Relational Width Collapses

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joint work with
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Constraint satisfaction problems and minimality

$\mathbb{A}$ - relational structure over finite signature

$\mathrm{CSP}(\mathbb{A})$ is the following computational problem:

Given:

- variable set V,
- pp-formulas $\phi_1, \ldots, \phi_n$ over the signature of $\mathbb{A}$ (*constraints*) with free variables from the set V.

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$\phi(x, y, z) \Rightarrow z \in \{0, 1\}$, $\psi(x, y, z) \Rightarrow z \in \{1, 2\}$
$\Rightarrow$ remove $z = 0$ from ϕ and $z = 2$ from ψ

Bounded width

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$\mathbb{A}$ has *relational width* (k,ℓ) if every (k,ℓ)-minimal instance of $\mathrm{CSP}(\mathbb{A})$ with satisfiable constraints is satisfiable.
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$$R_i := \{(a,b,c) \in (\mathbb{Z}_2)^3 \mid a + b + c = i\}$$

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    there exists $c \in \mathbb{Z}_2$ such that $a+b+c = i$.
    $\Rightarrow \Phi = R_0(x,y,z) \wedge R_1(x,y,z)$ is $(2,3)$-minimal
    and has non-empty constraints but is not satisfiable.

Collapse in the finite case

Theorem [Barto, 2016; Barto-Kozik, 2014].

Let $\mathbb{A}$ be a relational structure on a finite domain. TFAE:

- $\mathbb{A}$ has bounded width,
- $\mathbb{A}$ has relational width $(2, 3)$,
- $\mathbb{A}$ has an m-ary *weak near-unanimity* (WNU) polymorphism for all $m \geq 3$:

$$f(y, x, \ldots, x) \approx f(x, y, x, \ldots, x) \approx \ldots \approx f(x, \ldots, x, y).$$

Definition.

Let $k, \ell \geq 1$.

A relational structure $\mathbb{B}$ is k-*homogeneous* :$\leftrightarrow$
for all finite tuples a, b, a, b are in the same $\mathrm{Aut}(\mathbb{B})$-orbit
$\Leftrightarrow$ all k-subtuples of a, b are in the same $\mathrm{Aut}(\mathbb{B})$-orbit.

$\mathbb{B}$ is ℓ-bounded :$\leftrightarrow$ for every finite structure $\mathbb{X}$,
$\mathbb{X}$ embeds to $\mathbb{B}$ $\Leftrightarrow$ all substructures of $\mathbb{X}$ of size at most ℓ embed to $\mathbb{B}$.

Infinite-domain CSPs

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We are interested in structures with first-order definition
in a k-homogeneous ℓ-bounded structure $\mathbb{B}$ (*fo-reducts of* $\mathbb{B}$).

If $\mathbb{A}$ with domain $\{a_1, \ldots, a_n\}$ finite
$\Rightarrow$ $\mathbb{A}$ is a fo-reduct of $(\{a_1, \ldots, a_n\}; \{a_1\}, \ldots, \{a_n\})$
which is 1-homogeneous and 2-bounded.

Pseudo-Maltsev conditions

Let $\mathbb{A}$ be finite.

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  $\Leftrightarrow$ $\mathbb{A}$ has WNU polymorphisms of all arities ≥ 3
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- $\mathrm{CSP}(\mathbb{A})$ is in P (if $\mathrm{P} \neq \mathrm{NP}$) $\Leftrightarrow$ $\mathbb{A}$ has Siggers polymorphism
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Conjecture [Bodirsky-Pinsker, 2011; Barto-Pinsker, 2016].

Let $\mathbb{A}$ be a fo-reduct of a k-homogeneous ℓ-bounded structure $\mathbb{B}$ that is a core. Suppose that $\mathrm{P} \neq \mathrm{NP}$. TFAE:

- $\mathrm{CSP}(\mathbb{A})$ is in P,
- $\mathbb{A}$ has a pseudo-Siggers polymorphism modulo $\overline{\mathrm{Aut}(\mathbb{B})}$:

$$\alpha \circ s(x,y,x,z,y,z) \approx \beta \circ s(y,x,z,x,z,y)$$

for some $\alpha, \beta \in \overline{\mathrm{Aut}(\mathbb{B})}$.

Characterization of bounded width

What about bounded width? Can we take pseudo-WNUs,
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Bad news: For fo-reducts of $(\mathbb{Q}, <)$, no set of identities characterizing bounded width exists
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Good news: For fo-reducts of many other structures, pseudo-WNUs are sufficient - universal homogeneous graph, universal homogeneous tournament
(Mottet, Pinsker; 2020 - smooth approximations).
Reason: they have *canonical* pseudo-WNUs.

Canonical polymorphisms

Definition.

Let $\mathbb{A}$ be a fo-reduct of a k-homogeneous ℓ-bounded structure $\mathbb{B}$.
A polymorphism f of $\mathbb{A}$ is $\mathrm{Aut}(\mathbb{B})$-*canonical*
if it preserves the orbit-equivalence modulo $\mathrm{Aut}(\mathbb{B})$.

$\Leftrightarrow f$ induces an operation on the $\mathrm{Aut}(\mathbb{B})$-orbits of n-tuples for every $n \geq 1$.

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Canonical polymorphisms play a key role in all known complexity classification of infinite-domain CSPs.

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Relational width collapses

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Theorem. [Mottet, N., Pinsker, Wrona, 2021]

Let $k, \ell \geq 1$, and let $\mathbb{A}$ be a fo-reduct of a k-homogeneous ℓ-bounded structure $\mathbb{B}$. If $\mathbb{A}$ has canonical pseudo-WNU polymorphisms modulo $\overline{\mathrm{Aut}(\mathbb{B})}$ of all arities $n \geq 3$ then $\mathbb{A}$ has relational width $(2k, \max(3k, \ell))$.

stronger variant for pseudo-totally symmetric canonical polymorphisms

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- the universal homogeneous graph (relational width $(4, 6)$);

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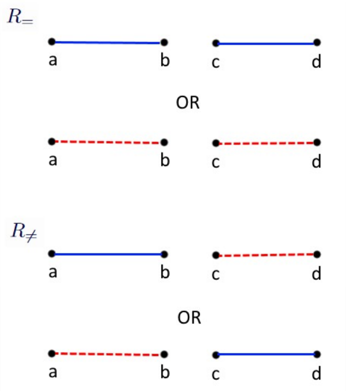
- the countably infinite equivalence relation with infinitely many equivalence classes ($(2, 3)$);
- the universal homogeneous partial order ($(2, 3)$).

Example

Let $\mathbb{B} := (A; E)$ be the universal homogeneous graph and let N be the non-edge relation.
Let $\mathbb{A} := (A; R_=, R_{\neq})$ be a fo-reduct of $\mathbb{B}$ with quaternary relations $R_=, R_{\neq}$, where:



$\Rightarrow$ the exact relational width of $\mathbb{A}$ is $(4, 6)$.

Further results

- Fo-reducts of a unary structure $\mathbb{B}$ that are cores:
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- The same for CSPs modeling model-checking problem for MMSNP-sentences.
  $\Rightarrow$ Datalog rewritability problem for MMSNP is decidable and 2NExpTime-complete.

Open questions

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- Which intermediate relational widths are possible for fo-reducts of a particular k-homogeneous ℓ-bounded structure?
- For which k-homogeneous ℓ-bounded structures the characterization of bounded width by (canonical) pseudo-WNUs applies?

Thank you for your attention!