Covering codes in Sierpiński graphs

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The research was supported in part by the Slovene-French projects Proteus 17934PB (*and Egide Scholarschip)

(a, b)-codes

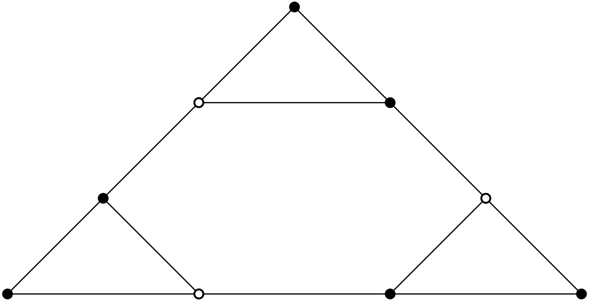
Definition

For integers a and b, an (a, b)-code is a set of vertices such that vertices in the code have exactly a neighbors in the code and all other vertices have exactly b neighbors in the code.

M.A. Axenovich, On multiple coverings of the infinite rectangular grid with balls of constant radius, Discrete Math. 268 (2003), 31–48.

P. Dorbec, S. Gravier and M. Mollard, Weighted codes in Lee metrics, submited.

An example (of (1,3)-code)



Covering codes

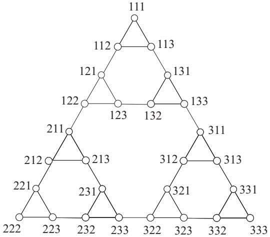
G.Cohen, I. Honkala, S. Lytsin and A. Lobstein, *Covering codes*, Elsevier, Amsterdam, 1997.

Sierpiński graphs

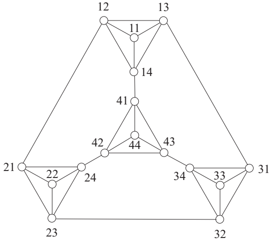
The graph $S(n,k)$ $(n,k \geq 1)$ is defined on the vertex set $\{1,2,\ldots,k\}^n$, two different vertices $u = (i_1, i_2, \ldots, i_n)$ and $v = (j_1, j_2, \ldots, j_n)$ being adjacent if and only if $u \sim v$. The relation $\sim$ is defined as follows: $u \sim v$ if there exists an $h \in \{1,2,\ldots,n\}$ such that

- $i_t = j_t$, for $t = 1, \ldots, h-1$;
- $i_h \neq j_h$; and
- $i_t = j_h$ and $j_t = i_h$ for $t = h+1, \ldots, n$.

Examples



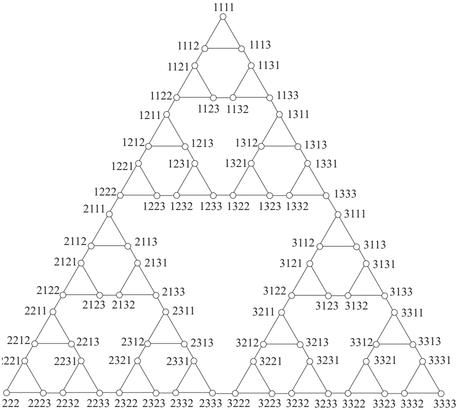
S(3,3)



S(2,4)

Examples

Figure: $S(4, 3)$



Some references

Introduced in...

S. Klavžar and U. Milutinović, Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem, Czechoslovak Math. J. 47(122) (1997), 95–104.

Motivation comes from topological theory of Fractals and Universal spaces...

S. L. Lipscomb anRd J. C. Perry, Lipscomb's $L(A)$ space fractalized in Hilbert's $l^2(A)$ space, Proc. Amer. Math. Soc. 115 (1992), 1157–1165.
U. Milutinović, Completeness of the Lipscomb space, Glas. Mat. Ser. III 27(47) (1992), 343–364.

and the fact that

$S(n, 3)$ is isomorphic to the Tower of Hanoi graph (with n disks).

Sierpiński graphs or Klavžar-Milutinović graphs

S. L. Lipscomb, *Fractals and Universal Spaces in Dimension Theory* (Springer Monographs in Mathematics), Springer-Verlag, Berlin, 2009.

Some more references

- S. Gravier, S. Klavžar and M. Mollard, Codes and $L(2,1)$-labelings in Sierpiński graphs, Taiwanese J. Math. 9 (2005), 671–681.
- S. Klavžar, Coloring Sierpiński graphs and Sierpiński gasket graphs, Taiwanese J. Math. 12 (2008), 513–522.
- S. Klavžar and M. Jakovac, Vertex-, edge-, and total-colorings of Sierpiński-like graphs, to appear in Discrete Math.
- S. Klavžar, U. Milutinović and C. Petr, 1-perfect codes in Sierpiński graphs, Bull. Austral. Math. Soc. 66 (2002), 369–384.
- S. Klavžar and B. Mohar, Crossing numbers of Sierpiński-like graphs, J. Graph Theory 50 (2005), 186–198.

One more basic definiton and the main observation

Two types of vertices

A vertex of the form $\langle ii \dots i \rangle$ of $S(n, k)$ is called an *extreme vertex*, the other vertices are called *inner*.
The extreme vertices of $S(n, k)$ are of degree $k - 1$ while the degree of the inner vertices is k.
Note that there k extreme vertices and that $|S(n, k)| = k^n$.

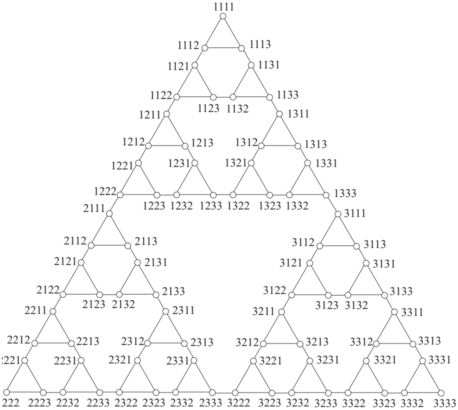
The main observation

Every vertex of $S(n, k)$ lies in a unique maximal k-clique (complete subgraph of size k).

More precisely

Extremal verices are simplicial vertices (their neighborhood induces complete subgraph), and closed neighborhoods of inner vertices induce complete graphs + an additional edge.

Figure: $S(4,3)$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Results

Easy observation

Codes in Sierpiński graphs $S(1, k) =$ complete graphs are clear, therefore from now on we always assume that $n \geq 2$.

Some necessary conditions

Lemma

Let C be an (a, b)-code in $S(n, k)$ then $a < k$ or $b = 0$.

Lemma

Let C be an (a, b)-code in $S(n, k)$ and K_k its clique. Then $|C \cap K_k| \leq a + 1$ and $|C \cap K_k| \geq b - 1$.

Lemma

Let C be an (a, b)-code of $S(n, k)$. Then $a \leq b$.

An immediate consequence of Lemmas 2 and 3 gives that the only possible (a, b)-codes are for $b = a, a + 1$ or $a + 2$.

Divisibility condition

Lemma

Let C be an (a,b)-code of $S(n,k)$ with d extremal vertices in C. Then

$$|C| \cdot (k - a + b) = bk^n + d\,.$$

Corollary

Let C be an (a,b)-code of $S(n,k)$ without extremal vertices. Then $(k - a + b) | bk^n$.

Handshaking Lemma gives...

Lemma

If a and k are odd then there is no (a, a)-code in $S(n, k)$.

From the divisibility condition it follows:

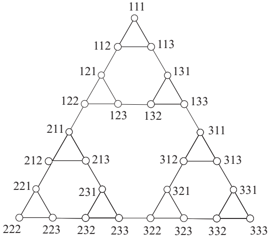
Lemma

If an $(a, a+2)$-code exists in $S(n, k)$, then $n = 2$ and $k = 2a + 1$.

Existence results

Lemma

Suppose there exist (a, b)-code in $S(2, k)$ that does not include any of the extreme vertices, then there also exists (a, b)-code in $S(n, k)$ for all $n \geq 3$.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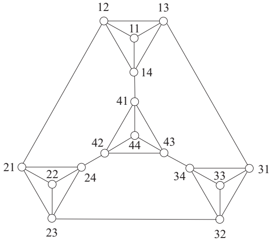
S(3,3)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S(2,4)

(a, a)-codes in $S(n, k)$

Lemma

An (a, a)-code of $S(n, k)$, $n \geq 2$, $a < k$, exist if and only if:

(i) *a is even or*

(ii) *a is odd and k is even.*

Corollary

Let C be an (a, a)-code in $S(n, k)$. Then $|C| = a \cdot k^{n-1}$.

(a, a)-codes in $S(2, k)$

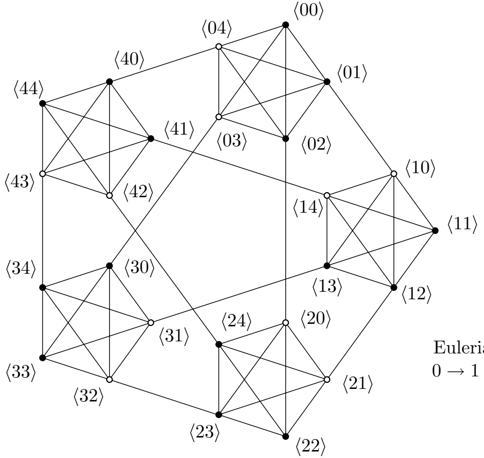
Lemma

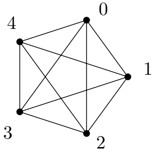
An $(a, a+2)$-code exists in $S(n, k)$ for $n = 2$ and $k = 2a + 1$.

Proof.

Use (an arbitrary) Eulerian tour of K_n and include alternatively vertices from the tour into the code.

(2,4)-code in $S(2,5)$





Eulerian tour in K_5:
$0 \to 1 \to 2 \to 3 \to 4 \to 0 \to 2 \to 4 \to 1 \to 3 \to 0$

Corollary

Let C be an $(a, a+2)$-code in $S(n, k)$, where $n = 2$ and $k = 2a+1$. Then $|C| = (a+1) \cdot k$.

Near codes

C is a near code if the condition from the definition of (a, b)-code is fulfilled for all inner vertices.

In this case extreme vertices miss at most one neighbor from the code.

We denote by $\bullet$ extremal vertices in the near code and $\circ$ extremal vertices in its complementary. Furthermore, we add the subscript $*$ for a vertex of weight 0 and $+$ for weight 1.

Special near codes

- SO^n is a near code with the additional conditions n is odd and there are $a+1$ extreme vertices $\bullet_*$ and the $k-a-1$ others are $\circ_*$.
- WO^n is a near code with the additional conditions n is odd and there are a extreme vertices $\bullet_+$ and the $k-a$ others are $\circ_+$
- SE^n is a near code with the additional conditions n is even and there are a extreme vertices $\bullet_+$ and the $k-a$ others are $\bullet_*$.
- WE^n is a near code with the additional conditions n is even and there are $a+1$ extreme vertices $\circ_+$ and the $k-a-1$ others are $\circ_*$

Theorem

Let $n \geq 2$, $a \geq 0$ and $k > a$ be integers. The near codes of $S(n,k)$ are precisely SO^n and WO^n if n is odd and SE^n and WE^n if n is even.

$(a, a+1)$-codes in $S(n,k)$

Corollary

Graphs $S(n,k)$ admits a $(a, a+1)$-code if and only if n is odd and $0 \leq a \leq k-1$.

Corollary

Let C be an $(a, a+1)$-code in $S(n,k)$, where n is an odd number. Then $|C| = (a+1) \cdot \frac{k^n+1}{k+1}$.

Main result

Theorem

The existing (a, b)-codes in $S(n, k)$ satisfy $0 \leq a < k$ and they are of three different types:

(i) *An (a, a)-code in $S(n, k)$ for $n \geq 2$ and k is even or a is even and k is odd.*

(ii) *An $(a, a + 1)$-code in $S(n, k)$ for k odd.*

(iii) *An $(a, a + 2)$-code in $S(n, k)$ for $n = 2$ and $k = 2a + 1$.*

Uniqueness of (a, b)-codes

Are all (a, b)-codes in Sierpiński graphs (up to the automorphisms) unique?

Another fractal type constructions

might be interesting to study different graph parameters.

T H A N K Y O U!