Between heaven and hell: The Bodirsky-Pinsker conjecture for hypergraphs

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joint work with
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Infinite-domain CSPs

$\mathbb{B}$ - finitely bounded, homogeneous
$\mathbb{A}$ - first-order definable in $\mathbb{B}$

$\mathrm{CSP}(\mathbb{A})$:

Input: $\Phi = \phi_1 \wedge \ldots \wedge \phi_k$ - conjunction of atomic formulas over the signature of $\mathbb{A}$
Question: Φ satisfiable?

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Finite formulation:
$\mathrm{maxarity}(\mathbb{B}) = k$, τ - signature of $\mathbb{B}$

Given:

- "values": $O_1, \ldots, O_m$ - k-orbits under $\mathrm{Aut}(\mathbb{B})$,
- "constraints": constraints given by Φ (quantifier-free τ-formulas) + $\mathcal{F} = \{F_1, \ldots, F_n\}$ - finite forbidden τ-structures

Want: assign to every k-tuple of free variables of Φ an orbit O_i
s.t. no $F_i \in \mathcal{F}$ embeds to the resulting structure
and s.t. Φ is satisfied

Bodirsky-Pinsker conjecture

Conjecture [Bodirsky-Pinsker, 2011].

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F-V conjecture:

- classification of CSPs over 2-element domain (Schaefer, 1978),
- $\ldots$,
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B-P conjecture:

- $\mathbb{B}$ with 2 injective k-orbits:
  - almost (?) done for $k = 1, 2$
    (unary structures, graphs, random tournament, $\mathbb{Q}$),
  - $k \geq 3 \rightsquigarrow$ hypergraphs

k-uniform hypergraphs, 1/2

$k \geq 3$,
X - set,
E - k-ary relation on X, injective and fully symmetric
$\Rightarrow (X; E)$ is a k-*uniform hypergraph*

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the *random* k*-uniform hypergraph* $\mathbb{H}$ - the Fraïssé limit of the class of all finite k-uniform hypergraphs

- every finite k-uniform hypergraph is a substructure of $\mathbb{H}$,
- $\mathcal{F} = \{$"E not symmetric“, "E not injective“$\}$,
- k-orbits: conjunctions of $\neq$ and $=$, E, N

$N :=$ injective k-tuples $\setminus E \subseteq H^k$

k-uniform hypergraphs, 2/2

Theorem [Mottet, N., Pinsker, 2023].

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Theorem [Mottet, N., Pinsker, 2024].

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- ”Unordered“ $\mathbb{B}$
  - unary structures, graphs, tournaments, MMSNP,...
  - *canonical* polymorphisms - preserving orbit-equivalence, i.e. $f(E, =) = N$ makes sense
    $\rightsquigarrow$ if $\mathrm{CSP}(\mathbb{A})$ not NP-complete, they satisfy nice identities
    $\rightsquigarrow$ reduction to finite (Bodirsky, Mottet, 2017)
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”This could be heaven or this could be hell.“

(Eagles, Hotel California, 1976)

Smooth approximations for the structures in heaven

$\mathbb{A}$ - fo-definable in unordered $\mathbb{B}$, does not construct $3-\mathrm{SAT}$

$\Rightarrow$ $\mathbb{A}$ has a *pseudo-Siggers polymorphism*

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$\Rightarrow$ $\mathrm{CSP}(\mathbb{A})$ in P

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Given: $f \in \mathrm{Pol}(\mathbb{A})$
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Smooth approximations + pact with the devil
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- $f(E, N, E, E, E, N)$ makes sense, $f(=\neq, E, N, \neq=, E, E)$ doesn't

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> ”Oh, I wish you well... a little bit of heaven, but a little bit of hell.“
>
> (Mika, Happy Ending, 2007)

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- solving injective instances $\rightsquigarrow$ linear equations
  - $\rightsquigarrow$ new algorithmic techniques needed
  - have to study the behaviour of some polymorphisms on ordered orbits

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  - does a reduction to a binary absorbing subuniverse always work? under which assumptions?
- More than 2 injective orbits $\Rightarrow$ can't use Post's classification
  - $\Rightarrow$ need to mix local consistency and linear equations
- New algorithmic techniques to get out of the purgatory
  - generalization of Zhuk's algorithm?

Thank you for your attention!