of meanders (where u encodes the final altitude and z encodes the length) are related by the formula

$$M(z,u) = \exp\left(\int_0^z \frac{W^+(t,u)-1}{t}\,dt\right) = \exp\left(\sum_{n\geq 1} \frac{w_n^+(u)}{n} t^n\right).$$

Proof (Spitzer/Sparre Andersen-like decomposition)

A non-empty walk $W^+(z,u)$ consists of a maximal meander $M(z,u)$ starting at the first minimum and a pointed prime walk $\phi_2.\phi_1$:

$$W^+(z,u) - 1 = M(z,u) z \frac{\partial}{\partial z}\left(1 - \frac{1}{M(z,u)}\right).$$

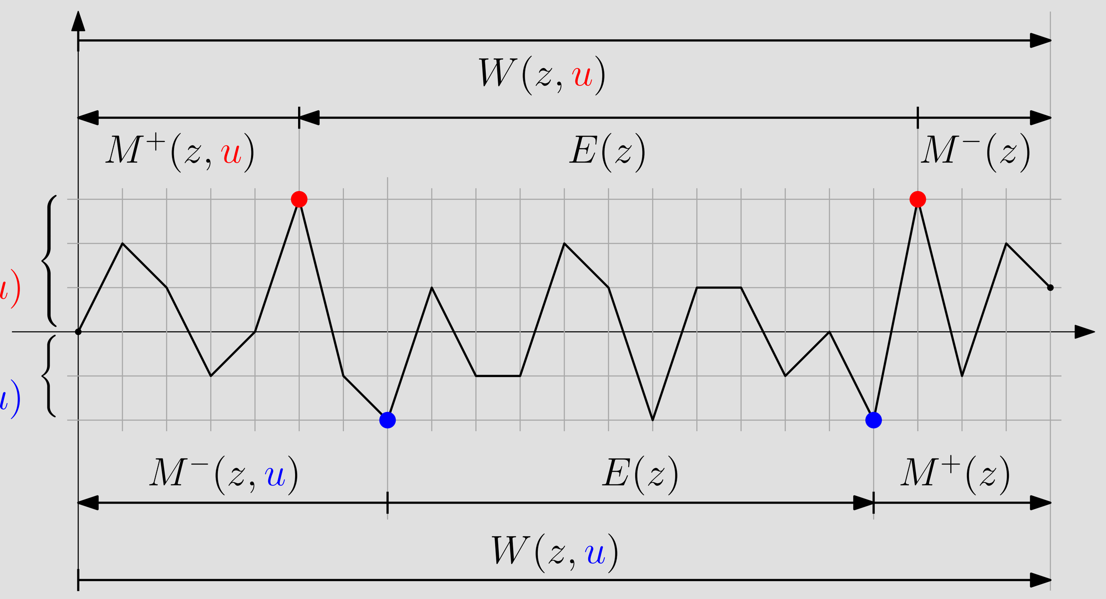


Theorem (Bivariate version of Wiener–Hopf formula)

The GFs $W_{+h}(z,u)$ and $W_{-h}(z,u)$ of walks (u marks the positive and negative height; not the altitude!) are related to the GFs $M^+(z,u)$ of positive and $M^-(z,u)$ of negative meanders (u marks the final altitude):

$$W_{+h}(z,u) = M^-(z)E(z)M^+(z,u) = \frac{-1}{s_d z}\prod_{j=1}^{c}\frac{1}{1-u_j(z)}\prod_{\ell=1}^{d}\frac{1}{u-v_\ell(z)},$$

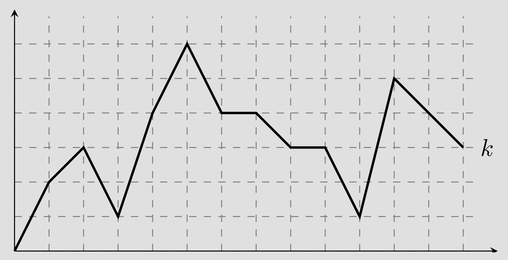
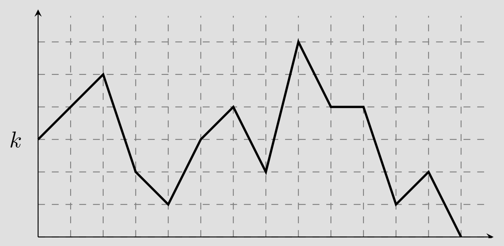
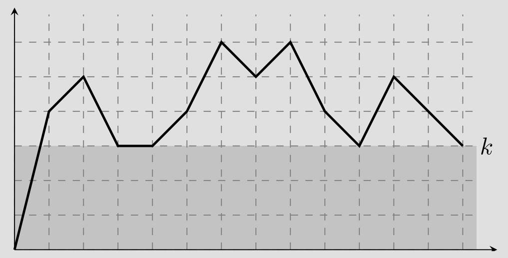
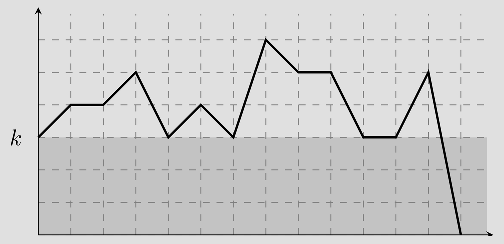
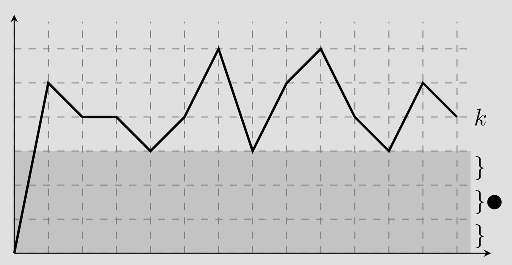
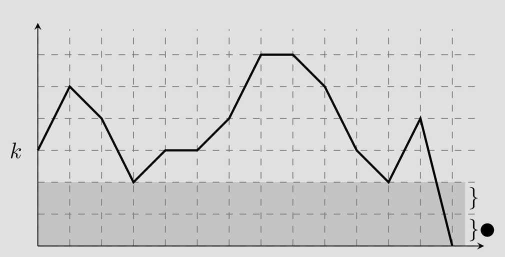
$$W_{-h}(z,u) = M^-(z,u)E(z)M^+(z) = \frac{-1}{s_d z}\prod_{j=1}^{c}\frac{1}{1-u_j(z)/u}\prod_{\ell=1}^{d}\frac{1}{1-v_\ell(z)}.$$



Symmetric polynomials of degree k in d variables

Complete hom. sym. pol.	$h_k(x_1,\dots,x_d) = \sum_{1\leq i_1\leq\dots\leq i_k\leq d} x_{i_1}\cdots x_{i_k}$
Elementary sym. pol.	$e_k(x_1,\dots,x_d) = \sum_{1\leq i_1<\dots<i_k\leq d} x_{i_1}\cdots x_{i_k}$
Power sum sym. pol.	$p_k(x_1,\dots,x_d) = \sum_{i=1}^{d} x_i^k$

Symmetric polynomials and new types of lattice paths

	from 0 to k	from k to 0
positive meander		
$M^+(z)$	$h_k\left(\frac{1}{v_1(z)},\dots,\frac{1}{v_d(z)}\right)$	$h_k(u_1(z),\dots,u_c(z))$
positive meander avoiding $(0,k)$		
$(-1)^{k-1}M^{\geq}(z)$	$e_k\left(\frac{1}{v_1(z)},\dots,\frac{1}{v_d(z)}\right)$	$e_k(u_1(z),\dots,u_c(z))$
positive meander marked below the minimum		
$M^\bullet(z)$	$p_k\left(\frac{1}{v_1(z)},\dots,\frac{1}{v_d(z)}\right)$	$p_k(u_1(z),\dots,u_c(z))$

Theorem (Asymptotics: explicit multiplicative constants)

The radius of convergence is $\rho := 1/S(\tau)$, s.t. $\tau > 0$ given by $S'(\tau) = 0$.

$$[z^n]M^+_{k,0}(z) = \alpha_1 \frac{S(\tau)^n}{2\sqrt{\pi n^3}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \alpha_1 = \frac{\partial h_k}{\partial x_1}(\tau, u_2(\rho), \dots, u_c(\rho)).$$

$$[z^n]M^{\geq}_{k,0}(z) = \alpha_2 \frac{S(\tau)^n}{2\sqrt{\pi n^3}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \alpha_2 = \frac{\partial e_k}{\partial x_1}(\tau, u_2(\rho), \dots, u_c(\rho)).$$

$$[z^n]M^{\bullet}_{k,0}(z) = \alpha_3 \frac{S(\tau)^n}{2\sqrt{\pi n^3}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \alpha_3 = \frac{\partial p_k}{\partial x_1}(\tau, u_2(\rho), \dots, u_c(\rho)).$$

References

[1] Banderier, C.; Flajolet, P.: *Basic analytic combinatorics of directed lattice paths.* Theoretical Computer Science, 2002.
[2] Bousquet-Mélou, M. *Discrete excursions.* Sém. Lothar. Combin., 2008.
[3] Duchon, P: *On the enumeration and generation of generalized Dyck words.* Discrete Mathematics, 2000.