Hypergraphs in the post-proof era

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joint work with
Antoine Mottet and Michael Pinsker

The Constraint Satisfaction Problem: Complexity and Approximability,
Dagstuhl

k-uniform hypergraphs

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X - set,
E - k-ary relation on X, injective and fully symmetric
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$N :=$ injective k-tuples $\setminus E \subseteq H^k$

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How to come from the purgatory to the heaven?

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$\mathbb{A}$ has relational width $(2k, 3k)$ (Mottet, N., Pinsker, Wrona, 2021)

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  - or $f(O, E) \neq f(O, N)$ for every order on O and for every fixed order in the second coordinate $\Rightarrow$ O is *non-deterministic*

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expanding by an order - pact with the devil
Price: the instance $\mathcal{I}$ has to go to the purgatory
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purify $\mathcal{I}$ until it has no sins
$\Rightarrow$ make $\mathcal{I}$ injective and send it to heaven - reduce to finite

eq-irreducibility

$\mathcal{I} = (V, D, \mathcal{C}) \rightsquigarrow \mathcal{I}_{\mathrm{eq}} = (V, D_{\mathrm{eq}}, \mathcal{C}_{\mathrm{eq}})$

$C_{eq} = \{\alpha\mathbf{t} \mid \mathbf{t} \in C, \alpha \in \mathrm{Sym}(H)\}$
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$\rightsquigarrow$ solving equality-CSP

Bad partitions, 1/2

$\mathcal{I}$ - without sins (1), (2)

$\mathbf{v}^1, \ldots, \mathbf{v}^n$ - k-tuples of variables,
partitions on $\mathrm{proj}_{\mathbf{v}^i}(\mathcal{I})$ with pp-definable classes $E_1^i, \ldots, E_t^i$ s.t.
E_1^i contains all injective orbits in $\mathrm{proj}_{\mathbf{v}^i}(\mathcal{I})$, no non-deterministic one

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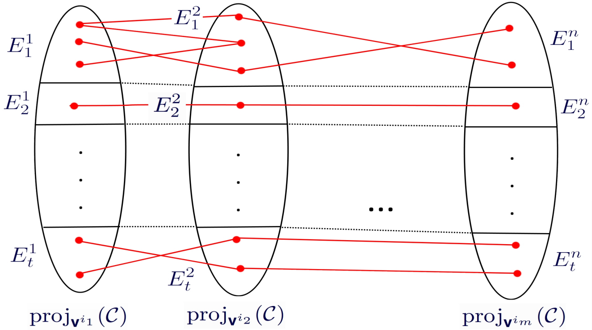

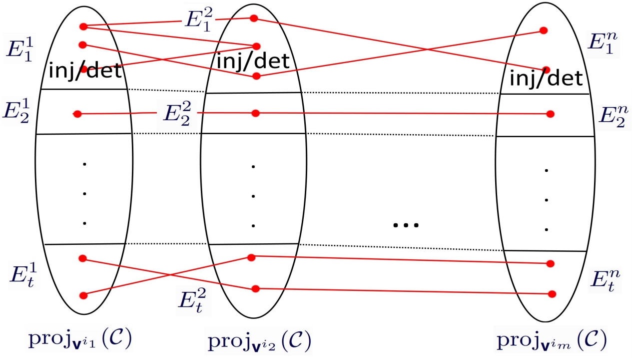

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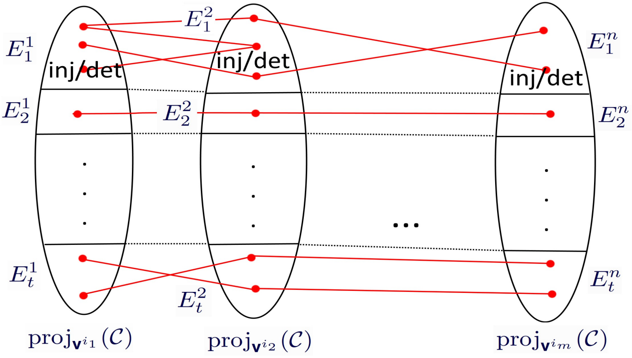

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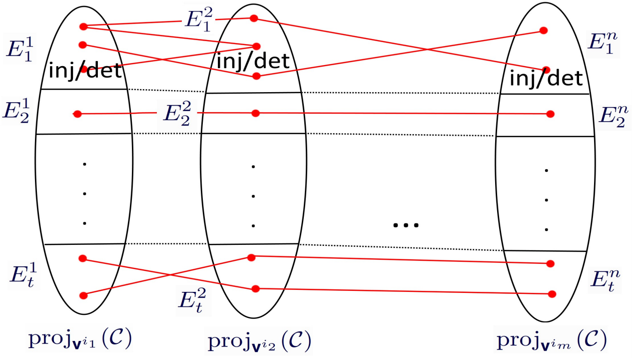

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not solvable? $\Rightarrow$ constrain every $\mathbf{v}^i$ by $\mathrm{proj}_{\mathbf{v}^i}(\mathcal{I}) \setminus E_1^i$

Thank you for your attention!

...and thank God for purifying the instances of their sins