Between heaven and hell: The Bodirsky-Pinsker conjecture for hypergraphs

Tomáš Nagy

TU Wien

joint work with
Antoine Mottet and Michael Pinsker

AAA 102, Szeged, 25th June 2022

B - finitely bounded, homogeneous
A - first-order definable in B

CSP(A):

Input: $\Phi = \phi_1 \wedge \dots \wedge \phi_k$ - conjunction of atomic formulas over the signature of A
Question: Φ satisfiable?

B - finitely bounded, homogeneous
A - first-order definable in B

<u>CSP(A)</u>:

Input: $= {}_1 \wedge \dots \wedge {}_k$ - conjunction of atomic formulas over the signature of A
Question: satisfiable?

<u>Finite formulation:</u>
maxarity(B) $= k$, - signature of B

Given:

- "values": $O_1, \dots, O_m$ - k-orbits under Aut(B),
- "constraints": constraints given by (quantifier-free -formulas) +
  $F = \{F_1, \dots, F_n\}$ - finite forbidden -structures

Want: assign to every k-tuple of free variables of an orbit O_i
s.t. no $F_i \in F$ embeds to the resulting structure
and s.t. is satisfied

Conjecture [Bodirsky-Pinsker, 2011].

B - finitely bounded homogeneous structure,
A - first-order definable in B
) CSP(A) is in P or NP-complete.

Conjecture [Bodirsky-Pinsker, 2011].

B - finitely bounded homogeneous structure,
A - first-order definable in B
$\Rightarrow$ CSP(A) is in P or NP-complete.

$B = (\{b_1, \ldots, b_n\}, \{b_1\}, \ldots, \{b_n\}) \Rightarrow$ Feder-Vardi conjecture

Conjecture [Bodirsky-Pinsker, 2011].

B - finitely bounded homogeneous structure,
A - first-order definable in B
$\Rightarrow$ CSP(A) is in P or NP-complete.

B = $(\{b_1, \ldots, b_n\}; \{b_1\}, \ldots, \{b_n\}) \Rightarrow$ Feder-Vardi conjecture

F-V conjecture:

- classification of CSPs over 2-element domain (Schaefer, 1978),
- $\ldots$,
- proof of the conjecture (Bulatov, Zhuk, 2017)

Conjecture [Bodirsky-Pinsker, 2011].

B - finitely bounded homogeneous structure,
A - first-order definable in B
$\Rightarrow$ CSP(A) is in P or NP-complete.

$B = (\{b_1, \dots, b_n\}, \{b_1\}, \dots, \{b_n\}) \Rightarrow$ Feder-Vardi conjecture

F-V conjecture:

- classification of CSPs over 2-element domain (Schaefer, 1978),
- $\dots$,
- proof of the conjecture (Bulatov, Zhuk, 2017)

B-P conjecture:

- B with 2 injective k-orbits:
  - almost (?) done for $k = 1, 2$ (unary structures, graphs, random tournament, $\mathbb{Q}$),
  - k 3 ; hypergraphs

k 3,
X - set,
E - k-ary relation on X, injective and fully symmetric
) $(X; E)$ is a k-*uniform hypergraph*

k 3,
X - set,
E - k-ary relation on X, injective and fully symmetric
) $(X; E)$ is a k-*uniform hypergraph*

the *random* k-*uniform hypergraph* H - the Fraïssé limit of the class of all finite k-uniform hypergraphs

- every finite k-uniform hypergraph is a substructure of H,
- F = f"E not symmetric", "E not injective"g,
- k-orbits: conjunctions of ≠ and =, E, N

N := injective k-tuples nE H^k

k

Theorem [Mottet, N., Pinsker, 2023].

k 3,
H - the *random* k-uniform hypergraph,
A - first-order definable in H
$)$ CSP(A) is in P or NP-complete.

Theorem [Mottet, N., Pinsker, 2024].

k 3,
H - the ?????? k-uniform hypergraph,
A - first-order definable in H
$)$ CSP(A) is in P or NP-complete.

How to confirm the BP-conjecture?

How to confirm the BP-conjecture?

”Unordered“ B

unary structures, graphs, tournaments, MMSNP,...
canonical polymorphisms - preserving orbit-equivalence,
i.e. $f(E; =) = N$ makes sense
; if CSP(A) not NP-complete, they satisfy nice identities
; reduction to finite (Bodirsky, Mottet, 2017)
) **Heaven!**

How to confirm the BP-conjecture?

”Unordered“ B

unary structures, graphs, tournaments, MMSNP,...
canonical polymorphisms - preserving orbit-equivalence,
i.e. $f(E; =) = N$ makes sense
; if CSP(A) not NP-complete, they satisfy nice identities
; reduction to finite (Bodirsky, Mottet, 2017)
) **Heaven!**

”Ordered B“

$(Q; <)$,
canonical polymorphisms satisfy NO identities!
; ad hoc algorithms
) **Hell!**

How to confirm the BP-conjecture?

"Unordered" B

unary structures, graphs, tournaments, MMSNP,...
canonical polymorphisms - preserving orbit-equivalence,
i.e. $f(E; =) = N$ makes sense
; if CSP(A) not NP-complete, they satisfy nice identities
; reduction to finite (Bodirsky, Mottet, 2017)
) **Heaven!**

"Ordered B"

$(Q; <)$,
canonical polymorphisms satisfy NO identities!
; ad hoc algorithms
) **Hell!**

> "This could be heaven or this could be hell."
>
> (Eagles, Hotel California, 1976)

A - fo-definable in unordered B, does not construct 3 SAT

) A has a *pseudo-Siggers polymorphism*

$$e \quad f(x;y;x;z;y;z) = g \quad f(y;x;z;x;z;y)$$

A - fo-definable in unordered B, does not construct 3 SAT

) A has a *pseudo-Siggers polymorphism*

$$e \quad f(x;y;x;z;y;z) = g \quad f(y;x;z;x;z;y)$$

Smooth approximations) A has a *canonical* pseudo-Siggers

) reduction to finite

) CSP(A) in P

A - fo-definable in an unordered B, does not construct 3-SAT

Given: f 2 Pol(A)
Want: f behaves canonically
often not possible to obtain this directly

$\mathbb{A}$ - fo-definable in an unordered $\mathbb{B}$, does not construct 3-SAT

Given: $f \in \mathrm{Pol}(\mathbb{A})$
Want: f behaves canonically
often not possible to obtain this directly

Pact with the devil:
extend $\mathbb{B}$ by a linear order $\rightsquigarrow$ f canonical on ordered orbits
$\Rightarrow$ does not satisfy any identities

A - fo-definable in an unordered B, does not construct 3-SAT

Given: f 2 Pol(A)
Want: f behaves canonically
often not possible to obtain this directly

Pact with the devil:
extend B by a linear order $;$ f canonical on ordered orbits
$)$ does not satisfy any identities

How to get a canonical polymorphism satisfying identities?

A - fo-definable in an unordered B, does not construct 3-SAT

Given: f 2 Pol(A)
Want: f behaves canonically
often not possible to obtain this directly

Pact with the devil:
extend B by a linear order ; f canonical on ordered orbits
) does not satisfy any identities

How to get a canonical polymorphism satisfying identities?

Compose with polymorphisms destroying the order
; obtain canonical polymorphisms

A - fo-definable in an unordered B, does not construct 3-SAT

Given: f 2 Pol(A)
Want: f behaves canonically
often not possible to obtain this directly

Pact with the devil:
extend B by a linear order ; f canonical on ordered orbits
) does not satisfy any identities

How to get a canonical polymorphism satisfying identities?

Compose with polymorphisms destroying the order
; obtain canonical polymorphisms

Is it always possible?

Is it always possible to destroy the order?

Is it always possible to destroy the order?

Why should an order that is not in there matter?

Is it always possible to destroy the order?

Why should an order that is not in there matter?

$\mathbb{A}$ - fo-definable in the random k-uniform hypergraph $\mathbb{H}$,
does not construct 3-SAT,
has pseudo-Siggers

Smooth approximations + pact with the devil
$\Rightarrow$ $\mathbb{A}$ has a pseudo-Siggers canonical on injective tuples

$f(E, N, E, E, E, N)$ makes sense, $f(\neq, E, N, \neq, E, E)$ doesn't

Is it always possible to destroy the order?

Why should an order that is not in there matter?

$\mathbb{A}$ - fo-definable in the random k-uniform hypergraph $\mathbb{H}$,
does not construct 3-SAT,
has pseudo-Siggers

Smooth approximations + pact with the devil
$\Rightarrow$ $\mathbb{A}$ has a pseudo-Siggers canonical on injective tuples

$f(E, N, E, E, E, N)$ makes sense, $f(=\neq, E, N, \neq=, E, E)$ doesn't

But: $\mathbb{A}$ does **not** necessarily have a canonical pseudo-Siggers

Is it always possible to destroy the order?

Why should an order that is not in there matter?

$\mathbb{A}$ - fo-definable in the random k-uniform hypergraph $\mathbb{H}$,
does not construct 3-SAT,
has pseudo-Siggers

Smooth approximations + pact with the devil
$\Rightarrow$ $\mathbb{A}$ has a pseudo-Siggers canonical on injective tuples

$f(E, N, E, E, E, N)$ makes sense, $f(=, \neq, E, N, \neq, =, E, E)$ doesn't

But: $\mathbb{A}$ does **not** necessarily have a canonical pseudo-Siggers

> "Oh, I wish you well... a little bit of heaven, but a little bit of hell."
>
> (Mika, Happy Ending, 2007)

- ⇒ have to pay for the pact with the devil

) have to pay for the pact with the devil

Have:

can solve an injective instance by a reduction to the finite, smooth approximations) binary injection g

) have to pay for the pact with the devil

Have:

can solve an injective instance by a reduction to the finite, smooth approximations) binary injection g

Want: Solve a general instance!

$\Rightarrow$ have to pay for the pact with the devil

Have:

can solve an injective instance by a reduction to the finite, smooth approximations $\Rightarrow$ binary injection g

Want: Solve a general instance!

Distinguish two cases:

injective instances solvable by local consistency

$\rightsquigarrow$ general instances solvable by local consistency, since injective k-tuples binary absorb A^k - witnessed by g

) have to pay for the pact with the devil

Have:

can solve an injective instance by a reduction to the finite, smooth approximations) binary injection g

Want: Solve a general instance!

Distinguish two cases:

injective instances solvable by local consistency

; general instances solvable by local consistency, since injective k-tuples binary absorb A^k - witnessed by g

solving injective instances ; linear equations

; new algorithmic techniques needed have to study the behaviour of some polymorphisms on ordered orbits

Understand what is wrong with the order...

when can’t it be destroyed?

Understand what is wrong with the order...

- when can't it be destroyed?

When can the CSP be reduced to solving injective instances?

- does a reduction to a binary absorbing subuniverse always work?
- under which assumptions?

Understand what is wrong with the order...

when can't it be destroyed?

When can the CSP be reduced to solving injective instances?

does a reduction to a binary absorbing subuniverse always work?
under which assumptions?

More than 2 injective orbits) can't use Post's classification

) need to mix local consistency and linear equations

Understand what is wrong with the order...

when can't it be destroyed?

When can the CSP be reduced to solving injective instances?

does a reduction to a binary absorbing subuniverse always work?
under which assumptions?

More than 2 injective orbits) can't use Post's classification

) need to mix local consistency and linear equations

New algorithmic techniques to get out of the purgatory

generalization of Zhuk's algorithm?

Thank you for your attention!

Rationals