Reachable endpoint sets and algebraic enumeration of nondeterministic walks

Enumerative Combinatorics Workshop – Oberwolfach

Élie de Panafieu
(Nokia, Bell Labs, FR)

Mohamed Lamine Lamali
(Univ. de Bordeaux, FR)

Michael Wallner
(TU Graz/TU Wien, AT)

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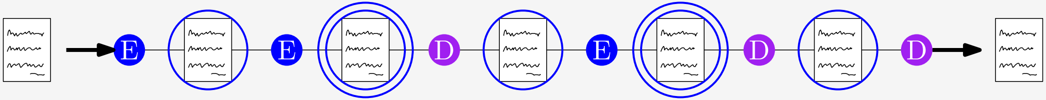
and the EU-funded RandNET project 101007705.)

I have an open PhD and Postdoc position to offer in my project "Universal Phenomena in Analytic Combinatorics" starting in Fall 2026. Feel free to contact me if you are interested!

Motivation: Nondeterministic Dyck walks model en- and decapsulation



Every node is capable of either

- Encapsulating, or
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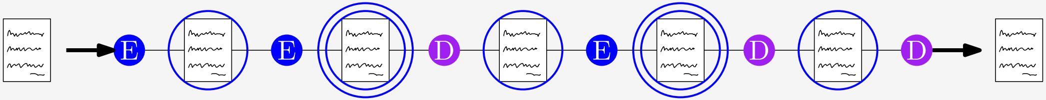

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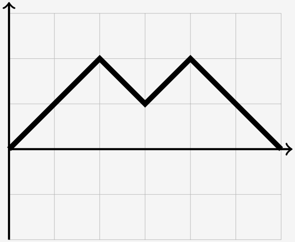
Dyck

Steps $\{-1, 1\}$



$$(1, 1, -1, 1, -1, -1)$$

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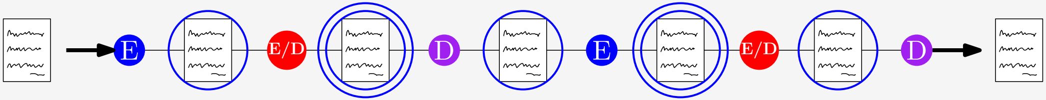

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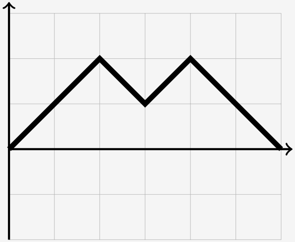
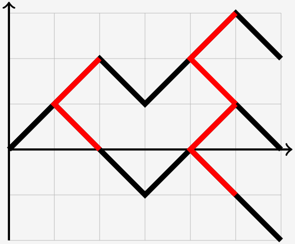


$(1, 1, -1, 1, -1, -1)$

N-Dyck

N-Steps $\{\{-1\}, \{1\}, \{-1, 1\}\}$

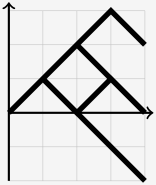


$(\{1\}, \{-1, 1\}, \{-1\}, \{1\}, \{-1, 1\}, \{-1\})$

Dyck N-walks

Definition

- Start at 0 and use N-steps $\{\{-1\},\{1\},\{-1,1\}\}$
- *N-walk*: sequence of N-steps (e.g., $(\{1\},\{-1,1\},\{-1,1\},\{-1\})$)
- *Length*: number of steps (above: length 4)



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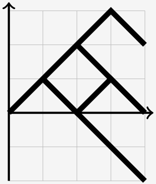
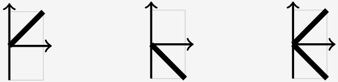


All Dyck-N walks of

- length 1



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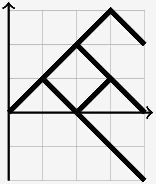

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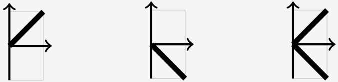
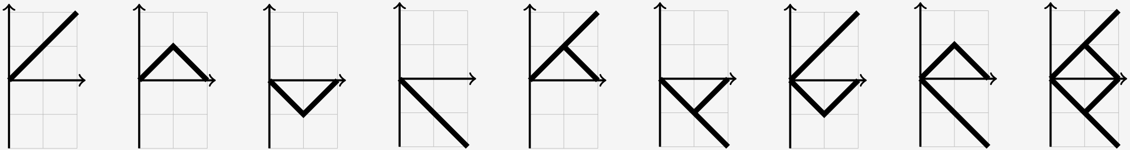


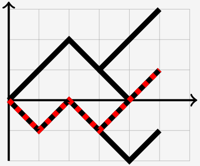
- length 2



Four types of N-Walks

	Classical		Nondeterministic
Walk:	unconstrained	**N-walk**:	contains a walk

N-walk



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Bridge:	ends at 0	**N-bridge**:	contains a bridge

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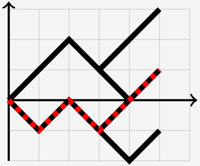

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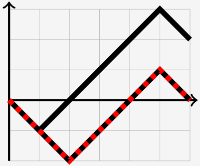

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Meander:	stays nonnegative	**N-meander**:	contains a meander

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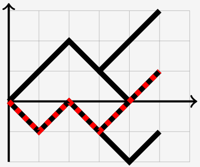

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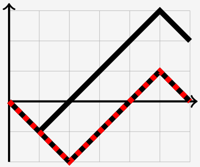
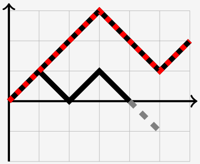


N-meander



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Bridge:	ends at 0	**N-bridge**:	contains a bridge
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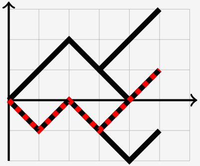
Excursion:	ends at 0 and stays nonneg.	**N-excursion**:	contains an excursion

N-walk



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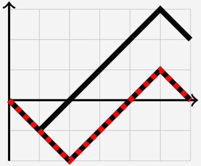

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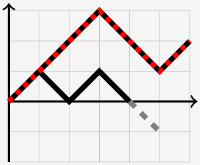
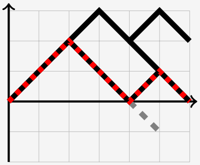


N-excursion



Reachable points for N-walks

For two sets A and B we define the **Minkowski sum** $+$ as

$$A + B := \{a + b \ : \ a \in A, \ b \in B\}.$$

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Dyck N-walks

- $\mathcal{S} = \{\{-1\}, \{1\}, \{-1, 1\}\}$
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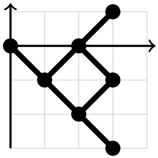

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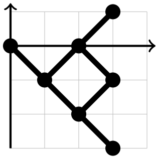

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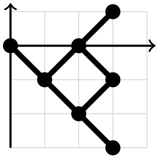

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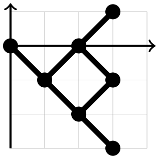

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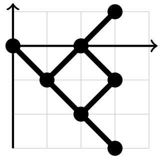

Generating function of Dyck N-walks

$$D(x, y; t) = \sum_{w} x^{\min(w)} y^{\max(w)} t^{|w|}$$

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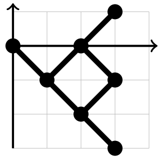

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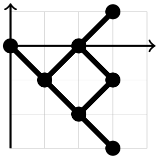

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$$D(x, y; t) = \sum_{w} x^{\min(w)} y^{\max(w)} t^{|w|} = \frac{1}{1 - t(\searrow + \nearrow + \diamond)} = \frac{1}{1 - t(x^{-1}y^{-1} + xy + x^{-1}y)}$$

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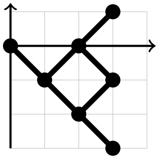

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- Obviously, there are 3^n Dyck N-walks of length n.
- In general, there are $|\mathcal{S}|^n$ many N-walks for a given N-step set $\mathcal{S}$.

Bijection to two-dimensional lattice paths

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Bijection to two-dimensional lattice paths

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Interpret change in min. and max. of reachable points, as a change in x- and y-direction:

$$\{-1\} \mapsto (-1,-1), \qquad \{1\} \mapsto (1,1), \qquad \{-1,1\} \mapsto (-1,1).$$

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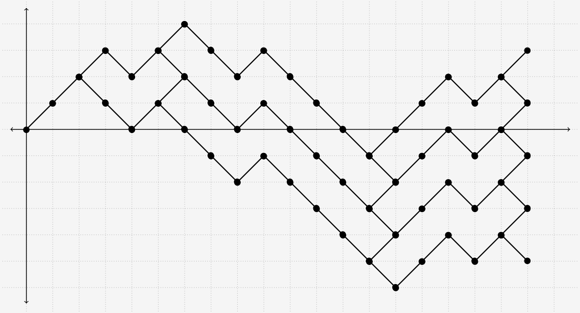
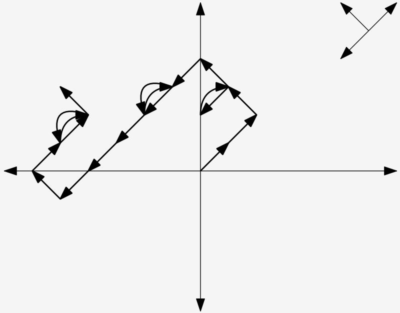


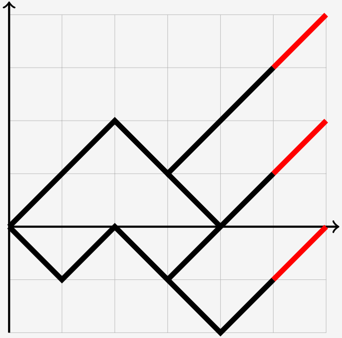


Reachable points for Dyck N-bridges

- N-step set $\mathcal{S} = \{\{-1\}, \{1\}, \{-1, 1\}\}$
- N-bridge is an N-walk containing a bridge (returns to 0)

Key observation

The reachable points are finite intervals of $2\mathbb{Z}$ or $2\mathbb{Z} + 1$.
$\Rightarrow$ uniquely characterized by $\min(w)$ and $\max(w)$!



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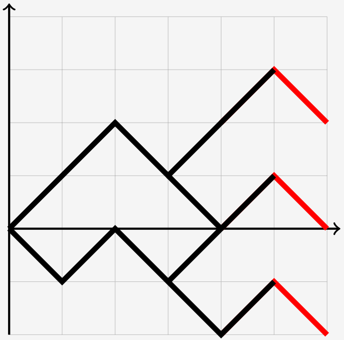

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The reachable points are finite intervals of $2\mathbb{Z}$ or $2\mathbb{Z}+1$.
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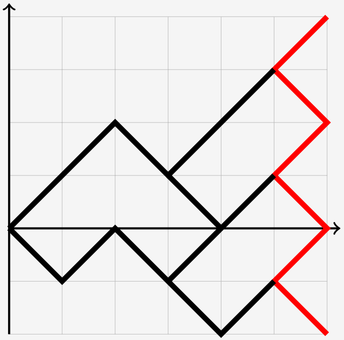

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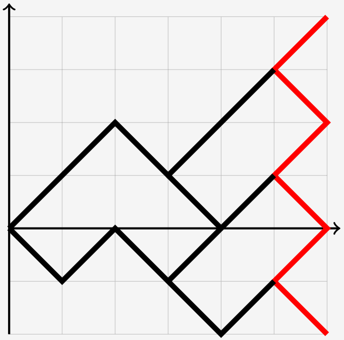
Characterizing N-bridges in N-walks

- N-bridges have even length and
- $\min(w) \leq 0$ and $\max(w) \geq 0$.



Dyck N-bridges

Theorem

The GF of Dyck N-bridges $B(x, y; t)$ is algebraic of degree 4.
Moreover, $B(1, 1; t)$ has degree 2:

$$B(1, 1, t) = \frac{1 - 6t^2}{\sqrt{1 - 8t^2}(1 - 9t^2)} = 1 + 7t^2 + 63t^4 + 583t^6 + 5407t^8 + \dots.$$

The number $[t^{2n}]B(1, 1, t)$ of Dyck N-bridges of even length is asymptotically equal to

$$3^{2n} - \frac{2}{\sqrt{\pi}} \frac{8^n}{\sqrt{n}} + \mathcal{O}\left(\frac{8^n}{n^{3/2}}\right).$$

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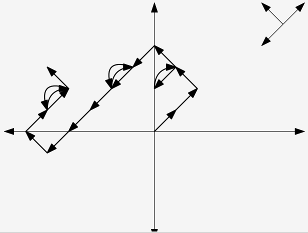
Proof:

- N-walks of **even length** have a rational generating function

$$D_2(x, y; t) = \frac{D(x, y; t) + D(x, y; -t)}{2}.$$

- We need $[x^{\leq 0} y^{\geq 0}] D_2(x, y; t)$ (Two coefficient extractions: D-finite but in general **not** algebraic!)



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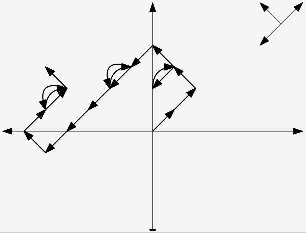
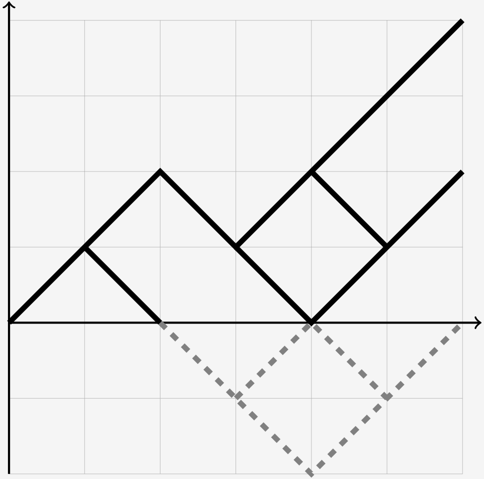


- We need $\mathbf{[x^{\leq 0} y^{\geq 0}] D_2(x,y;t)}$ (Two coefficient extractions: D-finite but in general **not** algebraic!)
- However,

$$B(x,y,t) = D_2(x,y;t) - [x^{>0}]D_2(x,y,t) - [y^{<0}]D_2(x,y,t).$$

Dyck N-meanders

- N-step set $\mathcal{S} = \{\{-1\}, \{1\}, \{-1, 1\}\}$
- N-meander is an N-walk containing a meander (staying non-negative)



$(\{1\}, \{-1, 1\}, \{-1\}, \{-1, 1\}, \{-1, 1\}, \{1\})$

Reachable points for Dyck N-meanders and N-excursions

For two sets A and B we define the **non-negative sum** $\oplus$ as

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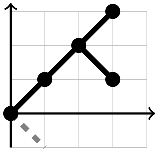

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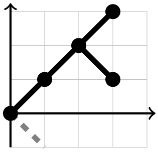
  $= \{1, 3\}$



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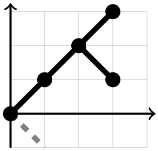

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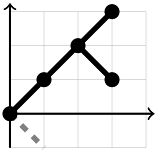

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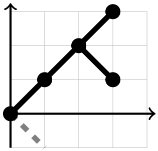

Generating function of Dyck N-meanders

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Theorem

The generating function $D^+(x,y;t)$ *of Dyck N-meanders is* *algebraic of degree* 4 *and equal to*

$$D^+(x,y;t) = \frac{x - X(y,t)}{1 - X(y,t)^2} \frac{y - xY(t) + (xy - Y(t))X(y,t)}{xy} D(x,y;t),$$

where

$$X(y,t) = \frac{1 - \sqrt{1 - 4(1+y^2)t^2}}{2yt} \qquad \text{and} \qquad Y(t) = \frac{1 - \sqrt{1 - 8t^2}}{4t}.$$

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Underlying decomposition:

$$\begin{aligned} D^+(x,y;t) = 1 &+ t(D^+(x,y;t) - D^+(0,y;t)) \left(\searrow + \nearrow + \swarrow\!\searrow \right) \\ &+ t(D^+(0,y;t) - D^+(0,0;t)) \left(\searrow + \nearrow + \swarrow\!\searrow \right) \\ &+ tD^+(0,0;t) \left(\nearrow + \swarrow\!\searrow \right). \end{aligned}$$

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Proof: We use the kernel method twice

- Rewrite functional equation into

$$K(x,y)D^+(x,y;t) = xy + t(x^2-1)(y^2+1)D^+(0,y;t) - tx^2D^+(0,0;t). \quad (1)$$

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Note that $Y(t)$ is chosen that $K(1,Y(t)) = 0$.
Hence, we get
$$D^+(0,0;t) = \frac{Y(t)}{t} \qquad \text{and} \qquad D^+(1,y;t) = \frac{y-Y(t)}{K(1,y)}.$$

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- Substituting this back into (1) and using the **kernel method again** in x, such that $K(X(y,t),y)=0$, the claim follows. □

The counting generating functions

For $x = y = 1$, the GFs of Dyck N-meanders, N-excursions, and N-excursions ending in $\{0\}$ are algebraic of degree 2:

$$D^{+}(1,1,t) = -\frac{1-4t-\sqrt{1-8t^2}}{4t(1-3t)} = 1 + 2t + 6t^2 + 16t^3 + 48t^4 + \dots, \qquad \text{(A151281)}$$

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Asymptotically, we get

$$[t^n]D^+(1,1,t) = \frac{3^n}{2} + \left(3\sqrt{2}(1+(-1)^n) + 4(1-(-1)^n)\right)\frac{8^{n/2}}{\sqrt{\pi n^3}} + \mathcal{O}\left(\frac{8^{n/2}}{n^{5/2}}\right),$$

$$[t^n]D^+(0,1,t) = \frac{1+(-1)^n}{2}\left(\frac{3^n}{4} + \sqrt{8}\frac{8^{n/2}}{\sqrt{\pi n^3}} + \mathcal{O}\left(\frac{8^{n/2}}{n^{5/2}}\right)\right),$$

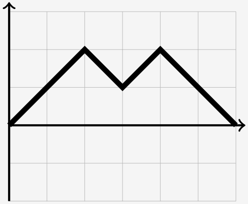
$$[t^n]D^+(0,0,t) = \sqrt{2}(1+(-1)^n)\frac{8^{n/2}}{\sqrt{\pi n^3}}\left(1 - \frac{9}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

Application in Networking

Networking: Classical excursions contained in N-excursions

Dyck

Steps $\{-1, 1\}$



N-Dyck

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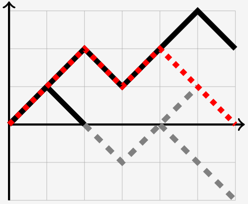

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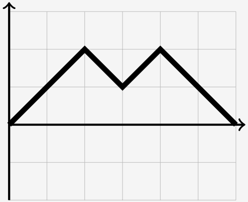

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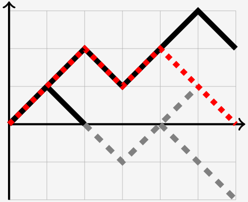

- Let c_{2n} be the total number of classical excursions contained in all N-excursions of length $2n$.
- Interpret every $\{-1, 1\}$-N-step either as a classical up- or down-step
- $\Rightarrow c_{2n} = 4^n \frac{1}{n+1}\binom{2n}{n}$.

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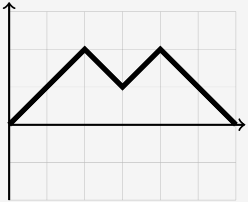

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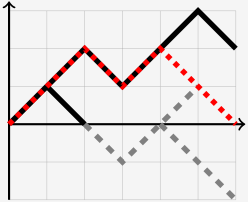

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Average number of classical excursions in all N-excursions of length $2n$

$$\frac{c_{2n}}{[t^{2n}]D^+(0,1,t)} \sim \frac{4}{\sqrt{\pi n^3}} \left(\frac{4}{3}\right)^{2n}.$$

Probability of a random N-walk to be an N-excursion

- Each N-step gets a probability

$$p_{-1}, p_1, p_{-1,1} \in [0,1] \qquad \text{such that} \qquad p_{-1} + p_1 + p_{-1,1} = 1.$$

- Weight of N-walk is product of its weights

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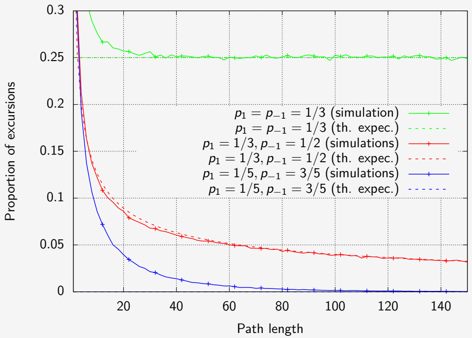

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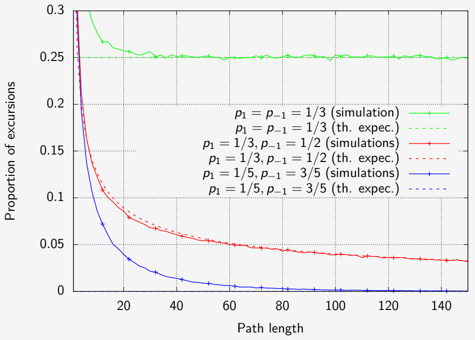

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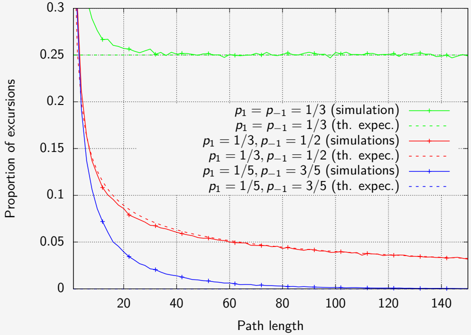
$$p_{-1}, p_1, p_{-1,1} \in [0,1] \qquad \text{such that} \qquad p_{-1} + p_1 + p_{-1,1} = 1.$$

- Weight of N-walk is product of its weights

Theorem

The probability of a random Dyck N-walk of length 2n to be an N-excursion is for $n \to \infty$ asymptotically equivalent to (where p_{-1} and p_1 are interchangeable):

$$\begin{cases} \frac{(1-2p_1)(1-2p_{-1})}{(1-p_1)(1-p_{-1})} & \text{if} \quad 0 \leq p_1 \leq p_{-1} < \frac{1}{2}, \\ \frac{1-2p_1}{(1-p_1)\sqrt{\pi n}} & \text{if} \quad 0 \leq p_1 < \frac{1}{2} \text{ and } p_{-1} = \frac{1}{2}, \\ \frac{1}{\sqrt{\pi n^3}} & \text{if} \quad p_1 = p_{-1} = \frac{1}{2}, \end{cases}$$



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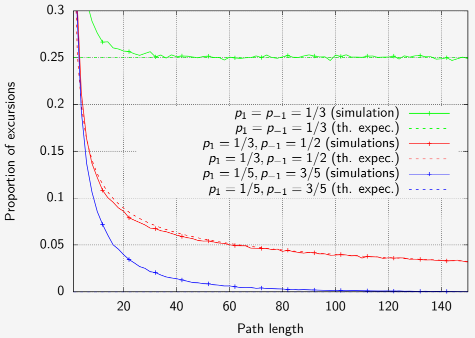

Limit laws and two-dimensional lattice paths

Dyck N-meanders and two-dimensional lattice paths

N-meanders admit again an interpretation in terms of two-dimensional lattice paths:

Previous bijection for N-walks plus **spatial constraints**

- Paths remain in the first quadrant;
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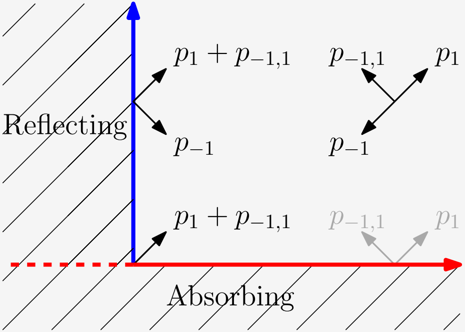
- y-axis as a reflecting boundaries.

In particular, N-excursions are mapped to walks that end on the nonnegative y-axis (since $\min^+ = 0$).



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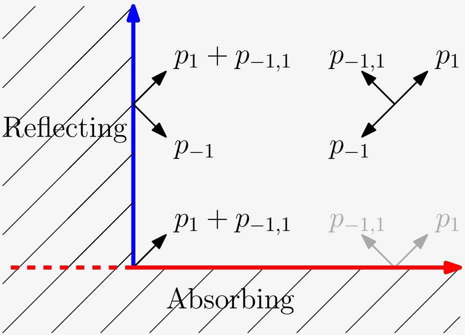

Both boundaries absorbing:

- GF is algebraic [Bousquet-Mélou, Mishna 2010]
- Walks ending on y-axis: distance to origin obeys a binomial distribution, i.e., normal in the limit

Limit law I: Final maximal point

We define the **x-drift** $\delta_x = \mathbb{E}(x)$ and the **y-drift** $\delta_y = \mathbb{E}(y)$. The **drift** is given by $\delta = (\delta_x, \delta_y)$ and for Dyck N-walks we have

$$\delta_x = p_1 - p_{-1,1} - p_{-1} = 2p_1 - 1,$$
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Theorem

For $p_{-1,1} \neq 0$ let X_n be the r.v. of the final maximal point of an N-excursion of length $2n$ drawn uniformly at random:

$$\mathbb{P}\left(X_n = k\right) := \frac{[t^{2n}y^{2k}]D^+(0,y;t)}{[t^{2n}]D^+(0,1;t)}.$$

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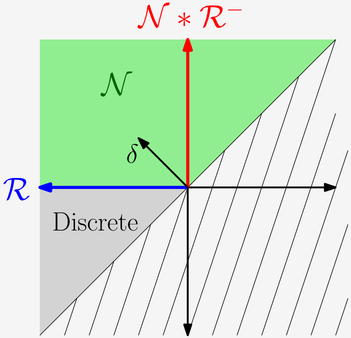
Then, X_n admits a limit distribution that depends on the drift δ: The limit law is either

- *discrete,*
- *normal $\mathcal{N}$,*
- *Rayleigh $\mathcal{R}$, or*
- *the convolution $\mathcal{N} * \mathcal{R}^-$ of $\mathcal{N}$ and $\mathcal{R}$ with negative support.*



Drift $\delta = (-1/3, 1/3)$ for $p_{-1} = p_1 = p_{-1,1} = \frac{1}{3}$.

Limit law I: Final maximal point (Proof)

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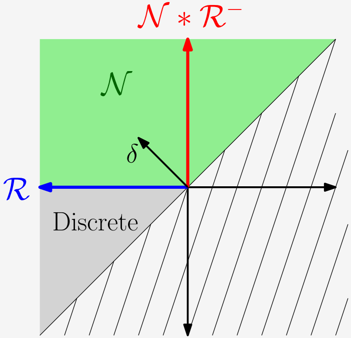

Proof:

- We start from the explicit shape of $D^+(0,y,t)$.
- Three candidates for the dominant singularity (polar and square-root type):

$$\rho_1 = \frac{1}{\sqrt{4p_{-1}(1-p_{-1})}}, \quad \rho_2(y) = \frac{1}{\sqrt{4p_1(p_{-1}+(1-p_1-p_{-1})y^2)}}, \quad \rho_3(y) = \frac{u}{(p_{-1}+(1-p_{-1})y^2)}.$$

- At most 2 coalesce.
- Methods: Singularity analysis [Flajolet, Odlyzko 90], quasi power-theorem [Hwang 98], square-root scheme on generating functions [Drmota, Soria 97]. □

Limit law II: Returns to $\{0\}$

Theorem

Let Y_n be the r.v. of the number of returns to $\{0\}$ in an N-excursion of length $2n$ drawn uniformly at random. Then, Y_n admits a discrete limit law of ***geometric, negative binomial, or mixed type****.*

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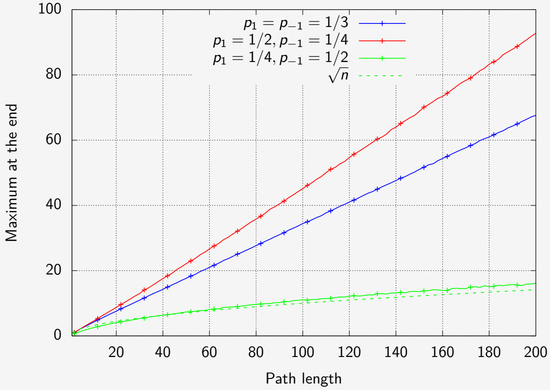
$$\mathbb{P}(Y_n = k) = \begin{cases} \frac{1}{D^+(0,0;1)}\left(1 - \frac{1}{D^+(0,0;1)}\right)^k & \text{if } 0 \le p_1 \le p_{-1} < \frac{1}{2}, \\ (1-p_{-1})p_{-1}^k & \text{if } 0 \le p_{-1} < \frac{1}{2} \text{ and } p_1 = \frac{1}{2}, \\ \frac{1}{2^{k+1}} & \text{if } 0 \le p_1 < \frac{1}{2} \text{ and } p_{-1} = \frac{1}{2}, \\ \frac{k}{2^{k+1}} & \text{if } p_1 + p_{-1} = 1, \\ \frac{1}{D^+(0,0;\rho_2)}\left(1 - \frac{1}{D^+(0,0;\rho_2)}\right)^k & \text{if } 0 \le p_{-1} < \frac{1}{2} < p_1 < 1 \text{ and } p_{-1} + p_1 < 1, \\ (1-\eta)\frac{1}{2^{k+1}} + \eta\frac{k}{2^{k+1}} & \text{if } 0 \le p_1 < \frac{1}{2} < p_{-1} < 1 \text{ and } p_{-1} + p_1 < 1, \end{cases}$$

where $\rho_2 = \frac{1}{4p_1(1-p_1)}$ *and* $\eta = \frac{p_{-1}(p_{-1}-p_1) - \sqrt{p_{-1}(1-p_{-1})(1-p_1-p_{-1})(p_{-1}-p_1)}}{p_{-1}(1-p_1)} \in [0,1]$.

Simulations of the limit laws: Expectations

Limit Law I: Final maximal point



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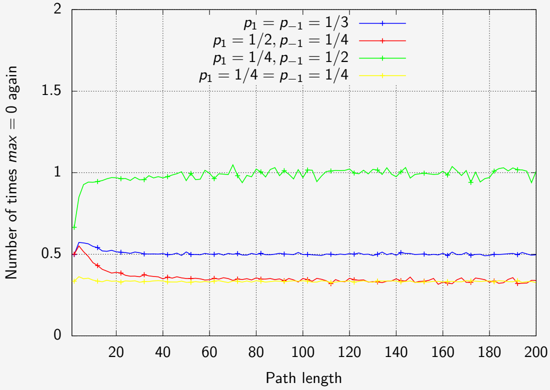


Depends on y-drift: $\delta_y = 1 - 2p_{-1}$:

- $\delta_y > 0$: linear
- $\delta_y = 0$: $\sqrt{n}$
- $\delta_y < 0$: constant

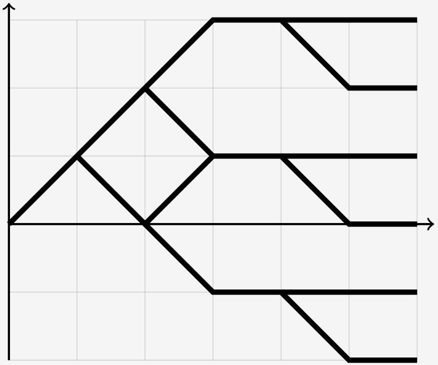
Other N-step sets

Motzkin N-steps

N-step set $\mathcal{S} = \Big\{\{-1\}, \{0\}, \{1\}, \{-1,0\}, \{-1,1\}, \{0,1\}, \{-1,0,1\}\Big\}$

Theorem

The generating functions of Motzkin N-bridges, N-meanders, and N-excursions are algebraic. *of degree at most* 16.



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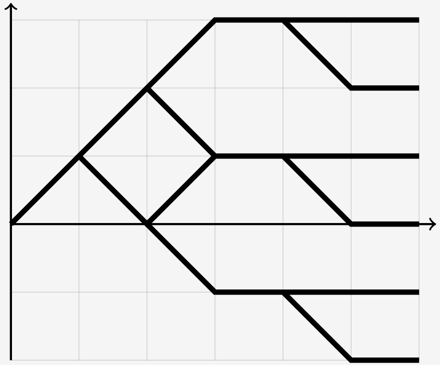

The **reachable point pattern changes**:

- 2-periodic: finite intervals in $2\mathbb{Z}$ or $2\mathbb{Z}+1$; or
- 1-periodic: finite intervals in $\mathbb{Z}$

Proof idea for Motzkin N-steps

- Reachable points have 2 types
  1. Type I: interval of $\mathbb{Z}$ (1-periodic)
  2. Type II: interval of $2\mathbb{Z}$ or $2\mathbb{Z}+1$ (2-periodic)

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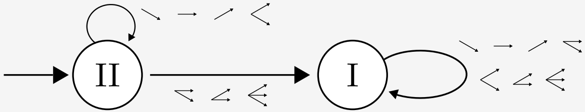
  - Walks and bridges



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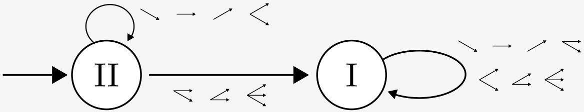

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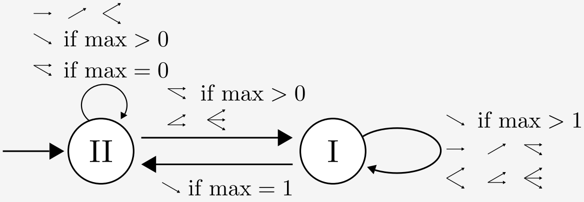

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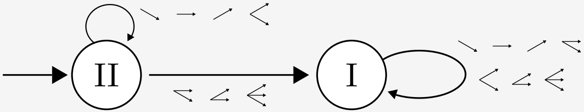

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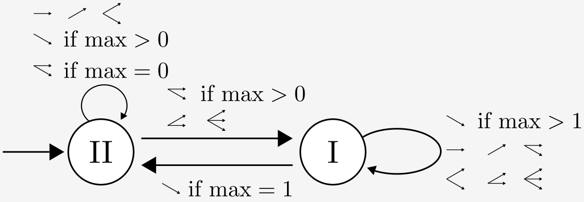

- Translate into system of generating of 2 generating functions
- Use vectorial extension of kernel method twice

N-Motzkin paths with arbitrary weights

The generating functions are algebraic wiht with arbitrary weights:

N-step set

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p_1	p_{-1}	p_0	$p_{-1,0}$	$p_{0,1}$	$p_{-1,1}$	$p_{-1,0,1}$	OEIS	Domain	Steps
1	0	0	1	0	0	1	A151281	Nonnegative line $\mathbb{N}$	$\{-1, 1_1, 1_2\}$
0	1	0	0	1	0	1			
1	0	1	1	0	0	1	A129637	Triangular lattice	$\{W, SE, SW, NW\}$
0	1	1	0	1	0	1			
1	0	0	1	1	0	1			
0	1	0	1	1	0	1			
1	0	1	1	1	0	1	A151251	First octant $\mathbb{N}^3$	$\{(0,0,1),(0,1,0),(1,1,0),(1,1,1),(-1,-1,0)\}$
0	1	1	1	1	0	1			

Table: N-Motzkin excursions related to (higher-dimensional) paths that start at the origin and remain in the given domain.

General N-bridges

Theorem

For any finite N-step set $\mathcal{S}$, the generating function $B(x, y; t)$ of N-bridges (with respect to length, minimal, and maximal reachable point) is algebraic.

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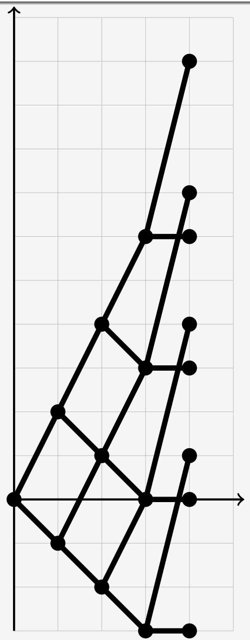

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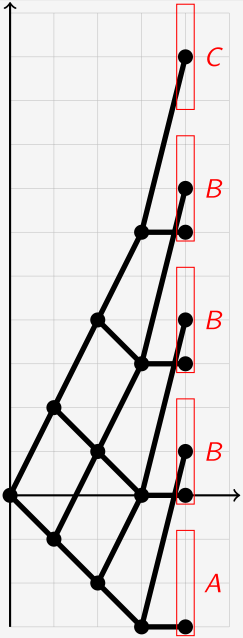

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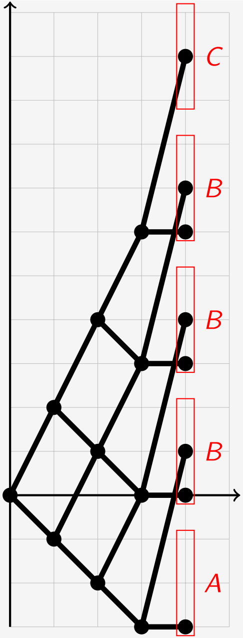

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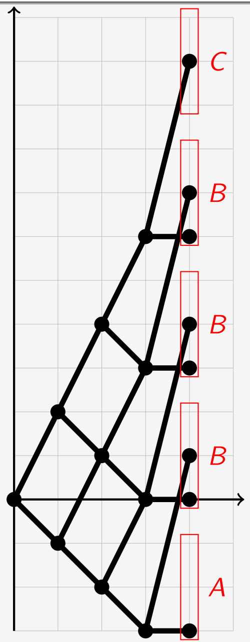
Proposition

For any finite subset $\mathcal{S} \subset \mathbb{Z}$, there is a ***finite set of types*** *$(A_i, B_i, C_i)_{1 \leq i \leq k}$ such that for any N-walk $w = (s_1, \dots, s_n) \in \mathcal{S}^n$, the sumset $s_1 + \dots + s_n$ belongs to type (A_i, B_i, C_i) for some $1 \leq i \leq k$.*



General N-meanders and N-excursions

Theorem (Algebraic subfamilies of N-meanders)

- The GF $D^{+}(1, y; t)$ (y marks the maximal reachable point and t the length) is algebraic.
- The GF $D^{+}(0, 0; t)$ (reachable point set $\{0\}$) is algebraic.

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Algebraicity Conjecture

For any N-step set, the generating function of N-excursions is algebraic.

- We developed a python package to experimentally find the types and the automaton
- We also implemented a Maple worksheet to analyze this data (e.g., guessing).

Example of an N-excursion with finitely many types

N-steps	States	Types
(red dashed) {-1}	$\{0\}$	$\text{type}(0, 0, \emptyset, \{0\}, \emptyset)$
(green dotted) {-1, 1}	$\{0, 2\}$	$\text{type}(0, 0, \emptyset, \{0, 2\}, \emptyset)$
(solid) {-2, 1}	I	$\text{type}(1, 6, \{1\}, \{0\}, \{1\})$
	I'	$\text{type}(1, 4, \emptyset, \{0\}, \{1\})$
	II	$\text{type}(2, 2, \emptyset, \{0\}, \emptyset)$
	III	$\text{type}(3, 1, \emptyset, \{0\}, \emptyset)$
	III'	$\text{type}(3, 1, \emptyset, \{0, 2\}, \emptyset)$
	III"	$\text{type}(3, 1, \{0\}, \{0, 2\}, \emptyset)$

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N-steps

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- black solid: {-2, 1}

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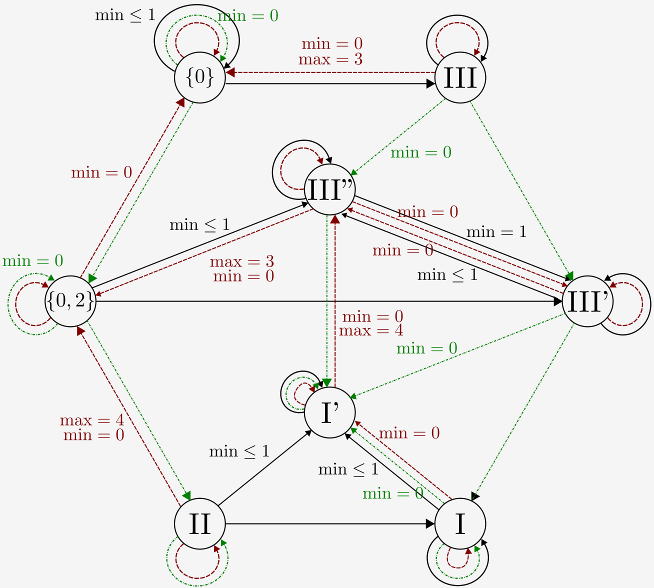

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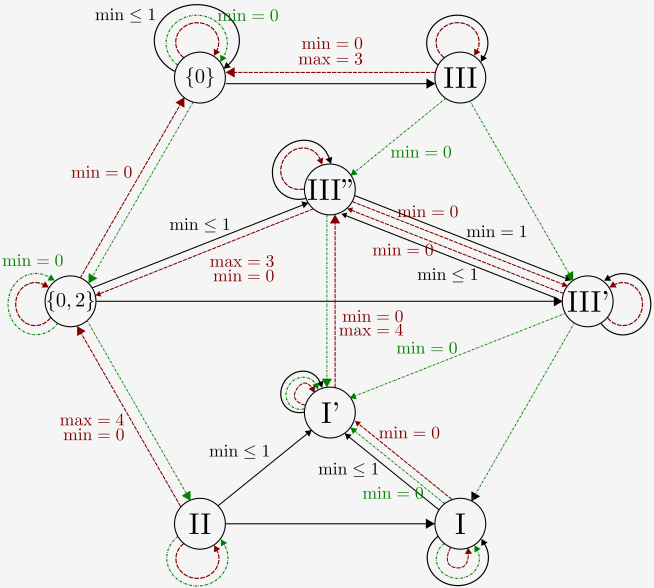

N-Excursions

$$D^{+}(0,1;t) = 1 + 4t^2 + 4t^3 + 28t^4 + \dots$$

Guess: algebraic of degree 4! Proof?

Context-free grammars and outlook

Nondeterminism and context-free grammars

The following holds for arbitrary N-step sets.

N-walks can be described by context-free grammars

- Context-free languages are recognized by (nondeterministic) pushdown automata with a single stack
- Use nondeterminism to follow all trajectories in parallel
- Use stack to track current altitude

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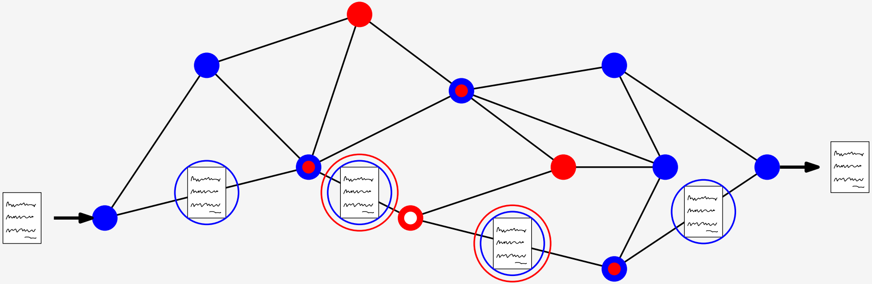
- Important result by [Chomsky, Schützenberger 63]: The GF of the number of words of unambiguous context-free grammars is algebraic
- Problems:
  - difficult to find grammar
  - difficult to solve associated system
- Our approach: **lattice paths** and the **kernel method**

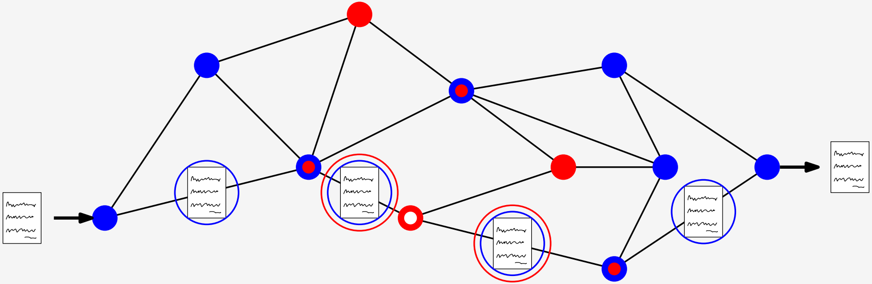
Future work

- **Other topologies:** e.g., series-parallel graphs
- **More protocols:** So far we considered only one protocol
- Underlying **context-free grammars**



Future work

- **Other topologies:** e.g., series-parallel graphs
- **More protocols:** So far we considered only one protocol
- Underlying **context-free grammars**



THANK YOU!

Backup

Interesting OEIS connections for Motzkin N-meanders

p_1	p_{-1}	p_0	$p_{-1,0}$	$p_{0,1}$	$p_{-1,1}$	$p_{-1,0,1}$	OEIS	Steps
1	1	0	0	1	1	0	A151162	$\{(-1,0,0),(1,0,0),(1,0,1),(1,1,0)\}$
1	1	0	0	1	0	1		
1	1	0	0	0	1	1		
0	1	0	0	1	1	1		
1	1	1	1	1	0	0	A151251	$\{(-1,-1,0),(0,0,1),(0,1,0),(1,1,0),(1,1,1)\}$
1	1	1	1	0	1	0		
0	1	1	1	1	1	0		
1	1	1	1	0	0	1		
0	1	1	1	1	0	1		
0	1	1	1	0	1	1		
1	1	1	0	1	1	0	A151253	$\{(-1,0,0),(0,0,1),(1,0,0),(1,0,1),(1,1,0)\}$
1	1	0	1	1	1	0		
1	1	1	0	1	0	1		
1	1	0	1	1	0	1		
1	1	1	0	0	1	1		
1	1	0	1	0	1	1		
0	1	1	0	1	1	1		
0	1	0	1	1	1	1		
1	1	0	0	1	1	1	A151254	$\{(-1,0,0),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$

Table: N-Motzkin meanders related to 3D paths that start at the origin and remain in the first octant $\mathbb{N}^3$.