Smooth Approximations and Relational Width Collapses

Antoine Mottet, Tomáš Nagy, Michael Pinsker and Michał Wrona

ICALP 2021, Glasgow, 13th July 2021

Constraint satisfaction problems

$\mathbb{A}$ - relational structure over finite signature

$\mathrm{CSP}(\mathbb{A})$ is the following computational problem:

Given:

- variable set V,
- pp-formulas $\phi_1, \ldots, \phi_n$ over the signature of $\mathbb{A}$ (*constraints*) with free variables from the set V.

Want: Is the *instance* $\Phi = \bigwedge_{i=1}^{n} \phi_i$ satisfiable in $\mathbb{A}$?

Constraint satisfaction problems

$\mathbb{A}$ - relational structure over finite signature

$\mathrm{CSP}(\mathbb{A})$ is the following computational problem:

Given:

- variable set V,
- pp-formulas $\phi_1, \ldots, \phi_n$ over the signature of $\mathbb{A}$ (*constraints*) with free variables from the set V.

Want: Is the *instance* $\Phi = \bigwedge_{i=1}^{n} \phi_i$ satisfiable in $\mathbb{A}$?

Minimality and bounded width

Let $1 \leq k \leq \ell$.

(k, ℓ)-minimality algorithm (produces a (k, ℓ)-minimal instance):

1. Adds constraint for every ℓ-element subset of V.
2. Reduces constraints by comparing projections of constraints on subsets of V of size $\leq k$.

Minimality and bounded width

Let $1 \leq k \leq \ell$.

(k, ℓ)-minimality algorithm (produces a (k, ℓ)-minimal instance):

1. Adds constraint for every ℓ-element subset of V.
2. Reduces constraints by comparing projections of constraints on subsets of V of size $\leq k$.

$\phi(x, y, z) \Rightarrow z \in \{0, 1\}$, $\psi(x, y, z) \Rightarrow z \in \{1, 2\}$

$\Rightarrow$ remove $z = 0$ from ϕ and $z = 2$ from ψ

Minimality and bounded width

Let $1 \leq k \leq \ell$.

(k,ℓ)-minimality algorithm (produces a (k,ℓ)-minimal instance):

1. Adds constraint for every ℓ-element subset of V.
2. Reduces constraints by comparing projections of constraints on subsets of V of size $\leq k$.

$\phi(x,y,z) \Rightarrow z \in \{0,1\}$, $\psi(x,y,z) \Rightarrow z \in \{1,2\}$
$\Rightarrow$ remove $z = 0$ from ϕ and $z = 2$ from ψ

Definition.

$\mathbb{A}$ has *relational width* (k,ℓ) if every (k,ℓ)-minimal instance of $\mathrm{CSP}(\mathbb{A})$ with satisfiable constraints is satisfiable.
$\mathbb{A}$ has *bounded width* if $\mathbb{A}$ has relational width (k,ℓ) for some $k \leq \ell$.

Minimality and bounded width

Let $1 \leq k \leq \ell$.

(k, ℓ)-minimality algorithm (produces a (k, ℓ)-minimal instance):

1. Adds constraint for every ℓ-element subset of V.
2. Reduces constraints by comparing projections of constraints on subsets of V of size $\leq k$.

$\phi(x, y, z) \Rightarrow z \in \{0, 1\}$, $\psi(x, y, z) \Rightarrow z \in \{1, 2\}$
$\Rightarrow$ remove $z = 0$ from ϕ and $z = 2$ from ψ

Definition.

$\mathbb{A}$ has *relational width* (k, ℓ) if every (k, ℓ)-minimal instance of $\mathrm{CSP}(\mathbb{A})$ with satisfiable constraints is satisfiable.
$\mathbb{A}$ has *bounded width* if $\mathbb{A}$ has relational width (k, ℓ) for some $k \leq \ell$.

equivalent definitions using e.g.

- logical definability
  (Datalog, infinitary logics with bounded number of variables),
- finite model theory (Spoiler-Duplicator games),
- treewidth of relational structures.

Bounded width, examples

$\mathbb{A}$ has bounded width $\Rightarrow \mathrm{CSP}(\mathbb{A})$ is in P

Bounded width, examples

$\mathbb{A}$ has bounded width $\Rightarrow \mathrm{CSP}(\mathbb{A})$ is in P

- $(\{0,1\};\neq,=)$ has relational width $(2,3)$ (computing transitivity),

Bounded width, examples

$\mathbb{A}$ has bounded width $\Rightarrow \mathrm{CSP}(\mathbb{A})$ is in P

- $(\{0,1\};\neq,=)$ has relational width $(2,3)$ (computing transitivity),
- $(\mathbb{Q};<)$ has relational width $(2,3)$ as well (transitivity again),

Bounded width, examples

$\mathbb{A}$ has bounded width $\Rightarrow \mathrm{CSP}(\mathbb{A})$ is in P

- $(\{0,1\};\neq,=)$ has relational width $(2,3)$ (computing transitivity),
- $(\mathbb{Q};<)$ has relational width $(2,3)$ as well (transitivity again),
- 2-SAT has relational width $(2,3)$,

Bounded width, examples

$\mathbb{A}$ has bounded width $\Rightarrow$ $\mathrm{CSP}(\mathbb{A})$ is in P

- $(\{0,1\}; \neq, =)$ has relational width $(2,3)$ (computing transitivity),
- $(\mathbb{Q}; <)$ has relational width $(2,3)$ as well (transitivity again),
- 2-SAT has relational width $(2,3)$,
- $(\mathbb{Z}_2; R_0, R_1)$ where

$$R_i := \{(a,b,c) \in (\mathbb{Z}_2)^3 \mid a+b+c = i\}$$

  does not have relational width $(2,3)$.
  - **Why?**

Bounded width, examples

$\mathbb{A}$ has bounded width $\Rightarrow$ $\mathrm{CSP}(\mathbb{A})$ is in P

- $(\{0,1\}; \neq, =)$ has relational width $(2,3)$ (computing transitivity),
- $(\mathbb{Q}; <)$ has relational width $(2,3)$ as well (transitivity again),
- $2\text{-}\mathrm{SAT}$ has relational width $(2,3)$,
- $(\mathbb{Z}_2; R_0, R_1)$ where

$$R_i := \{(a,b,c) \in (\mathbb{Z}_2)^3 \mid a+b+c = i\}$$

  does not have relational width $(2,3)$.
  - **Why?**
    For every $i \in \mathbb{Z}_2$ and for every $a, b \in \mathbb{Z}_2$
    there exists $c \in \mathbb{Z}_2$ such that $a+b+c = i$.

Bounded width, examples

$\mathbb{A}$ has bounded width $\Rightarrow \mathrm{CSP}(\mathbb{A})$ is in P

- $(\{0,1\}; \neq, =)$ has relational width $(2,3)$ (computing transitivity),
- $(\mathbb{Q}; <)$ has relational width $(2,3)$ as well (transitivity again),
- 2-SAT has relational width $(2,3)$,
- $(\mathbb{Z}_2; R_0, R_1)$ where

$$R_i := \{(a,b,c) \in (\mathbb{Z}_2)^3 \mid a+b+c = i\}$$

  does not have relational width $(2,3)$.
  - **Why?**
    For every $i \in \mathbb{Z}_2$ and for every $a, b \in \mathbb{Z}_2$
    there exists $c \in \mathbb{Z}_2$ such that $a+b+c = i$.
    $\Rightarrow \Phi = R_0(x,y,z) \wedge R_1(x,y,z)$ is $(2,3)$-minimal
    and has non-empty constraints but is not satisfiable.

Collapse in the finite case

Theorem [Barto, 2016; Barto-Kozik, 2014].

Let $\mathbb{A}$ be a relational structure on a finite domain. TFAE:

- $\mathbb{A}$ has bounded width,
- $\mathbb{A}$ has relational width $(2, 3)$,
- $\mathbb{A}$ has an m-ary *weak near-unanimity* (WNU) polymorphism for all $m \geq 3$:

$$f(y, x, \dots, x) \approx f(x, y, x, \dots, x) \approx \dots \approx f(x, \dots, x, y).$$

Infinite-domain CSPs

Definition.

Let $k, \ell \geq 1$.

A relational structure $\mathbb{B}$ is *k-homogeneous* $:\leftrightarrow$
for all tuples a, b of finite length, a, b are in the same $\mathrm{Aut}(\mathbb{B})$-orbit
$\Leftrightarrow$ all k-subtuples of a, b are in the same $\mathrm{Aut}(\mathbb{B})$-orbit.

$\mathbb{B}$ is ℓ-bounded $:\leftrightarrow$ for every finite structure $\mathbb{X}$,
$\mathbb{X}$ embeds to $\mathbb{B}$ $\Leftrightarrow$ all substructures of $\mathbb{X}$ of size at most ℓ embed to $\mathbb{B}$.

finitely bounded homogeneous $\Rightarrow$ k-homogeneous l-bounded with
$k :=$ maximal arity of a relation, $l :=$ size of the largest bound

Definition.

Let $k, \ell \geq 1$.

A relational structure $\mathbb{B}$ is *k-homogeneous* $:\leftrightarrow$
for all tuples a, b of finite length, a, b are in the same $\mathrm{Aut}(\mathbb{B})$-orbit
$\Leftrightarrow$ all k-subtuples of a, b are in the same $\mathrm{Aut}(\mathbb{B})$-orbit.

$\mathbb{B}$ is ℓ-bounded $:\leftrightarrow$ for every finite structure $\mathbb{X}$,
$\mathbb{X}$ embeds to $\mathbb{B} \Leftrightarrow$ all substructures of $\mathbb{X}$ of size at most ℓ embed to $\mathbb{B}$.

finitely bounded homogeneous $\Rightarrow$ k-homogeneous l-bounded with
$k :=$ maximal arity of a relation, $l :=$ size of the largest bound

We are interested in structures with first-order definition
in a k-homogeneous ℓ-bounded structure $\mathbb{B}$ (*fo-reducts of* $\mathbb{B}$).

If $\mathbb{A}$ with domain $\{a_1, \ldots, a_n\}$ finite
$\Rightarrow$ $\mathbb{A}$ is a fo-reduct of $(\{a_1, \ldots, a_n\}; \{a_1\}, \ldots, \{a_n\})$
which is 1-homogeneous and 2-bounded.

Infinite-domain CSPs

Definition.

Let $k, \ell \geq 1$.

A relational structure $\mathbb{B}$ is *k-homogeneous* $:\leftrightarrow$
for all tuples a, b of finite length, a, b are in the same $\mathrm{Aut}(\mathbb{B})$-orbit
$\Leftrightarrow$ all k-subtuples of a, b are in the same $\mathrm{Aut}(\mathbb{B})$-orbit.

$\mathbb{B}$ is ℓ-bounded $:\leftrightarrow$ for every finite structure $\mathbb{X}$,
$\mathbb{X}$ embeds to $\mathbb{B}$ $\Leftrightarrow$ all substructures of $\mathbb{X}$ of size at most ℓ embed to $\mathbb{B}$.

finitely bounded homogeneous $\Rightarrow$ k-homogeneous l-bounded with
$k :=$ maximal arity of a relation, $l :=$ size of the largest bound

We are interested in structures with first-order definition
in a k-homogeneous ℓ-bounded structure $\mathbb{B}$ (*fo-reducts of* $\mathbb{B}$).

If $\mathbb{A}$ with domain $\{a_1, \ldots, a_n\}$ finite
$\Rightarrow$ $\mathbb{A}$ is a fo-reduct of $(\{a_1, \ldots, a_n\}; \{a_1\}, \ldots, \{a_n\})$
which is 1-homogeneous and 2-bounded.

Infinite-domain CSPs: digraph-acyclicity, linear programming,
reasoning problems in AI.

Pseudo-Maltsev conditions

Let $\mathbb{A}$ be finite.

- $\mathbb{A}$ has bounded width
  $\Leftrightarrow$ $\mathbb{A}$ has WNU polymorphisms of all arities ≥ 3
  (Barto, Kozik; 2014);

Pseudo-Maltsev conditions

Let $\mathbb{A}$ be finite.

- $\mathbb{A}$ has bounded width
  $\Leftrightarrow$ $\mathbb{A}$ has WNU polymorphisms of all arities ≥ 3
  (Barto, Kozik; 2014);
- $\mathrm{CSP}(\mathbb{A})$ is in P (if $\mathrm{P} \neq \mathrm{NP}$) $\Leftrightarrow$ $\mathbb{A}$ has Siggers polymorphism
  (Bulatov, Zhuk; 2017):

$$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y).$$

Pseudo-Maltsev conditions

Let $\mathbb{A}$ be finite.

- $\mathbb{A}$ has bounded width
  $\Leftrightarrow$ $\mathbb{A}$ has WNU polymorphisms of all arities ≥ 3
  (Barto, Kozik; 2014);
- $\mathrm{CSP}(\mathbb{A})$ is in P (if $\mathrm{P} \neq \mathrm{NP}$) $\Leftrightarrow$ $\mathbb{A}$ has Siggers polymorphism (Bulatov, Zhuk; 2017):

$$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y).$$

Conjecture [Bodirsky-Pinsker, 2011; Barto-Pinsker, 2016].

Let $\mathbb{A}$ be a fo-reduct of a k-homogeneous ℓ-bounded structure $\mathbb{B}$ that is a core. Suppose that $\mathrm{P} \neq \mathrm{NP}$. TFAE:

- $\mathrm{CSP}(\mathbb{A})$ is in P,
- $\mathbb{A}$ has a pseudo-Siggers polymorphism modulo $\overline{\mathrm{Aut}(\mathbb{B})}$:

$$\alpha \circ s(x, y, x, z, y, z) \approx \beta \circ s(y, x, z, x, z, y)$$

  for some $\alpha, \beta \in \overline{\mathrm{Aut}(\mathbb{B})}$.

Characterization of bounded width

What about bounded width? Can we take pseudo-WNUs,
i.e. $e_1 \circ f(y, x, \ldots, x) \approx \ldots \approx e_n \circ f(y, x, \ldots, x)$?

Characterization of bounded width

What about bounded width? Can we take pseudo-WNUs, i.e. $e_1 \circ f(y, x, \ldots, x) \approx \ldots \approx e_n \circ f(y, x, \ldots, x)$?

Bad news: For fo-reducts of $(\mathbb{Q}, <)$, no set of identities characterizing bounded width exists
(Bodirsky, Pakusa, Rydval, 2020).

Characterization of bounded width

What about bounded width? Can we take pseudo-WNUs, i.e. $e_1 \circ f(y, x, \ldots, x) \approx \ldots \approx e_n \circ f(y, x, \ldots, x)$?

Bad news: For fo-reducts of $(\mathbb{Q}, <)$, no set of identities characterizing bounded width exists
(Bodirsky, Pakusa, Rydval, 2020).

Good news: For fo-reducts of many other structures, pseudo-WNUs are sufficient - universal homogeneous graph, universal homogeneous tournament
(Mottet, Pinsker; 2020 - smooth approximations).
Reason: they have *canonical* pseudo-WNUs.

Canonical polymorphisms

Definition.

Let $\mathbb{A}$ be a fo-reduct of a k-homogeneous ℓ-bounded structure $\mathbb{B}$.
A polymorphism f of $\mathbb{A}$ is $\mathrm{Aut}(\mathbb{B})$-*canonical*
if it preserves the orbit-equivalence modulo $\mathrm{Aut}(\mathbb{B})$.

$\Leftrightarrow$ f induces an operation on the $\mathrm{Aut}(\mathbb{B})$-orbits of n-tuples
for every $n \geq 1$.

Canonical polymorphisms

Definition.

Let $\mathbb{A}$ be a fo-reduct of a k-homogeneous ℓ-bounded structure $\mathbb{B}$.
A polymorphism f of $\mathbb{A}$ is $\mathrm{Aut}(\mathbb{B})$-*canonical*
if it preserves the orbit-equivalence modulo $\mathrm{Aut}(\mathbb{B})$.

$\Leftrightarrow$ f induces an operation on the $\mathrm{Aut}(\mathbb{B})$-orbits of n-tuples for every $n \geq 1$.

Canonical polymorphisms play a key role in all known complexity classifications of infinite-domain CSPs.

Relational width collapses

There is no collapse of the relational width hierarchy
for fo-reducts of k-homogeneous ℓ-bounded structures (Grohe, 1994).

Relational width collapses

There is no collapse of the relational width hierarchy
for fo-reducts of k-homogeneous ℓ-bounded structures (Grohe, 1994).

Theorem. [Mottet, N., Pinsker, Wrona, 2021]

Let $k, \ell \geq 1$, and let $\mathbb{A}$ be a fo-reduct of a k-homogeneous ℓ-bounded structure $\mathbb{B}$. If $\mathbb{A}$ has canonical pseudo-WNU polymorphisms modulo $\overline{\mathrm{Aut}(\mathbb{B})}$ of all arities $n \geq 3$ then $\mathbb{A}$ has relational width $(2k, \max(3k, \ell))$.

stronger variant for pseudo-totally symmetric canonical polymorphisms

Collapses for particular structures

Collapse of relational width hierarchy for fo-reducts of:

- the universal homogeneous graph (relational width $(4, 6)$);

Collapses for particular structures

Collapse of relational width hierarchy for fo-reducts of:

- the universal homogeneous graph (relational width $(4, 6)$);
- the universal homogeneous tournament ($(4, 6)$);

Collapses for particular structures

Collapse of relational width hierarchy for fo-reducts of:

- the universal homogeneous graph (relational width $(4, 6)$);
- the universal homogeneous tournament ($(4, 6)$);
- the universal homogeneous K_n-free graph, where $n \geq 3$ ($(2, n)$);

Collapses for particular structures

Collapse of relational width hierarchy for fo-reducts of:

- the universal homogeneous graph (relational width $(4, 6)$);
- the universal homogeneous tournament ($(4, 6)$);
- the universal homogeneous K_n-free graph, where $n \geq 3$ ($(2, n)$);
- $(\mathbb{N}; =)$ ($(2, 3)$);

Collapses for particular structures

Collapse of relational width hierarchy for fo-reducts of:

- the universal homogeneous graph (relational width $(4, 6)$);
- the universal homogeneous tournament ($(4, 6)$);
- the universal homogeneous K_n-free graph, where $n \geq 3$ ($(2, n)$);
- $(\mathbb{N}; =)$ ($(2, 3)$);
- the countably infinite equivalence relation with infinitely many equivalence classes ($(2, 3)$);

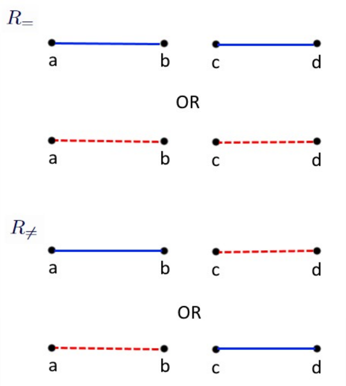
Collapses for particular structures

Collapse of relational width hierarchy for fo-reducts of:

- the universal homogeneous graph (relational width $(4,6)$);
- the universal homogeneous tournament ($(4,6)$);
- the universal homogeneous K_n-free graph, where $n \geq 3$ ($(2,n)$);
- $(\mathbb{N};=)$ ($(2,3)$);
- the countably infinite equivalence relation with infinitely many equivalence classes ($(2,3)$);
- the universal homogeneous partial order ($(2,3)$).

Example

Let $\mathbb{B} := (A; E)$ be the universal homogeneous graph and let N be the non-edge relation.
Let $\mathbb{A} := (A; R_=, R_{\neq})$ be a fo-reduct of $\mathbb{B}$ with quaternary $R_=, R_{\neq}$:



$R_=(a,b,c,d) \wedge R_{\neq}(a,b,c,d)$ is $(3,k)$-minimal for $k \geq 3$.
The exact relational width of $\mathbb{A}$ is $(4,6)$.

Further results

- Fo-reducts of a unary structure $\mathbb{B}$ that are cores:
  Bounded width is characterized by canonical pseudo-WNUs modulo $\overline{\mathrm{Aut}(\mathbb{B})}$, relational width at most $(4,6)$.

Further results

- Fo-reducts of a unary structure $\mathbb{B}$ that are cores:
  Bounded width is characterized by canonical pseudo-WNUs modulo $\overline{\mathrm{Aut}(\mathbb{B})}$, relational width at most $(4, 6)$.
- The same for CSPs modeling model-checking problem for MMSNP-sentences.
  $\Rightarrow$ Datalog rewritability problem for MMSNP is decidable and 2NExpTime-complete.

Open questions

- Does the ”collapse” hold for all fo-reducts of k-homogeneous, ℓ-bounded structures?

Open questions

- Does the ”collapse” hold for all fo-reducts of k-homogeneous, ℓ-bounded structures?
- Which intermediate relational widths are possible for fo-reducts of a particular k-homogeneous ℓ-bounded structure?

Open questions

- Does the ”collapse” hold for all fo-reducts of k-homogeneous, ℓ-bounded structures?
- Which intermediate relational widths are possible for fo-reducts of a particular k-homogeneous ℓ-bounded structure?
- For which k-homogeneous ℓ-bounded structures the characterization of bounded width by (canonical) pseudo-WNUs applies?

Thank you for your attention!