INTERIOR COMPONENTS OF A TILE ASSOCIATED TO A QUADRATIC CANONICAL NUMBER SYSTEM

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Abstract. Let \( \alpha = -2 + \sqrt{-1} \) be a root of the polynomial \( p(x) = x^2 + 4x + 5 \). It is well known that the pair \((p(x), \{0, 1, 2, 3, 4\})\) forms a canonical number system, i.e., that each \( x \in \mathbb{Z}[\alpha] \) admits a finite representation of the shape \( x = a_0 + a_1\alpha + \cdots + a_E\alpha^E \) with \( a_i \in \{0, 1, 2, 3, 4\} \). The set \( T \) of points with integer part 0 in this number system

\[
T := \left\{ \sum_{i=1}^{\infty} a_i\alpha^{-i}, \ a_i \in \{0, 1, 2, 3, 4\} \right\}
\]

is called the fundamental domain of this canonical number system. It has been studied extensively in the literature. Up to now it is known that it is a plane continuum with nonempty interior which induces a tiling of the \( \mathbb{C} \). However, its interior is disconnected. In the present paper we describe some of (the closures of) the components of its interior as attractors of graph-directed self-similar sets.

1. Introduction and basic definitions

We are interested in describing the topology of a plane self-similar set with disconnected interior that is related to a quadratic canonical number system (see Figure 1). More precisely, we want to describe the closure of some components of its interior by a graph directed self-similar set. We start with the necessary definitions.

![Figure 1. Tile associated to the base \(-2 + \sqrt{-1}\) with interior component containing 0.](image)

It is known (see [9, 13, 14]) that the root \( \alpha = -2 + \sqrt{-1} \) of the polynomial \( x^2 + 4x + 5 \) together with \( \mathcal{N} := \{0, 1, 2, 3, 4\} \) forms a canonical number system (or CNS) \((\alpha, \mathcal{N})\), i.e., each element \( x \in \mathbb{Z}[\alpha] \) has a unique representation

\[
x = \sum_{i=0}^{E(x)} a_i\alpha^i
\]
for some non-negative integer \( \ell(x) \) and \( a_i \in \mathcal{N} \) with \( a_{\ell(x)} \neq 0 \) for \( x \neq 0 \). We define the natural embedding

\[
\Phi : \mathbb{C} \rightarrow \mathbb{R}^2 \\
x \mapsto (\Re(x), \Im(x)).
\]

Then the multiplication by \( \alpha \) can be represented by the \( 2 \times 2 \) matrix

\[
A := \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix},
\]

i.e., for every \( x \in \mathbb{C} \),

\[
\Phi(\alpha x) = A \Phi(x).
\]

The set \( T \) of points of integer part zero in the base \( \alpha \) embedded into the plane is defined by

\[
T := \left\{ \Phi(\alpha^{-i} a_i), (a_i)_{i \in \mathbb{N}} \in \mathcal{A}^\mathbb{N} \right\} = \left\{ \sum_{i=1}^{\infty} A^{-i} \Phi(a_i), (a_i)_{i \in \mathbb{N}} \in \mathcal{A}^\mathbb{N} \right\}
\]

and is depicted in Figure 1. \( T \) is often called the \textit{fundamental domain} of the number system \((\alpha, \mathcal{N})\). Thus each point of this set can be represented by an infinite string \( w = (a_1, a_2, a_3, \ldots) \) with \( a_i \in \mathcal{N} \). The set \( T \) satisfies the equation

\[
T = \bigcup_{i=0}^{4} \psi_i(T),
\]

where \( \psi_i, (i = 0, \ldots, 4) \) are contractions defined via the matrix \( A \) and the embedding \( \Phi \) by

\[
\psi_i(x) = A^{-1} (x + \Phi(i)), \quad x \in \mathbb{R}^2 \quad (0 \leq i \leq 4).
\]

\( T \) is a self-similar connected compact set (or \textit{continuum}) with nonempty interior (see [12]). It induces a \textit{tiling} of the plane by its translates. We recall that a \textit{tiling} (cf. [10, 24]) of the plane is a decomposition of \( \mathbb{R}^2 \) into sets whose interiors are pairwise disjoint (so-called \textit{non-overlapping} sets), each set being the closure of its interior and having a boundary of Lebesgue measure zero. Properties of tiles and tilings can be found for instance in [3, 7, 15, 22, 25]. It was shown in [12] that the family of sets

\[
\{ T + \Phi(\omega), \ \omega \in \mathbb{Z}[\alpha] \}
\]

is a tiling of the plane. We call \( T \) the \textit{central tile} of this tiling.

\textbf{Remark 1.1.} This tile is an example of the large class of tiles associated to a root \( \beta \) of a quadratic polynomial \( p(x) = x^2 + Ax + B \). Indeed, if \( p(x) \) satisfies the conditions

\[
-1 \leq A \leq B, \quad B \geq 2,
\]

then \((\beta, \{0, 1, \ldots, B - 1\})\) is a CNS (see e.g. [5, 6, 13, 14]). One can associate a tile to these CNS in the same way as we did for the special case \( A = 4, B = 5 \). It is shown in [1] that for \( 2A - B \geq 3 \), which is also the case for our tile \( T \), these tiles have disconnected interior.

Some research on the structure of the components of the interior of self-similar and self-affine tiles has already been done. In Bailey \textit{et al.} [2] investigate the interior of the Lévy dragon, which is a self-affine continuum with disconnected interior providing a tiling of the plane. They stated many conjectures concerning the geometrical shape of the connected components of the interior. Ngai and Nguyen [18] study the components of the Heighway dragon. Moreover, Ngai and Tang [19, 20] gave general results on components of the interior of self-affine tiles. As an example they consider our tile \( T \) in [19]. They prove that the closure of each component of its interior is homeomorphic to a closed disk.

The aim of our paper is to describe the closure \( C_0 \) of the connected component containing \( \Phi(0) \) (or 0, for short) of the interior of \( T \). To make this precise we need some notations: a \textit{finite digit string} is a finite sequence \( w = (a_1, \ldots, a_n) \) with \( n \in \mathbb{N} \) and \( a_i \in \mathcal{N} \). The integer \( n \) is then called
the length of the string \(w\) (we write \(|w| = n\)). For a finite string \(w = (a_1, \ldots, a_n)\) we define the map \(\psi_w\) by

\[
\psi_w(x) := \psi_{a_1} \circ \cdots \circ \psi_{a_n}(x) = A^{-n}x + \sum_{i=1}^{n} A^{-i} \Phi(a_i), \quad x \in \mathbb{R}^2.
\]

The set \(\psi_w(T)\) is called an \(n\)-th level subpiece of \(T\). So by definition, it contains all the points represented by an infinite string of the shape \((a_1, \ldots, a_n, d_1, d_2, \ldots)\) with \(d_i \in \mathcal{N}\).

Note that iterating (1.2) we have for every \(n \geq 1\) the subdivision principle

\[
T = \bigcup_{w, |w| = n} \psi_w(T).
\]

Our description of \(C_0\) will be in terms of \(n\)-th level subpieces with \(n \geq 0\). Indeed, it will be shown that \(C_0\) can be obtained as the closure of the union of such subpieces; the strings \(w\) involved in this union will be read off from a graph \(G\) presented in the next section. The set \(C_0\) can be viewed as the attractor of a graph-directed construction (see Definition 2.1).

A similar description will be obtained in a forthcoming paper for a large class of tiles associated to quadratic number systems; nevertheless, this class will not contain the present example.

2. Statement of the main result

As indicated at the end of the previous section, the description of \(C_0\) will be given via a graph \(G\). We will present this graph in the present section, explain how the set \(C_0\) can be derived from this graph, and state our main theorem together with a sketch of the proof.

2.1. Graph \(G\). This graph is depicted in Figure 2. For the so-called accepting state \(\circ\), there is by convention an edge \(\circ \xrightarrow{a} \circ\) for every \(a \in \mathcal{N}\).

We state some definitions and make some remarks about this graph. The graph \(G\) is right resolving, i.e., each walk of \(G\) is uniquely defined by its starting state together with its labeling. Thus we will write \(w = (A; a_1, \ldots, a_n)\) for a walk \(w\) starting in \(A\) with labeling \((a_1, \ldots, a_n)\). For subsets of the walks in \(G\) we adopt the following notations:

\[
\begin{align*}
p & \quad \text{set of all walks in } G, \\
p_n & \quad \text{set of all walks in } G \text{ having length } n, \\
p(A_1) & \quad \text{set of walks in } p \text{ starting at node } A_1, \\
p_n(A_1) & \quad \text{set of walks in } p_n \text{ starting at node } A_1, \\
p(A_1, A_2) & \quad \text{set of walks in } p(A_1) \text{ ending at node } A_2, \\
p_n(A_1, A_2) & \quad \text{set of walks in } p_n(A_1) \text{ ending at node } A_2.
\end{align*}
\]

If \(w\) is a walk in \(G\) with labeling \((a_1, \ldots, a_n)\), then we denote the walk which corresponds to \(w\) in the transposed graph \(G^T\) by \(w^T\) (backwards walk). Its labeling is obviously \((a_n, \ldots, a_1)\). The terminal state of a walk \(w\) in \(G\) shall be denoted by \(t(w)\). If \(w_1\) and \(w_2\) are two walks in \(G\) and \(w_2\) starts at the terminal state of \(w_1\) then we write \(w_1 \& w_2\) for the concatenation of these two walks. If we emphasize on the labeling \((a_1, \ldots, a_n)\) of a walk \(w\) we will write \(w = (a_1, \ldots, a_n)\). For instance, if we concatenate \(w_1 = (A_1; a_1, \ldots, a_n)\) and \(w_2 = (A_2; b_1, \ldots, b_m)\) we will often write \((A_1, a_1, \ldots, a_n) \& (b_1, \ldots, b_m)\) because the starting state of \(w_2\) is defined via \(w_1\). For a walk \(w\) of length \(n\) and \(k \leq n\) we denote by \(w|_k\) the walk consisting of the first \(k\) edges of \(w\), i.e., \((a_1, \ldots, a_n)|_k = (a_1, \ldots, a_k)\). If \(w = w|_k\) we write \(v < w\).

If \(A\) is a state of \(G\), we call \(A^*\) its dual. By convention we set \(F' = F\), \(\circ = \circ'\) and \(A' = A\) for all the other states of \(G\). Note that in \(G\), every edge \(A_1 \xrightarrow{a} A_2\) has a dual edge \(A_1' \xrightarrow{A^{-1}a} A_2'\).
Figure 2. Graph $\mathcal{G}$ of the closure of the interior component of $\mathcal{T}$ containing 0.

2.2. Graph directed sets. The graph $\mathcal{G}$ can be used to describe the set $C_0$. We recall to this matter the notion of graph directed self-affine (self-similar) sets.

Definition 2.1. A geometric graph directed construction of $\mathbb{R}^d$ consists of

1. finitely many compact subsets $J_1, \ldots, J_q$ of $\mathbb{R}^d$ such that each $J_i$ has nonempty interior;
2. a directed graph $G(V,E)$ with set of vertices $V = \{1, \ldots, q\}$ and to each edge $e \in E$ a uniform contraction $T_e$ having the following properties:
   a) Each vertex has outgoing edges.
   b) Let $E_{ij}$ be the set of edges leading from $i$ to $j$. Then $\bigcup_j T_e(J_j) \mid e \in E_{ij}$ is a non-overlapping family and

\[
J_i \supset \bigcup_j T_e(J_j) \mid e \in E_{ij} \quad (i \in \{1, \ldots, q\}).
\]
A very similar definition can be found in Mauldin and Williams [16] (cf. also [4] and [8]). Despite the definition of geometric graph directed construction in [16] is more restrictive in some regards the following result is still valid with the same proof.

**Proposition 2.2** ([16, Theorem 1]). There exists a unique vector \((K_1, \ldots, K_q)\) of compact subsets of \(\mathbb{R}^d\) such that for each \(i \in \{1, \ldots, q\}\)
\[
K_i = \bigcup_j (T_e(K_j) \mid e \in E_{ij})
\]
holds. \((K_1, \ldots, K_q)\) is called a system of graph directed sets. If the \(K_i\) are affinities the system is called self-affine, if they are similarities, it is called self-similar.

By attaching the contraction \(\psi_a\) defined in (1.3) to each edge labelled by \(a\) in \(G\), the graph \(G\) leads to a system of graph directed sets. For each state \(A\) of \(G\) let
\[
M(A) := \left\{ x = \sum_{i \geq 1} A^{-i}\Phi(a_i), \ w = (a_1, a_2, \ldots) \text{ infinite walk of } p(A) \right\}.
\]
Then we have the following result.

**Proposition 2.3.** The vector \(\{M(A), A \in G\}\) together with the graph \(G\) defines a system of graph directed sets. It is even a system of self-similar graph directed sets.

**Proof.** We have to verify the conditions in Definition 2.1 and Proposition 2.2. \(M(A)\) is obviously bounded. The fact that it is closed follows by a Cantor diagonal argument very similar to the one used in Kátai [11].

The family \(\bigcup_B \{\psi_e(M(B)) \mid e \in E_{AB}\}\) is non-overlapping because \(M(B) \subset T\) and \(G\) is right resolving (note that \((\alpha, \mathcal{N})\) admits unique representations). Furthermore, it is easy to see that \(\{M(A) \mid A \in G\}\) fulfills (2.2). \(\square\)

In particular,
\[
M := M(F)
\]
is a compact set and \(M \subset T\). This paper aims at showing that \(M = C_0\), as stated in our main result.

2.3. Main result.

**Theorem 2.4.** Let \((\alpha = -2 + \sqrt{-1}, \mathcal{N} = \{0, 1, 2, 3, 4\})\) be the quadratic canonical number system related to the polynomial \(x^2 + 4x + 5\). Let \(T\) be the fundamental domain associated to \((\alpha, \mathcal{N})\). Then \(\text{Int}(M)\) is the component of \(\text{Int}(T)\) containing 0. Moreover, \(M\) is the closure of its interior, hence \(M = C_0\), the closure of the component of \(\text{Int}(T)\) containing 0.

**Remark 2.5.** Note that the above mentioned result of Ngai and Tang [19] implies that \(C_0\) is homeomorphic to a closed disk.

For the proof of this theorem we will consider approximations of the set \(M\) in terms of finite walks of the graph \(G\).

**Definition 2.6.** For some \(n \in \mathbb{N}\) and some state \(A\) of \(G\) let \(W \subset p_n(A)\) be a set of walks. Then we set
\[
\mathcal{M}(W) := \bigcup_{w \in W} \psi_w(T).
\]
Here, according to (1.5), \(\psi_w(T)\) is the subpiece associated to the labeling of \(w\). The approximating sets are obtained by taking for \(W\) the sets
\[
G_n := p_n(F, \circ) \quad (n \geq 3)
\]
It is easy to see that for every \( n > 3 \), \( \mathcal{M}(G_{n-1}) \subset \mathcal{M}(G_n) \subset M \). Note that there exists no walk in \( p_n(F, \circ) \) if \( n < 3 \). This means that there are no subpieces \( \psi_w(T) \) with \( |w| < 3 \) entirely contained in \( M \).

We will show that \( M = C_0 \) in the following way. First we show that the interior of \( M \) is connected and contained in the interior of \( T \), hence \( \text{Int}(M) \subset \text{Int}(C_0) \). In a second step we show that its boundary lies on the boundary of \( T \), which implies that \( C_0 \subset M \). By proving that \( M \) is the closure of its interior, \( \text{i.e.}, \bar{\text{Int}(M)} = M \), this will finally yield \( M = C_0 \).

The connectivity of \( \text{Int}(M) \) will be obtained by considering the approximations \( \mathcal{M}(G_n) \) for \( n \geq 3 \), since we will show that

\[
\text{Int}(M) \subset \bigcup_{n \geq 3} \mathcal{M}(G_n).
\]

Using this fact, we will be able to connect a point from \( \text{Int}(M) \) to \( 0 \) by a path going from subpiece to subpiece with increasing size \( \text{i.e.}, \) over subpieces \( \psi_w(T) \) with decreasing \( |w| \) within \( \text{Int}(T) \).

The remaining part of this paper is organized as follows. In Section 3 we present an automaton \( B_0 \) which is helpful to determine whenever subpieces intersect each other. This leads to the definition of an action of \( B_0 \) on \( G \) in a way that is stated in this section too. Sections 4 and 5 will be helpful for the proof of the connectivity of \( \text{Int}(M) \). Section 6 shows that the boundary of \( M \) is contained in the boundary of \( T \). Section 7 is devoted to the construction of connected paths within the interior of \( T \) together with some of its neighbors, that will be also used to show the connectivity of \( \text{Int}(M) \) inside \( \text{Int}(T) \). Section 8 eventually contains the proof of Theorem 2.4.

3. Counting automaton \( B_0 \) and its action on the graph \( G \)

We define a counting automaton and an action of this automaton on the preceding graph \( G \).

3.1. Counting automaton \( B_0 \). The counting automaton \( B_0 \) is given in [21, 23] for bases of quadratic canonical number systems in general and reproduced in Figure 3 for \( \alpha = -2 + \sqrt{3} \).

![Figure 3. Counting automaton \( B_0 \).](image)

Its states are defined by

\[
\pm P \ := \ \pm \Phi(1), \quad \pm Q \ := \ \pm \Phi(3 + \alpha), \quad \pm R \ := \ \pm \Phi(-4 - \alpha),
\]
and \( \circ \) denotes the accepting state 0.

The edges of \( B_0 \) are defined as follows. There exists an edge from a state \( S \) to a state \( S' \) in \( B_0 \) labelled by \( a|a' \) with \( a, a' \in \mathcal{N} \) if and only if
\[
S + \Phi(a) = A S' + \Phi(a').
\]
In particular, since \( \circ \) denotes 0, there is an edge \( \frac{a|a'}{a'|a} \circ \) for each \( a \in \mathcal{N} \) (these edges are not represented in Figure 3).

The numbers \( a \) and \( a' \) in a label \( a|a' \) are called input and output digits respectively.

**Remark 3.1.** Note that \( B_0 \) is right resolving: to any state \( S \) and any input digit \( a \in \mathcal{N} \), there is exactly one state \( S' \) and one output digit \( a' \) such that the addition in the graph can be performed, i.e., such that \( S \frac{a|a'}{a'|a} S' \in B_0 \).

Thus the automaton \( B_0 \) can also perform the addition of \( S + \sum_{i=0}^{n-1} A^i \Phi(a_{n-i}) \) for \( S \in B_0 \) and \( a_i \in \mathcal{N} \), simply by feeding \( B_0 \) with the input digit string \((a_n, \ldots, a_1)\) from left to right starting from \( S \) and collecting the output digit string \((a'_n, \ldots, a'_1)\) and the landing state \( S' \) to
\[
S \xrightarrow{a_{n-1}|a_n} S_1 \xrightarrow{a_{n-2}|a_{n-1}} \ldots \xrightarrow{a_1|a'_n} S'
\]
corresponds the addition
\[
S + \sum_{i=0}^{n-1} A^i \Phi(a_{n-i}) = \sum_{i=0}^{n-1} A^i \Phi(a'_{n-i}) + A^n S'.
\]

Note that for \( S' = 0 \), i.e., for a walk leading from \( S \) to 0 in \( B_0 \), the term \( A^n S' \) vanishes. In this case the automaton produces from the string \((a_n, \ldots, a_1)\) corresponding to the “A-adic” expansion of \( z = \sum_{i=0}^{n-1} A^i \Phi(a_{n-i}) \) the string \((a'_n, \ldots, a'_1)\) which is the string of the “A-adic” expansion of \( z + S \).

**Remark 3.2.** By Remark 3.1, the outputs \( S' \) and \((a'_n, \ldots, a'_1)\) are uniquely defined by the inputs \( S \) and \((a_n, \ldots, a_1)\).

The automaton emerging from \( B_0 \) by leaving away the accepting state is called \( B \). It is helpful in order to characterize the boundary of \( T \), as the following results show.

**Proposition 3.3** (Scheicher and Thuswaldner [22]). The following equation holds for the boundary \( \partial T \) of \( T \):
\[
\partial T = \bigcup_{S \in B} \{ T \cap (T + S) \}.
\]

Thus, even if \( T \) has more neighbors than the six presented here (see [1]), these neighbors are sufficient to describe the whole boundary.

**Proposition 3.4** (Müller et al. [17]). For \( S \in \{ \pm P, \pm Q, \pm R \} \) let \( B_S := T \cap (T + S) \). Then \( B_S \neq \emptyset \). Furthermore, if there exists an infinite walk
\[
S \xleftarrow{a_1|a'_1} S_1 \xleftarrow{a_2|a'_2} \ldots
\]
in \( B \) such that \( x = \sum_{i \geq 1} A^{-i} \Phi(a_i) \) then \( x \in B_S \).

As a consequence of these propositions and of the definition of \( B_0 \), we have the following way to characterize that two \( n \)-th level subpieces of \( T \) have common points.

**Characterization 3.5.** Let \( n \in \mathbb{N} \) and \( w = (a_1, \ldots, a_n) \), \( w' = (a'_1, \ldots, a'_n) \) be two strings of length \( n \). If there is a walk
\[
S_n \xrightarrow{a_n|a'_n} S_{n-1} \ldots \xrightarrow{a_1|a'_1} \circ
\]
in \( B_0 \), then
\[
\psi_w(T) \cap \psi_{w'}(T) \neq \emptyset.
\]
3.2. Graph action of $\mathcal{B}_0$ on $\mathcal{G}$. The structure of $\mathcal{M}$ will be understood with the help of the following graph action of $\mathcal{B}_0$ on the graph $\mathcal{G}$.

**Definition 3.6.** Let $S$ be a state in $\mathcal{B}_0$, $A$ a state of $\mathcal{G}$ and let $w = (A; a_1, \ldots, a_n) \in p_n(A)$. Take $(a_n, \ldots, a_1)$ as the input string for the automaton $\mathcal{B}_0$ with starting state $S$ and denote the output string by $(a'_n, \ldots, a'_1)$. Then we define $\Psi_S(w) := (A; a'_1, \ldots, a'_n)$. $\Psi_S$ is called the addition of $S$. If the automaton $\mathcal{B}_0$ rests in $\sigma$ after reading $(a_n, \ldots, a_1)$ and if $\Psi_S(w) \in p_n(A)$ then we say that the addition of $S$ is admissible for $w$. Note that for a walk $w = (A)$ of length zero only $\Psi_\sigma(w) = w$ is admissible.

**Remark 3.7.** By Characterization 3.5, the admissible addition of $S$ to a string $w$ produces a string $w' := \Psi_S(w)$ such that $\psi_w(T) \cap \psi_{w'}(T) \neq \emptyset$.

**Definition 3.8.** Fix $n \in \mathbb{N}$, a state $A$ of $\mathcal{G}$ and let $w_1, w_2 \in W \subseteq p_n(A)$. Let $\psi_{w_1}(T)$ and $\psi_{w_2}(T)$ be the corresponding subsets of $T$. We say that $w_1$ and $w_2$ are $W$-equivalent to each other, if there exist finitely many states $S_1, \ldots, S_m$ of $\mathcal{B}$ such that the following conditions hold with admissible additions $\Psi_{S_j}$,

$$
\Psi_{S_m} \circ \cdots \circ \Psi_{S_1}(w_1) = w_2 \quad \text{and} \quad \Psi_{S_j} \circ \cdots \circ \Psi_{S_1}(w_1) \in W \quad (1 \leq j \leq m).
$$

We denote this by $w_1 \sim w_2 (W)$ or simply by $w_1 \sim w_2$ if the underlying set $W$ is clear from the context. In this case we also call the corresponding sets $\psi_{w_1}(T)$ and $\psi_{w_2}(T)$ $W$-equivalent and use the same notation $\psi_{w_1}(T) \sim \psi_{w_2}(T)$.

If $w_1 = \Psi_S(w_2)$ we also write in a slight abuse of notation $w_1 \sim_S w_2$ or $w_2 \sim w_1$.

It is easy to check that $\sim$ is an equivalence relation.

**Remark 3.9.** We want to give some comments on these definitions.

1) Let $w = (A; a_1, \ldots, a_n) \in p_n(A)$ be a walk and assume that $\Psi_{S_n}$ with $S_n \in \mathcal{B}_0$ is an admissible addition for $w$. Then from Definition 3.6 it follows that there exists a walk

$$
\mathcal{B}_0\left(\frac{a_n}{a'_n}, S_{n-1}; \frac{a_{n-1}}{a'_{n-1}}, \ldots; \frac{a_2}{a'_2}, S_1; \frac{a_1}{a'_1}\right)
$$

in $\mathcal{B}_0$ such that $w' = (A; a'_1, \ldots, a'_n) \in p_n(A)$. Furthermore, we can perform this addition "digit wise", i.e.,

$$
\Psi_{S_n}(A; a_1, \ldots, a_n) = \Psi_{S_n}(A; a_1, \ldots, a_j) \circ (a'_{j+1}, \ldots, a'_n) = (A; a'_1, \ldots, a'_n).
$$

Note that from this we easily see that

$$
\psi_{\Psi_{S_n}(w)}(T) = \psi_w(T + S_n).
$$

We even have, if $S_n^{(1)}, \ldots, S_n^{(m)}$ are $m$ states of $\mathcal{B}_0$ such that the additions $\Psi_{S_n^{(1)}} \circ \cdots \circ \Psi_{S_n^{(m)}}(w)$ are admissible for $1 \leq j \leq m$, that:

$$
\psi_{\Psi_{S_n^{(m)}} \circ \cdots \circ \Psi_{S_n^{(1)}}(w)}(T) = \psi_w(T + S_n^{(m)} + \ldots + S_n^{(1)}).
$$

2) Let $w_1, w_2$ be $W$-equivalent for some $W \subseteq p_n(A)$. Then there exist $v_1, \ldots, v_m$ with $v_1 := w_1$ and $v_m := w_2$ such that

$$
\psi_{v_j}(T) \cap \psi_{v_{j+1}}(T) \neq \emptyset \quad (1 \leq j \leq m - 1).
$$

This follows immediately from Remark 3.7 together with Definition 3.8.

3) Let $W \subseteq p_n(A)$ and let $k \leq n$ be integers. Let $w_1, w_2 \in W$ such that $w_1|k = w_2|k =: \sigma$. Then there exist

$$
\tau_1, \tau_2 \in W_\sigma := \{ \tau \mid \sigma \& \tau \in W \}
$$

such that $w_i = \sigma \& \tau_i \ (i = 1, 2)$. If $\tau_1$ and $\tau_2$ are equivalent in $W_\sigma$ then $w_1$ and $w_2$ are equivalent in $W$. This follows from the following fact together with Definition 3.8. Let $\tau, \tau' \in W_\sigma$ and $S \in \mathcal{B}_0$. Then

$$
\Psi_S(\tau) = \tau' \implies \Psi_S(\sigma \& \tau) = \Psi_\sigma(\sigma) \& \tau' = \sigma \& \tau'.
$$
This implies that
\[ \tau \sim \tau'(W_\sigma) \implies \sigma \& \tau \sim \sigma \& \tau'(W), \]
and this means that in order to examine equivalences of walks it often suffices to examine equivalences of their tails.

**Definition 3.10.** Fix \( n \in \mathbb{N} \) and let \( W \subset p_n(A) \) be a set of strings. Then \( W \) and the set
\[ \mathcal{M}(W) := \{ \psi_w(T) \mid w \in W \} \]
are called transitive if we have \( w_1 \sim w_2(W) \) for each two \( w_1, w_2 \in W \).

**Remark 3.11.** Since the subpiece \( \psi_w(T) \) is arcwise connected for every string \( w \) (remember that \( T \) is arcwise connected), by definition of the equivalence relation in \( W \), a transitive set \( W \subset p_n(F, \circ) \) yields an arcwise connected subset \( \mathcal{M}(W) \) of \( \mathcal{M} \).

We end this section with a last definition.

**Definition 3.12.** Let \( A \) be a node of \( \mathcal{G} \) and \( S \) a state of \( \mathcal{B}_0 \). If \( \Psi_S \) is admissible for all walks in \( p(F, A) \) then we call \( \Psi_S \) an admissible graph action for \( A \) on \( \mathcal{G} \), or an A-action, for short.

If \( \Psi_S \) is an A-action then we call
\[ F(\Psi_S, A) := \{ t(w') \mid w' = \Psi_S(w) \text{ for a walk } w \in p(F, A) \} \]
the ending set of \( (\Psi_S, A) \).

**Remark 3.13.** Let the assertion : \( \Psi_S \) is an A-action with ending set \( F(\Psi_S, A) \), then we define the dual assertion : \( \Psi_{-S} \) is an A'-action with ending set \( \{ Z', Z \in F(\Psi_S, A) \} \).

4. Admissibility of all the additions for a class of walks in \( \mathcal{G} \)

This section is devoted to the proof of the following result.

**Proposition 4.1.** Let \( w \) be a finite walk in \( p(F, \circ) \). Then for each state \( S \) of \( \mathcal{B} \) the addition \( \Psi_S(w) \) is admissible for \( w \).

**Remark 4.2.** By Remarks 3.7 and 3.9.1, this implies that for \( w \in p_n(F, \circ) \), all the sets \( \psi_w(T + S) \) with \( S \in \mathcal{B}_0 \) are subpieces of \( T \) that have non-empty intersection with \( \psi_w(T) \).

Suppose that \( w \) is a finite walk in \( p(F, \circ) \). Then \( w \) is of the shape
\[ (4.1) \quad F \xrightarrow{a_1} A_1 \xrightarrow{a_2} \ldots \xrightarrow{a_k} A_k \xrightarrow{a_{k+1}} \circ \xrightarrow{a_{k+2}} \ldots \xrightarrow{a_n} \circ \quad (A_k \neq \circ) \]
for some \( 2 \leq k < n \). Let \( S_n := S \). Note that \( S_n \) together with the labels \( (a_1, \ldots, a_n) \) defines uniquely the walk
\[ (4.2) \quad S_n \xrightarrow{a_1} S_{n-1} \xrightarrow{a_{n-1}} S_{n-2} \ldots \xrightarrow{a_{k+1}} S_{k+1} \xrightarrow{a_{k+2}} S_k \xrightarrow{a_k} \ldots \xrightarrow{a_1} S_0 \]
in \( \mathcal{B}_0 \) (recall that \( \mathcal{B}_0 \) is right resolving by Remark 3.1). By the definition of \( \Psi_S \) this walk yields the identities
\[ (4.3) \quad \Psi_S(w) = \Psi_S(a_1, \ldots, a_j)(a_{j+1}', \ldots, a_n') = \Psi_S(a_1, \ldots, a_k)(a_{k+1}', \ldots, a_n'). \]
We want to show that \( \Psi_S(w) \) is a walk in \( p(F) \) and that \( S_0 = \circ \) for all states \( S \) of \( \mathcal{B} \). We first need the following lemma.

**Lemma 4.3.** \( \Psi_S \) is an A-action in the following cases :

(\( L, Q \)): \( \Psi_{-Q} \) is an L-action with \( F(\Psi_{-Q}, L) \subset \{ \circ, L', K' \} \).

(\( L, R \)): \( \Psi_R \) is an L-action with \( F(\Psi_R, L) \subset \{ \circ, I', J', L', M' \} \).

(\( L, F \)): \( \Psi_F \) is an L-action with \( F(\Psi_F, L) \subset \{ L, M, I, N' \} \).

(\( L, P \)): \( \Psi_P \) is an L-action with \( F(\Psi_P, L) \subset \{ L, I, J \} \).

(\( M, Q \)): \( \Psi_{-Q} \) is an M-action with \( F(\Psi_{-Q}, M) \subset \{ \circ, J' \} \).

(\( M, R \)): \( \Psi_R \) is an M-action with \( F(\Psi_R, M) \subset \{ \circ, L' \} \).
\((M,P)\): \(\Psi_P\) is an \(M\)-action with \(F(\Psi_P, M) \subset \{o\}\).

\((M',P)\): \(\Psi_{-P}\) is an \(M\)-action with \(F(\Psi_{-P}, M) \subset \{L\}\).

\((H',Q)\): \(\Psi_{-Q}\) is an \(H\)-action with \(F(\Psi_{-Q}, H) \subset \{I', K\}\).

\((H',P)\): \(\Psi_P\) is an \(H\)-action with \(F(\Psi_P, H) \subset \{I, L\}\).

\((H',-P)\): \(\Psi_{-P}\) is an \(H\)-action with \(F(\Psi_{-P}, H) \subset \{G\}\).

\((I,-Q)\): \(\Psi_{-Q}\) is an \(I\)-action with \(F(\Psi_{-Q}, I) \subset \{H', J\}\).

\((I,R)\): \(\Psi_{R}\) is an \(I\)-action with \(F(\Psi_{R}, I) \subset \{o', L', K', I'\}\).

\((I,-P)\): \(\Psi_{-P}\) is an \(I\)-action with \(F(\Psi_{-P}, I) \subset \{H, L\}\).

\((J,Q)\): \(\Psi_Q\) is an \(J\)-action with \(F(\Psi_Q, J) \subset \{I, K'\}\).

\((J,Q)\): \(\Psi_{-Q}\) is an \(J\)-action with \(F(\Psi_{-Q}, J) \subset \{o, J', M'\}\).

\((J,R)\): \(\Psi_{R}\) is an \(J\)-action with \(F(\Psi_{R}, J) \subset \{o', L\}\).

\((J,P)\): \(\Psi_{-P}\) is an \(J\)-action with \(F(\Psi_{-P}, J) \subset \{L\}\).

\((K,Q)\): \(\Psi_Q\) is an \(K\)-action with \(F(\Psi_Q, J) \subset \{H, J'\}\).

\((K,-Q)\): \(\Psi_{-Q}\) is an \(K\)-action with \(F(\Psi_{-Q}, K) \subset \{o, L'\}\).

\((K,P)\): \(\Psi_{P}\) is an \(K\)-action with \(F(\Psi_{P}, K) \subset \{o, J\}\).

Moreover, the duals of these assertions are also true, that is to say: if for some pair \(S \in \mathcal{B}\) and \(A \in \mathcal{G}\)

\((A,S)\): \(\Psi_S\) is an \(A\)-action with \(F(\Psi_S, A) \subset \{A_1, \ldots, A_k\}\)

holds then also the dual statement

\((A',-S)\): \(\Psi_{-S}\) is an \(A'\)-action with \(F(\Psi_{-S}, A') \subset \{A'_1, \ldots, A'_k\}\)

holds.

**Remark 4.4.** Note that \(\Psi_o\) is an \(A\)-action (with set of ending states \(\{A\}\)) for every \(A \in \mathcal{G}\).

**Proof.** The statement will be proved by induction on the length of the walks \(w \in p(F)\). The assertion \((A,S)_n\) stands for: \((A,S)\) holds for all walks up to length \(n\). If there is no walk in \(p_k(F,A)\) for \(k \leq n\), then \((A,S)_n\) is true.

For \(n \leq 1\) the statements \((A,S)_n\) for the pairs \((A,S)\) in the proposition are all true. Suppose now that \((A,S)_{n-1}\) is true for all pairs \((A,S)\) of the proposition and their duals. We show that \((A,S)_n\)'s and their duals are also true. We will show how to proceed for the case of \((L, -Q)_n\) and sum up the results for all the cases in a table.

We have to show that \(\Psi_{-Q}\) is an \(L\)-action for all walks \(w\) of length at most \(n\) with ending set \(F(\Psi_{-Q}, L) \subset \{o', L', K'\}\). Let \(w = (a_1, \ldots, a_n)\) such that there is a walk

\[ A \xrightarrow{a_1} A_1 \xrightarrow{a_2} \ldots \xrightarrow{a_{n-2}} A_{n-2} \xrightarrow{a_{n-1}} A_{n-1} \xrightarrow{a_n} L \]

in \(\mathcal{G}\). Then by Remark 3.1 the input digits \((a_n, \ldots, a_1)\) define a unique walk in \(\mathcal{B}_0\) starting from \(-Q\):

\[-Q \xrightarrow{a_n|a'_n} S_{n-1} \xrightarrow{a_{n-1}|a'_{n-1}} S_{n-2} \xrightarrow{a_{n-2}|a'_{n-2}} \ldots \xrightarrow{a_1|a'_1} S_0.\]

First suppose that \(a_n \in \{0, 1, 2\}\), i.e., by \(\mathcal{B}_0\), \(S_{n-1} = Q\). Then

\[\Psi_{-Q}(w) = \Psi_Q(w|_{n-1}) \& (a'_n).\]

According to \(\mathcal{G}\), since \(w\) ends up in \(L\) and \(a_n \in \{0, 1, 2\}\), \(w|_{n-1}\) can end up in \(H', K', I', L', M'\) or \(M'\), i.e., \(A_{n-1} \in \{H', K', I', L', M'\}\). If \(A_{n-1} = H'\), then \(a_n = 2\), because the edge leading from \(H'\) to \(L\) in \(\mathcal{G}\) has only the labels \(\{2, 3\}\), and we assumed \(a_n \in \{0, 1, 2\}\). Thus \(a'_n = 4\), as indicated by the edge \(-Q \xrightarrow{2|4} Q\) of \(\mathcal{B}_0\). Moreover, by \((H', Q)_{n-1}\), which is the dual of \((H, -Q)_{n-1}\), we have \(Q \xrightarrow{w|_{n-1}} o\) in \(\mathcal{B}_0\) and \(\Psi_Q(w|_{n-1})\) ends up in \(\{I, K'\}\). Thus \(-Q \xrightarrow{w^{T}} o\), i.e., \(S_0 = o\), and \(\Psi_{-Q}(w)\)
| \((A,S)\) | \(a_n\) | \(A_{n-1} (A_{n-2})\) | \(a_n' (a_n'-1)\) | \(S_{n-1} (S_{n-2})\) | end of \(\Psi_{n-2} (w|_{n-2})\) | end of \(\Psi_\Sigma (w)\) |
|--------|--------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \((L,-Q)\)_\(_n\) | 0, 1, 2, 3 | \(H'\) | 4 | \(Q\) | I', K' | o |
| \(K'\) | 2 | \(\circ, L\) | \(\circ, L'\) |
| \(I'\) | 4 | \(H, J'\) | \(K'\) |
| \(L'\) | 4 | \(\circ, L, K\) | \(\circ, L'\) |
| \(M'\) | 4 | \(\circ, J\) | o |
| \(H'\) | 0 | \(-P\) | I', L' | o |
| \(L'\) | 0 | I', L', M', N | \(\circ, L'\) | o |
| \(M'\) | 0, 1 | | | o |
| \(N' (I')\) | 1 (2) | \(-P (\circ)\) | I' | L' |
| \((L,R)\)_\(_n\) | 0, 1, 2, 3 | \(H'\) | 3, 4 | \(Q\) | I', K' | \(\circ, J'\) |
| \(K'\) | 1 | \(\circ, L\) | \(\circ, L'\) |
| \(L'\) | 3, 4 | \(\circ, L, K\) | \(\circ, J', M', L'\) |
| \(I'\) | 3 | \(H, J'\) | \(\circ, J'\) |
| \(M'\) | 3, 4 | \(\circ, J\) | o |
| \(M'\) | 0 | \(-P\) | o |
| \(N'\) | 0 (2) | \(-P (\circ)\) | I' | I' |
| \((L,P)\)_\(_n\) | 0, 1, 2, 3 | \(H'\) | 3, 4 | \(\circ\) | H' | I, L |
| \(K'\) | 1 | \(K'\) | M |
| \(L'\) | 3, 4 | \(L'\) | L, I |
| \(I'\) | 3 | \(I'\) | N' |
| \(M'\) | 3, 4 | \(M'\) | L |
| \((L,-P)\)_\(_n\) | 1, 2, 3, 4 | \(H'\) | 1, 2 | \(\circ\) | H' | J, L |
| \(L'\) | 1, 2 | \(L'\) | J, L |
| \(I'\) | 1 | \(I'\) | J |
| \(M'\) | 1, 2, 3 | \(M'\) | J, L |
| \(N'\) | 3 | \(N'\) | H |
| \(K' (I')\) | 4 (3) | \(-R (-Q)\) | I', K | L |
| \((M,-Q)\)_\(_n\) | 1 | \(K'\) | 3 | \(Q\) | \(\circ, L\) | \(\circ, J'\) |
| \((M,R)\)_\(_n\) | 1 | \(K'\) | 2 | \(Q\) | \(\circ, L\) | \(\circ, L'\) |
| \((M,P)\)_\(_n\) | 1 | \(K'\) | 2 | \(\circ\) | \(K'\) | o |
| \((M,-P)\)_\(_n\) | 1 | \(K'\) | 0 | \(\circ\) | \(K'\) | L |
| \((H,-Q)\)_\(_n\) | 3 | \(N' (I')\) | 0 (2) | \(-P (\circ)\) | I' | I' |
| \(G' (F,G,N)\) | 0 (0; 1) | \(-P (\circ)\) | \(I', L'\) | J |
| \((H,P)\)_\(_n\) | 3 | \(N', G'\) | 4 | \(\circ\) | \(N', G'\) | L, I |
| \((H,-P)\)_\(_n\) | 3 | \(N', G'\) | 2 | \(\circ\) | \(N', G'\) | G |
| \((I,-Q)\)_\(_n\) | 4 | \(H'\) | 1 | \(-P\) | \(I', L'\) | J |
| \(L'\) | 1 | \(I', M', L', N\) | J, H' |
| \(G' (F,G,N)\) | 1 (0; 1) | \(-P (\circ)\) | \(F, G, N\) | H', J |
| \((I,R)\)_\(_n\) | 4 | \(H'\) | 0 | \(-P\) | \(I', L'\) | \(\circ, L'\) |
| \(L'\) | 0 | \(I', M', L', N\) | \(\circ, L'\) |
| \(G' (F,G,N)\) | 0 (0; 1) | \(-P (\circ)\) | \(F, G, N\) | \(I', K\) |
| \((I,-P)\)_\(_n\) | 4 | \(G', L', H'\) | 3 | \(\circ\) | \(G', L', H'\) | H, L |
| \((J,Q)\)_\(_n\) | 1 | \(H'\) | 4 | \(P\) | \(G'\) | I |
| \(M'\) | 4 | \(L'\) | I |
| \(I'\) | \(H', L'\) | I |
| \(L'\) | \(H', L', J'\) | I, K' |
| \((J,-Q)\)_\(_n\) | 1 | \(H'\) | 3 | \(Q\) | \(K', I\) | \(\circ, J'\) |
| \(M'\) | \(\circ, J\) | \(\circ, J'\) |
| \(I'\) | \(H, J'\) | \(\circ, J'\) |
| \(L'\) | \(\circ, L, K\) | \(\circ, J', M'\) |

Table 1. Proof of Lemma 4.3.
ends up in \( \{ \circ \} \), because of the edges \( I \xrightarrow{1} \circ \) and \( K' \xrightarrow{1} \circ \) in \( G \). We can argue along the same lines if \( A_{n-1} \in \{ K', I', L' \} \). All these cases lead to walks \( \Psi(-Q)(w) \) ending in \( \circ \), \( K' \) or \( L' \).

Secondly, suppose that \( a_n \in \{ 3, 4 \} \), i.e., by \( \mathcal{B}_0 \), \( S_{n-1} = -P \). Then

\[
\Psi(-Q)(w) = \Psi(-P)(w|_{n-1}) \& (a_n')
\]

According to \( G \), since \( w \) ends up in \( L \) and \( a_n \in \{ 3, 4 \} \), \( w|_{n-1} \) can only end up in \( H', L', M' \) or \( N' \), i.e., \( A_{n-1} \in \{ H', L', M', N' \} \). The first three cases can be treated as above and lead to \( \Psi(-Q)(w) \) ending in \( \circ \) or \( L' \), so let us assume that \( A_{n-1} = N' \). Then, the only edge in \( G \) leading from \( N' \) to \( L \) being \( N' \xrightarrow{3} L \), we have \( a_n = 4 \), and since there is only one edge landing in \( N' \) (\( I' \xrightarrow{3} N' \)), we even have \( A_{n-2} = I' \) and \( a_{n-2} = 3 \). Thus, by the edges \( -Q \xrightarrow{4|1} -P \xrightarrow{3|2} \circ \) of \( \mathcal{B}_0 \), we read \( a_n' = 1 \), \( a_{n-1}' = 2 \) and \( S_{n-2} = \circ \). Consequently, we have \( \circ \xrightarrow{w|_{n-2}} \circ \) and \( \Psi_S(w|_{n-2}) = w|_{n-2} \) ends up in \( \{ I' \} \). Thus again \( S_0 = \circ \) and

\[
\Psi(-Q)(w) = \Psi_S(w|_{n-2}) \& (a_{n-1}', a_n') = w|_{n-2} \& (2, 1)
\]

ends up in \( \{ L' \} \), as it can be checked on \( G \) by considering the edges \( I' \xrightarrow{2} L \xrightarrow{1} L' \).

Thus in all cases \( \Psi(-Q)(w) \) is a walk in \( p(F) \) ending up in \( \{ \circ, L', K' \} \). Thus \( (L, -Q)_n \) is true and we are done.

All the other assertions can be treated likewise. The occurring cases are summed up in Tables 1 and 2 from which the complete proof can be read off easily. In these tables, \( A_{n-2}, a_{n-1}' \) and \( S_{n-2} \) are given if they are needed, and in this case we use \( i = 2 \) in the 6-th column, otherwise \( i = 1 \).

Proposition 4.1 will be proved inductively: let \( w \) be a finite walk in \( p(F, \circ) \) and \( j \geq k + 1 \), where \( k \) is defined by (4.1). We will show that for every \( j \geq k + 1 \), the addition \( \Psi_S(w|_{j}) \) is admissible for each state \( S \) of \( \mathcal{B} \) taking for \( j \) the length \( n \) of \( w \) will yield the result. Lemma 4.5 will contain the induction start, Lemma 4.6 the induction step.

**Lemma 4.5.** Suppose that \( w \) is a walk of the shape (4.1). Then the following assertions hold.

(i) \( \Psi_P(w|_{k+1}) \) ends in \( \{ \circ, M', K', J \} \).

(ii) \( \Psi_Q(w|_{k+1}) \) ends in \( \{ \circ, J, K, L \} \).

(iii) \( \Psi_R(w|_{k+1}) \) ends in \( \{ \circ, L', J', I', K \} \).

(iv) \( w|_{k+1} \) ends in \( \circ \).

Their associated duals also hold (“\( \Psi_S(w|_{k+1}) \) ends in the set of states \( A \)” has the dual “\( \Psi_S(w|_{k+1}) \) ends in \( A' \)”.

| \((A,S)_n\) | \(a_n\) | \(A_{n-1}(A_{n-2})\) | \(a_{n-1}'(a_{n-2}')\) | \(S_{n-1}(S_{n-2})\) | end of \(\Psi_{S_{n-1}}(w|_{n-1})\) | end of \(\Psi_S(w)\) |
|---|---|---|---|---|---|---|
| \((J,K)\_n\) | 1 | \(H'\) | 2 | \(Q\) | \(K', I\) | \(\circ, L'\) |
| | | \(M'\) | | | \(\circ, J\) | |
| | | \(I'\) | | | \(H, J'\) | \(\circ, L'\) |
| | | \(L'\) | | | \(\circ, L, K\) | \(\circ, L'\) |
| \((J,P)\_n\) | 1 | \(H', M', I', L'\) | 2 | \(\circ\) | \(H', M', I', L'\) | \(L\) |
| \((J,-P)\_n\) | 1 | \(H', M', I', L'\) | 0 | \(\circ\) | \(H', M', I', L'\) | \(\circ, K\) |
| \((K,Q)\_n\) | 0 | \(H'\) | 3 | \(P\) | \(G'\) | \(H\) |
| | | \(J\) | | | \(L\) | \(J'\) |
| \((K,-Q)\_n\) | 0 | \(H', J\) | 2 | \(Q\) | \(K', I\) | \(\circ, L'\) |
| \((K,-R)\_n\) | 0 | \(H'\) | 4 | \(P\) | \(G'\) | \(I\) |
| | | \(J\) | | | \(L\) | \(\circ\) |
| \((K,P)\_n\) | 0 | \(H', J\) | 1 | \(\circ\) | \(H', J\) | \(\circ, J\) |

**Table 2.** Proof of Lemma 4.3: end of the preceding table.
\[
\begin{array}{|c|c|c|c|c|}
\hline
S & a_{k+1} & S' & A_k & a'_{k+1} \\
\hline
P & 0, 1, 2, 3 & o & L' & L' & 1 & J \\
& & J & J & 2, 3, 4 & o \\
& & J' & J' & 1, 2, 3, 4 & o, K' \\
& K & K & K' & 1, 2, 3 & o, M' \\
& M' & M' & M' & 3, 4 & o \\
& 4 & R & J & o, L' \\
& I & o, I', K, L' \\
& L & o, J', L', M', J' & 0 & o \\
& K' & o, I' \\
& M & o, L' \\
\hline
Q & 0, 1 & P & K & o, J \\
& L' & H', J, L' & 3 & o, L \\
& J & L & 4 & o \\
& M' & L' & 3 & L \\
& I' & o, K' & 3, 4 & L \\
& J' & o, K' & 3, 4 & o \\
\hline
& 2, 3, 4 & -Q & K' & H', J \\
& L & o, L', K' & 2 & o, L \\
& K & o, L' & 0 & o \\
& J' & K', I' & 0, 1 & o \\
& I & H', J & 2 & o, L \\
& J & o, M', J' & 0, 1, 2 & o, J, L \\
& M & o, J' & 2 & o \\
\hline
& 0, 1, 2, 3 & Q & K & H, J, J' \\
& L' & o, L, K & 1 & o, L' \\
& J & I, K' & 2, 3, 4 & o, L', J' \\
& J' & o, M, J & 1, 2, 3, 4 & o, J', L' \\
& I' & H, J' & 1 & o, L' \\
& M' & o, J & 1 & o \\
\hline
& 4 & -P & K' & o, J' \\
& L & H, J, L & I', K \\
& J & o, K & 0 & o \\
& I & H, L & I' \\
& M & L & I' \\
\hline
\end{array}
\]

Table 3. Proof of Lemma 4.5.

In particular, \( \Psi_S(w_{k+1}) \) is a walk in \( p(F) \) for all \( S \in B_0 \). Moreover, \( S \xrightarrow{w_{k+1}} \circ \) for all \( S \in B_0 \).

That is to say, \( \Psi_S(w_{k+1}) \) is admissible for all \( S \in B_0 \).

\textbf{Proof.} Let \( S \in B_0 \). Note that the following edges exist:

\[ (4.4) \quad A_k \xrightarrow{a_{k+1}} \circ \quad \text{in } G \text{ by definition of } k \text{ and} \]

\( S \xrightarrow{a_{k+1}} S' \text{ in } B \text{ for some } S' \in B_0 \)

(the second edge is uniquely defined by \( S \) and \( a_{k+1} \)). We recall the identity:

\[ (4.5) \quad \Psi_S(w_{k+1}) = \Psi_{S'}(w_{k}) & (a'_{k+1}). \]
To (i): $S = P$. Depending on $a_{k+1}$, the edge $P \xrightarrow{a_{k+1} | a'_{k+1}} S'$ of (4.4) in $B_0$ implies that $S' = \circ$ or $R$, which fix the range of $a'_{k+1}$ ($a'_{k+1} = 0$ if $S' = R$ and $a'_{k+1} \in \{1, \ldots, 4\}$ if $S' = \circ$). The possible states $A_k$ are also determined by $a_{k+1}$ via the existence of the edge $A_k \xrightarrow{a_{k+1}} \circ$ in $G$ (see (4.4)). Using the corresponding assertion ($A_k, S'$) of Lemma 4.3 it is then easy to get the possible endings of $\Psi_{S'}(w_k)$. Now if $Y \in G$ is such an ending, then, by (4.5), with the range of $a'_{k+1}$ one obtains the possible endings $Z$ of $\Psi_S(w_{k+1})$ by looking for all edges $Y \xrightarrow{a'_{k+1}} Z$ in $G$. Let us consider an example: if $a_{k+1} = 4$, we are considering the edge $P \xrightarrow{4|0} R$ in $B_0$, thus $S' = R$ and $a'_{k+1} = 0$. Moreover, $A_k \in \{K', L, J, I, M\}$ because these states are the only starting states of edges in $G$ labelled by 4 and leading to $\circ$. For $A_k = K'$, using ($K', R$) of Lemma 4.3 we get that $\Psi_R(w_{k+1})$ ends up in $\circ$ or $I'$. Consequently, since $a'_{k+1} = 0$, $\Psi_F(w_{k+1})$ ends up in $\circ$: indeed, we have $\circ \xrightarrow{0} \circ$ and $I' \circ \xrightarrow{0} \circ$ in $G$. Note that ($K', R$) also implies $R \xrightarrow{w_{k+1}^T} \circ$ in $B_0$, thus $P \xrightarrow{w_{k+1}^T} \circ$. The results for the other values of $a_{k+1}$ are given in Table 3.

The proof is the same for the other cases ($S = Q, S = R$) and their duals, it is summed up in Table 3 for $S = Q$ and $S = R$. Item (iv) is clear.

Again, let $w$ be a finite walk in $p(F, \circ)$ and $j \geq k + 1$, where $k$ is defined by (4.1). We call ($B_j$) the assertion: for all $S \in B_0$, $\Psi_S(w_j)$ is a walk in $p(F)$ with possible ending states given in Table 4. Moreover, $S \xrightarrow{w_j^T} S_0 = \circ$ in $B_0$. In particular, ($B_j$) states that $\Psi_S(w_j)$ is admissible for all $S \in B_0$.

By Lemma 4.5, ($B_{k+1}$) already holds. The induction step is contained in the following lemma.

<table>
<thead>
<tr>
<th>walk</th>
<th>possible ending states</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_P(w_j)$</td>
<td>${\circ, M', K', J}$</td>
</tr>
<tr>
<td>$\Psi_Q(w_j)$</td>
<td>${\circ, J, K, L, M}$</td>
</tr>
<tr>
<td>$\Psi_R(w_j)$</td>
<td>${\circ, L', J', I', M', K}$</td>
</tr>
</tbody>
</table>

**Table 4.** Table of statement ($B_j$).

**Lemma 4.6.** If ($B_j$) holds for some $j \geq k + 1$, then ($B_{j+1}$) holds too.

**Proof.** First we deal with the case of $S = P$. Then there is an edge $P \xrightarrow{a_{j+1} | a'_{j+1}} S'$ in $B_0$, thus $S' \in \{\circ, R\}$. Remember that $\Psi_F(w_{j+1}) = \Psi_S(w_j) \& a'_{j+1}$.

By assumption ($B_j$), $S' \xrightarrow{w_{j+1}^T} \circ$, thus $P \xrightarrow{a_{j+1} | a'_{j+1}} S' \xrightarrow{w_{j+1}^T} \circ$, i.e., $P \xrightarrow{w_{j+1}^T} \circ$ in $B_0$.

Moreover, if $S' = \circ$, then $\Psi_P(w_{j+1}) = w_{j+1} \& a'_{j+1}$ is a walk in $p(F)$ that ends at $\circ$: indeed, $w_j$ is in $p(F)$ and $j \geq k + 1$, so by (4.1) $w_j$ already ends at $\circ$: its concatenation with $a'_{j+1}$ remains in $p(F)$, because $\circ \xrightarrow{a'_{j+1}} \circ$.

If $S' = R$, then the edge $P \xrightarrow{a_{j+1} | a'_{j+1}} R$ in $B_0$ indicates that $a'_{j+1} = 0$. Hence $\Psi_P(w_{j+1}) = \Psi_R(w_j) \& a'_{j+1}$, with $\Psi_R(w_j)$ walk in $p(F)$ ending at $Y \in \{\circ, L', J', I', M', K\}$ by assumption ($B_j$) (see Table 4 for the endings). Now we can check on $G$ that there is an edge starting from each $Y \in \{\circ, L', J', I', M', K\}$ and labelled by 0. They all lead to $\circ$.

The other cases ($S = Q, S = R$) as well as their duals are treated in a similar way (see Table 5 for $S = Q, S = R$).
concentrate on the tails of the walks. Moreover, the lemmata correspond to the following classes of walks via the state of walks:

| $S_j$ | $a_{j+1}$ | end of $\Psi_{S_j}(w|_{j})$ | end of $\Psi_{S_{j+1}}(w|_{j+1})$ |
|-------|----------|---------------------------|-------------------------------|
| $P$  | $\circ$  | $1,2,3,4$                | $\circ$                       |
|       | $R$      | $0$                       | $\circ, J', L', I', M', K$    |
| $Q$  | $P$      | $3,4$                     | $\circ, J, K, M'$            |
|       | $-Q$     | $0,1,2$                   | $\circ, J', K, M'$           |
| $R$  | $Q$      | $1,2,3,4$                 | $\circ, L, K, J, M$          |
|       | $-P$     | $0$                       | $\circ, J', K, M$            |

Table 5. Proof of Lemma 4.6.

Proof of Proposition 4.1. Let $w$ be a walk in $p_n(F, \circ)$ of the shape (4.1) and the resulting walk in $B_0$ given by (4.2). If $n = k + 1$, then $w|_{k+1} = w$ and Lemma 4.5 gives the result immediately. Otherwise, starting from the same lemma and going on with Lemma 4.6 from $j = k + 1$ to $j = n$, we also obtain the statement of Proposition 4.1.

$\square$

5. Equivalences of paths in $p(F, \circ)$

The main result of this section, Proposition 5.2, will be used in Section 8 to construct arcs inside $\text{Int}(\mathbf{M})$ from arbitrary points contained in a subpiece $\psi_w(T)$, where $w \in p(F, \circ)$, to the point $0$ (contained in $\psi(0,0,0)(T)$).

In the following, the equivalences of walks from the set $G_k$ defined as in (2.4) for some $k \in \mathbb{N}$ will take place in $G_k$. First we note the following fact about the walks of length 3.

Remark 5.1. We have $G_3 = \{(F;0,0,0),(F;1,4,4)\}$ and these walks are equivalent in $G_3$. Namely, $(F;1,4,4) = \Psi_{-p}(F;0,0,0)$. In other words, $G_3$ is transitive.

Proposition 5.2. Let $n \geq 4$ and $w \in G_n$. Then there is a walk $w' \in G_{n+1}$ such that $w'|_{n-1} \in G_{n-1}$ and $w & d \sim w'$ for some $d \in \{0,\ldots,4\}$.

In view of Remark 3.9.2 this proposition says the following. Let $w$ be a walk in $G_n$ and let $\psi_w(T)$ be the associated subset of $T$. Then $\psi_w(T)$ contains a subpiece $\psi_{w&d}(T)$ with the following property. There exist walks $w&d = v_1, v_2, \ldots, v_k = w'$ in $G_{n+1}$ such that the associated subpieces satisfy

$$\psi_{v_j}(T) \cap \psi_{v_{j+1}}(T) \neq \emptyset \quad (1 \leq j \leq k - 1).$$

Since $w'' := w'|_{n-1} \in G_{n-1}$, this means that one can draw an arc from each piece $\psi_w(T)$ of $\mathcal{M}(G_n)$ to a piece $\psi_{w''}(T)$ of $\mathcal{M}(G_{n-1})$. By induction on $n$ this will lead to a proof of the connectivity of $\mathbf{M}$ because it allows to draw arcs from each point of $\mathbf{M}$ to the connected set $\psi_{\text{close}}(T) \subset \mathbf{M}$.

Remark 5.3. 1) If $w & d \sim w'$ for $w \in G_n$ and $d \in \{0,\ldots,4\}$, then by using $\Psi_{\pm p}$ we even have $w & d \sim w'$ for every $d \in \{0,\ldots,4\}$.

2) If two walks $w$ and $w'$ of $G_n$ are equivalent, then there exist $d, d' \in \{0,\ldots,4\}$ with $w & d \sim w' & d'$. This means that two intersecting pieces $\psi_w(T)$ and $\psi_{w'}(T)$ (i.e., such that $\psi_w(T) \cap \psi_{w'}(T) \neq \emptyset$) contain intersecting subpieces $\psi_{w&d}(T)$ and $\psi_{w'd'}(T)$. In particular it is sufficient to find a walk $w' \in G_n$ with $w'|_{n-1} \in G_{n-1}$ and $w \sim w'$. In this case $w$ will automatically fulfill Proposition 5.2.

3) For $p \in \mathbb{N}$, we introduce the notation $w_{sp} \sim w'$: this means that $w'$ is obtained after applying $\Psi_S$ to $w$ for $p$ times.

Proposition 5.2 will be shown via the following lemmata. By Remark 3.9.3, we just have to concentrate on the tails of the walks. Moreover, the lemmata correspond to the following classes of walks:

$$E_n(A) := \{w \in G_n, w \text{ contains the edge } A \rightarrow \circ \} \quad (a \in \mathcal{G} \setminus \{\circ\}).$$

Sloppily spoken the walks contained in $E_n(A)$ are those walks of $G_n$ which reach the accepting state via the state $A$. Note that

$$G_n = \bigcup_{A \in \mathcal{G}} E_n(A).$$
Lemma 5.4. Let \( n \geq 4 \) and \( w \in E_n(K) \cup E_n(K') \). Then there is a walk \( w' \in G_{n+1} \) such that \( w'|_{n-1} \in G_{n-1} \) and \( w \& d \sim w' \) for some \( d \in \{0, \ldots, 4\} \).

Proof. Let us consider \( w \in E_n(K) \). If \( w'|_{n-1} \in E_{n-1}(K) \), we are ready. We suppose it is not the case. We have the following cases for the tails \( \tau = \sigma \& \tau \) (\( \sigma \) is fixed by \( w \) and \( \tau \)).

1. \( \tau = (J; 0; a) \) with \( a \in \{0, 1, 2\} \). Then
   \[
   (J; 0, 2)_p \sim (J; 0, 1)_p \sim (J; 0, 0)_q \sim (J; 1, 3) \sim (J; 1),
   \]
   which ends at \( \circ \). Thus \( w \sim \sigma \& (J; 1, 3) \) with \( \sigma \& (J; 1) \in G_{n-1} \), and we are ready by Remark 5.3.2.

2. \( \tau = (A; d, 1, 0, a) \) with \( (A, d) \in C := \{(F, 0), (G', 2), (N', 2), (I, 1)\} \) and \( a \in \{0, 1, 2\} \). Then for all constellations \( (A, d) \in C \), we have
   \[
   (A; d, 1, 0, 0)_p \sim (A; d, 1, 0, 1)_p \sim (A; d, 1, 0, 2)_q \sim (A; d + 1, 4, 2, 0)_q \sim (A; d + 1, 4, 3, 3) \sim (A; d + 1, 4, 4, 3) \sim (A; d + 1, 4, 4),
   \]
   which ends at \( \circ \). Thus \( w \sim \sigma \& (A; d + 1, 4, 4, 3) \) with \( \sigma \& (A; d + 1, 4, 4) \in G_{n-1} \), and we are ready by Remark 5.3.2.

One can proceed identically for \( w \in E_n(K') \) by considering the dual walks of the previous ones. Thus Lemma 5.4 is proved.

Lemma 5.5. Let \( n \geq 4 \) and \( w \in E_n(J) \cup E_n(J') \). Then there is a walk \( w' \in G_{n+1} \) such that \( w'|_{n-1} \in G_{n-1} \) and \( w \& d \sim w' \) for some \( d \in \{0, \ldots, 4\} \).

Proof. Let us consider \( w \in E_n(J) \). If \( w'|_{n-1} \in E_{n-1}(J) \), we are ready. We suppose it is not the case. Then the tail \( \tau = \sigma \& \tau \) has the form \( \tau = (A; 1, a) \) with \( A \in A := \{L', M', I', H'\} \) and \( a \in \{1, 2, 3, 4\} \). We have the following equivalences for \( A \in A \):

\[
(A; 1, 1)_p \sim (A; 1, 2)_p \sim (A; 1, 3)_p \sim (A; 1, 4)_q \sim (A; 0, 1).
\]

For \( A \in \{L', M', I', H'\} \), \( (A; 0, 1) \sim (A; 0, 0) \), which ends at \( \circ \). Thus \( w \sim \sigma \& (A; 0, 1) \) with \( \sigma \& (A; 0) \in G_{n-1} \), and we are ready by Remark 5.3.2.

For \( A = H' \), we have \( w = \sigma \& (H'; 0, 1) \) which is now a walk belonging to \( E_n(K) \), thus we obtain the required result by using Lemma 5.4.

One can proceed identically for \( w \in E_n(J') \), thus Lemma 5.5 is proved.

Lemma 5.6. Let \( n \geq 4 \) and \( w \in E_n(L) \cup E_n(L') \). Then there is a walk \( w' \in G_{n+1} \) such that \( w'|_{n-1} \in G_{n-1} \) and \( w \& d \sim w' \) for some \( d \in \{0, \ldots, 4\} \).

Proof. Let us consider \( w \in E_n(L') \). If \( w'|_{n-1} \in E_{n-1}(L') \), we are ready. We suppose it is not the case. Then \( w \) belongs to one of the following classes of tails \( \tau = \sigma \& \tau \) (\( \sigma \) is fixed by \( w \) and \( \tau \)).

1. \( \tau = (L; 2, 0) \). Then \( (L; 2, 0)_q \sim (L; 3, 3) \) which leads to a walk \( \sigma \& (L; 3, 3) \) equivalent to \( w \) which belongs to \( E_n(J) \), treated in Lemma 5.5.

2. \( \tau = (L; 1, 0) \). We subdivide this class into the following smaller classes:
   A. \( (A; d, 1, 0) \) with \( (A, d) \in C_A := \{(K', 0), (H', a), (L', a); a \in \{2, 3\}\} \).
   B. \( (K; 3, d, 1, 0) \) with \( d \in \{2, 3, 4\} \).
   C. \( (I'; 3, 4, 1, 0) \).
   D. \( (A; d, 0, (3, 4, 0)^p, 2, 1, 0) \) for some \( p \in \mathbb{N} \) and \( (A, d) \in C_D := \{(F, 0), (K', 0), (I', 2), (M', 4), (G', a), (N', a), (H' a), (L', a), (M', a); a \in \{2, 3\}\} \).
Here $(3, 4, 0)\,p$ inside the walk means that the sequence of digits $(3, 4, 0)$ has to be read $p$ times before going on to the digit 2. This corresponds to the cycle $I' \rightarrow N' \rightarrow L \rightarrow I'$ in the graph of Figure 2.

A. We have for $(A, d) \in C_A$: $(A; d, 1, 0)_{-Q} \sim (A; d + 1, 4, 2) < (A; d + 1, 4)$, which ends at $\circ$.

B. We have for $d = 3, 4$ that $(K; 3, d, 1, 0)_{-P} \sim (K; 2, d - 3, 0, 4) < (K; 2)$, which ends at $\circ$, and $(K; 3, 2, 1, 0)_{-Q} \sim (K; 3, 3, 4, 2) < (K; 3, 3, 4)$, which ends at $\circ$.

C. We have $(I'; 3, 4, 1, 0)_{-P} \sim (I'; 2, 1, 0, 4) < (I'; 2, 1, 0)$, which ends at $\circ$ too.

D. The following chain holds for every $(A, d) \in C_D \setminus \{(M', 4)\}$ and $p \geq 0$:

$$(A; d, 0, (3, 4, 0)^p, 2, 1, 0)_{-P} \sim (A; d + 1, (3, 4, 0)^p, 3, 4, 0, 4) \sim (A; d + 1, (3, 4, 0)^p, 3, 4, 0, 4, 4)$$

$$\sim (A; d + 1, (3, 4, 0)^p, 3, 4, 0, 3, 1)_{P^2} \sim (A; d + 1, (3, 4, 0)^p, 3, 4, 0, 3, 3)$$

$$\sim (A; d + 1, (3, 4, 0)^p, 3, 4, 0, 2, 0)_{(-Q)^2} \sim (A; d, 0, (3, 4, 0)^p, 2, 2, 0, 2)$$

which is of type $(L; 2, 0)$ treated in Item (1.i).

If $(A, d)$ happens to be $(M', 4)$, we go into smaller classes by considering the tails $(A'; 1, 0, 3, 4, 0, (3, 4, 0)^p, 2, 1, 0)$ with $A' \in \mathcal{A} := \{G, N, H', M', L', I'\}$ and we have the similar chain for $A' \in \mathcal{A}$:

$$(A'; 1, 0, 3, 4, 0, (3, 4, 0)^p, 2, 1, 0)_{-P} \sim (A'; 2, 3, 4, 0, (3, 4, 0)^p, 3, 4, 0, 4)$$

$$\sim (A'; 2, 3, 4, 0, (3, 4, 0)^p, 3, 4, 0, 4, 4)$$

$$\sim (A'; 2, 3, 4, 0, (3, 4, 0)^p, 3, 4, 0, 3, 1)$$

$$\sim (A'; 2, 3, 4, 0, (3, 4, 0)^p, 3, 4, 0, 3, 3)$$

$$\sim (A'; 2, 3, 4, 0, (3, 4, 0)^p, 3, 4, 0, 2, 0)$$

$$\sim (A'; 1, 0, 3, 4, 0, (3, 4, 0)^p, 2, 2, 0, 2)$$

which is again of type $(L; 2, 0)$ treated in Item (1.i).

(2) $\tau \in \{(I, 2), (M, 0), (M, a), (H, a), a \in \{1, 2\}\}$. Then

$$(A; 2, 0)_{Q} \sim (S; 3, 3) \in E_n(J')$$

for $A \in \{I, M, H\}$, and for the other cases one can consider the smaller classes:

- for $A \in \{G, N\}$,

$$(A; 3, 1, 0)_{Q} \sim (A; 4, 4, 2) \sim (A; 4, 4)$$

which ends at $\circ$.

- for $d \in \{0, 1\}$,

$$(K'; 1, d, 0)_{-Q} \sim (K'; 2, d + 3, 2) \sim (K'; 2)$$

which ends at $\circ$. 

(3) $\tau = (N; 0, 0)$. Then $u$ must end in the form $(I; 1, 0, 0)$, and the following chain holds:

$$(I; 1, 0, 0) \sim (I; 1, 0, 0, 0)_{Q} \sim (I; 1, 0, 1, 3)_{P} \sim (I; 1, 0, 1, 4)$$

$$\sim (I; 2, 3, 3, 2)$$

$$\sim (I; 2, 3, 3, 0)_{Q} (I; 2, 3, 4, 3) \sim (I; 2, 3, 4)$$

which is of type $(L'; 3, 4)$: this is the dual of the tail $(L; 1, 0)$, thus it can be treated as in Item (1.ii).
(4) \( \tau = (K; 4, 0) \). Then the walk \( w \) ends in the following way: \((A; d, 1, 0, (4, 1, 0)^p, 4, 0)\) for some \( p \in \mathbb{N} \) and 
\[
(A, d) \in \mathcal{C} := \{(F, 0), (G', 2), (N', 2), (I, 1),
(G, 1), (N, 1), (K, 3), (G, 0), (L, 0), (H, 0),
(I, 2), (N, 0), (M, 0), (M, a), (L, a), (H, a), a \in \{1, 2\}\}.
\]

For \((A, d) \in \mathcal{C}\), the following chain holds:
\[
(A; d, 1, 0, (4, 1, 0)^p, 4, 0, 0) \sim_p (A; d, 1, 0, (4, 1, 0)^p, 4, 1, 3)
\]
\[
(-p)^2 \sim (A; d + 1, (4, 1, 0)^p, 4, 2, 2, 3, 0) \sim (A; d + 1, (4, 1, 0)^p, 4, 2, 2, 4, 3).
\]
If now \( p = 0 \) and \((A, d) \in \{(G', 2), (N', 2), (I, 2), (M, 2), (L, 2), (H, 2)\}\), then we have
\[
(A; d + 1, (4, 1, 0)^p, 4, 2, 2, 4, 3) \sim (A; d + 1, 4, 2),
\]
which ends at 0; otherwise
\[
(A; d + 1, (4, 1, 0)^p, 4, 2, 2, 4) \sim (A; d + 1, (4, 1, 0)^p, 4, 2, 2, 4),
\]
which is of type \((L'; 3, 4)\), i.e., of the dual of the tail \((L; 1, 0)\), that can be treated similarly as in Item (1.x).

Proceeding identically for \( w \in E_n(L) \), we obtain Lemma 5.6.

\[\textbf{Lemma 5.7.}\] Let \( n \geq 4 \) and \( w \in E_n(M) \cup E_n(M') \). Then there is a walk \( w' \in G_{n+1} \) such that
\[w|_{n-1} \in G_{n-1} \text{ and } w \& d \sim w' \text{ for some } d \in \{0, \ldots, 4\}.
\]

\[\textbf{Proof.}\] Let us consider \( w \in E_n(M') \). If \( w|_{n-1} \in E_n(M') \), we are ready. We suppose it is not the case. Then \( w \) belongs to the following classes of tails \( \tau \) of \( w = \sigma \& \tau: (A; d, 1, 0, (4, 1, 0)^p), 3, 0 \) for some \( p \in \mathbb{N} \) and 
\[
(A, d) \in \mathcal{C} := \{(F, 0), (G', 2), (N', 2), (I, 1),
(G, 1), (N, 1), (K, 3), (G, 0), (L, 0), (H, 0),
(I, 2), (N, 0), (M, 0), (M, a), (L, a), (H, a), a \in \{1, 2\}\}.
\]

For \((A, d) \in \mathcal{C}\), we have the equivalence
\[
(A; d, 1, 0, (4, 1, 0)^p), 3, 0 \sim_p (A; d + 1, (4, 1, 0)^p, 4, 2, 2, 4).
\]
If now \( p = 0 \) and \((A, d) \in \{(G', 2), (N', 2), (I, 2), (M, 2), (L, 2), (H, 2)\}\), then we have
\[
(A; d + 1, (4, 1, 0)^p, 4, 2, 2, 4) \sim (A; d + 1, 4, 2),
\]
which ends at 0, otherwise the tail of the equivalent walk is of type \((L; 4)\) which was treated in Lemma 5.6.

We can proceed similarly for the dual case, and Lemma 5.7 is proved.

\[\textbf{Lemma 5.8.}\] Let \( n \geq 4 \) and \( w \in E_n(I) \cup E_n(I') \). Then there is a walk \( w' \in G_{n+1} \) such that
\[w|_{n-1} \in G_{n-1} \text{ and } w \& d \sim w' \text{ for some } d \in \{0, \ldots, 4\}.
\]

\[\textbf{Proof.}\] Let us consider \( w \in E_n(I) \). If \( w|_{n-1} \in E_{n-1}(I) \), we are ready. We suppose it is not the case. Then \( w \) belongs to the following classes of tails \( \tau \) of \( w = \sigma \& \tau: (A; d, 4, (1, 0, 4)^p), 4 \) for some \( p \in \mathbb{N} \) and 
\[
(A, d) \in \mathcal{C} := \{(G, 1), (N, 1), (F, 1), (G, 2), (N, 2)
(I, 2), (K, 4), (M, 0), (M, a), (L, a), (H, a), a \in \{1, 2\}\}.
\]

For \((A, d) \in \mathcal{C} \setminus \{(M, 0)\}\), we have the equivalence
\[
(A; d, 4, (1, 0, 4)^p), 4 \sim_p (A; d - 1, (1, 0, 4)^p), 0, 0).
\]
We consider the following cases:

- for \( p = 0 \) and \((A, d) = (F, 1)\), then \( w = (F; 1, 4, 4) \in G_3 \) has length \( n = 3\);
• for \( p = 0 \) and \((A, d) \in \{(G, 2), (N, 2)\}\), \((A; d - 1, 0, 0)\) is the tail of a walk belonging to \(E_n(K)\), treated in Lemma 5.4;
• for \( p = 0 \) and \((A, d) = (I, 2), (I; 1, 0, 0)\) is the tail of a walk in \(E_n(L')\), treated in Lemma 5.6;
• otherwise \((A; d - 1, (1, 0, 4)^p, 0, 0) \sim (A; d - 1, (1, 0, 4)^p, 0)\), which ends at \(\circ\).

For \((A, d) = (M, 0)\), we go into the smaller classes \((A'; 3, 4, 1, 0, 4, (1, 0, 4)^p, 4)\) with \(p \geq 0\) and \(A' \in \{G', X', I, M, L, H\}\). In these cases,
\[
(A'; 3, 4, 1, 0, 4, (1, 0, 4)^p, 4) \sim (A'; 2, 1, 0, 4, (1, 0, 4)^p, 0, 0) \sim (A'; 2, 1, 0, 4, (1, 0, 4)^p, 0),
\]
which ends at \(\circ\).

Dealing with the walks of \(E_n(J')\) in the same way, we obtain Lemma 5.8. \(\Box\)

Proposition 5.2 now follows from Lemmata 5.4 to 5.8 together with the equation (5.1).

6. Boundary of \(M\)

As will be seen later, the last two sections assure the connectivity of the subset of \(M\) consisting of the union of the subpieces \(\psi_w(T)\) where \(w\) is a walk of \(G\) starting at \(F\) and ending at the accepting state \(\circ\). By definition, this subset is dense in \(M\). The present section now uses the walks of \(p(F)\) that do not end at \(\circ\) to prove that the boundary \(\partial M\) of \(M\) lies in the boundary of \(T\).

**Proposition 6.1.** Let \(w_0 := (a_1, \ldots, a_n)\) be a walk in \(p(F)\) which does not end at \(\circ\). Then
\[
\psi_{w_0}(T) \cap \partial T \neq \emptyset.
\]

For \(w_0\) as in the proposition, we will show that \(w := (0, 4)w_0\) satisfies \(\psi_w(T) \cap B_Q \neq \emptyset\). Note that then \(\psi_w(T) = \psi_0 \circ \psi_4 \circ \psi_{w_0}(T)\); we will see that this piece stays in contact with \(\partial T\) after application of the inverse of \(\psi_0 \circ \psi_4\). We need the following lemma.

For \(A\) a state of \(G\), \(S\) a subset of \(B\) and \(n \geq 3\), let \((B_n)\) be the following assertion:

If \(w_0\) is a walk of length \(n - 2\) in \(p(F)\) ending at \(A \neq \circ\), then \(w^T := ((0, 4)w_0)^T\) is the labelling of a walk in \(B\) starting at \(S\) and ending at \(Q\) for every \(S \in S\), where \((A, S)\) are given in the Table 6 (the duals have to be added, they associate \(A'\) to \(\neg S\)).

**Lemma 6.2.** The assertion \((B_n)\) holds for every \(n \geq 3\).

**Proof.** For \(n = 3\) we have \(w = (0, 4, 0)\) and \(A = G\), or \(w = (0, 4, 1)\) and \(A = G'\). It is easily seen on \(B_0\) that \(S \xrightarrow{w^T} Q\) for all \(S\) in the corresponding \(S\).

Let us suppose \((B_n)\) to be true for an \(n \geq 3\). We show that \((B_{n+1})\) also holds. Let \(w = (0, 4, a_3, \ldots, a_{n+1})\) with \((F; a_3, \ldots, a_{n+1}) =: w_0 \in p(F)\).

<table>
<thead>
<tr>
<th>(A)</th>
<th>(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G)</td>
<td>({Q, \neg Q, R, \neg R})</td>
</tr>
<tr>
<td>(H)</td>
<td>({Q, R, \neg R})</td>
</tr>
<tr>
<td>(I)</td>
<td>({P, Q})</td>
</tr>
<tr>
<td>(J)</td>
<td>({\neg R})</td>
</tr>
<tr>
<td>(K)</td>
<td>({\neg P})</td>
</tr>
<tr>
<td>(L)</td>
<td>({Q, \neg R})</td>
</tr>
<tr>
<td>(M)</td>
<td>({Q})</td>
</tr>
<tr>
<td>(N)</td>
<td>({Q, \neg Q})</td>
</tr>
</tbody>
</table>

**Table 6.** Table for assertion \((B_n)\).
If $w_0$ ends up in $A = G$, then $u := (a_3, \ldots, a_n)$ ends up in $G'$ or $N'$ and $a_{n+1} = 2$, because $G' \not \to G$ and $N' \not \to G$ are the only edges of $G$ leading to $G$ (the case $F \not \to A$ is not possible, since there would be no edge labelled by $a_n$ leading to $F$). If $u$ ends up in $G'$, then by assumption we have $S^w_\to Q$ for all $S \in \{Q, -Q, R, -R\}$. Since $R \not \to S = Q$, $-Q \not \to S = Q$ and $-R \not \to S = -Q$, $Q \not \to S = -Q$ all exist in $B$, we obtain for every $S' \in \{\pm Q, \pm R\}$: $S' \not \to S \not \to S \not \to Q$, i.e., $S' \not \to Q$. If $u$ ends up in $N'$, then by assumption we have $S^w_\to Q$ for all $S \in \{Q, -Q, \}$. Thus one can use the preceding walks in $B$: $R \not \to S = Q$, $-Q \not \to S = Q$ and $-R \not \to S = -Q$, $Q \not \to S = -Q$ all exist in $B$, hence for every $S' \in \{\pm Q, \pm R\}$, $S' \not \to S \not \to Q$, i.e., $S' \not \to Q$.

The results for the other possible endings $A$ of $w_0$ are summed up in Table 7 (the duals can be treated likewise).

\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
$A$ & end of $w|_n$ & $a_{n+1}$ & $S' \stackrel{a_{n+1}}{\rightarrow} S$ \\
\hline
$G$ & $G', N'$ & 2 & $R, -Q \rightarrow Q$ \\
& & & $-R, Q \rightarrow -Q$ \\
\hline
$H$ & $G', N'$ & 3 & $R \rightarrow Q$ \\
& & & $-R, Q \rightarrow -Q$ \\
\hline
$I$ & $H', G', L'$ & 4 & $P \rightarrow R$ \\
& & & $Q \rightarrow -Q$ \\
\hline
$J$ & $H', M', I'$ & 1 & $-R \rightarrow -Q$ \\
\hline
$K$ & $H', J$ & 0 & $-P \rightarrow -R$ \\
\hline
$L$ & $H', L'$ & 2, 3 & $Q \rightarrow -Q$ \\
& & & $-R \rightarrow -Q$ \\
& $M'$ & 2, 3, 4 & $Q \rightarrow -Q$ \\
& & & $-R \rightarrow -Q$ \\
& $N'$ & 4 & $-R \rightarrow -Q$ \\
& & & $Q \rightarrow P$ \\
& $K'$ & 0 & $-R \rightarrow P$ \\
\hline
$M$ & $K'$ & 1 & $Q \rightarrow P$ \\
\hline
$N$ & $I$ & 1 & $Q \rightarrow P$ \\
& & & $-Q \rightarrow Q$ \\
\hline
\end{tabular}
\end{center}
\end{table}

Table 7. Proof of Lemma 6.2.

Proof of Proposition 6.1. Let $w := (0, 4)\&w_0$ with $w_0$ a walk in $p(F)$ that does not end at 0. Then
\[
\psi_w(T) \cap B_Q \neq \emptyset.
\] Indeed, by Proposition 3.4, it suffices to show that there exists a walk $Q \not \to S$ in $B$. This is what Lemma 6.2 does. Now recall that $\psi_w(T) = \psi_0 \circ \psi_4 \circ \psi_w(T)$. Thus again by Proposition 3.4 there exists an infinite walk $Q \not \to S \not \to S' \not \to \ldots$ in $B$. This implies that $S' \in \{F, Q, -R\}$, as can be checked on $B_0$. Thus there is an infinite walk $S' \not \to \ldots$ in $B$ with $S' \in B$, and therefore $\psi_w(T) \cap \partial T \neq \emptyset$, as assured by Proposition 3.4.

In what follows we will use the following notations. We fix a metric $\text{dist}(\cdot, \cdot)$ on $\mathbb{R}^2$, and denote the diameter of a compact set $C$, defined as the maximal distance between two points of $C$, by $\text{diam}(C)$.

Proposition 6.3. The boundary of $M$ is contained in the boundary of $T$.

Proof. Let $x \in \partial M$. We will show that for every $\varepsilon > 0$, we have $\text{dist}(x, \partial T) < \varepsilon$. This will imply that $x \in \partial T$, since $\partial T$ is a closed set. We consider two cases.
Case 1. For every \( n \geq 3 \), \( x \notin \mathcal{M}(G_n) \) (see Definition 2.6). The element \( x \) belonging to \( \mathcal{M} \), we can write \( x = \sum_{i=1}^{\infty} \mathbf{A}^{-i} \Phi(a_i) \) with \( w_n := (a_1, \ldots, a_n) \in p_n(F) \). In our assumption, for every \( n \geq 3 \), we have \( w_n \notin p_n(F, \circ) \). Let \( \varepsilon > 0 \) and \( n_0 \) such that for \( n \geq n_0 \), \( \text{diam}(\psi_w(T)) < \varepsilon \) for every \( w \) of length \( |w| = n \). Then

\[
x \in \psi_{w_{n_0}}(T) \quad \text{with} \quad \begin{cases} w_{n_0} \in p_{n_0}(F) & \text{(by definition)} \\
0 \neq w_{n_0} \notin p_{n_0}(F, \circ) & \text{(by assumption)}
\end{cases}.
\]

By Proposition 6.1, \( \psi_{w_{n_0}}(T) \cap \partial T \neq \emptyset \), thus \( \text{dist}(x, \partial T) < \varepsilon \) since \( \text{diam}(\psi_{w_{n_0}}(T)) < \varepsilon \).

Case 2. There is an \( n_0 \geq 3 \) with \( x \in \mathcal{M}(G_{n_0}) \). Because of Equation (1.2), we even have \( x \in \mathcal{M}(G_n) \) for all \( n \geq n_0 \). We denote by \( B_r(0) \) the open ball \( \{ y \in \mathbb{R}^2, \text{dist}(0, y) < r \} \). By compactness of \( T \), it is possible to find \( r_1 > 0 \) such that

\[
T \subset B_{r_1}(0).
\]

Since \( \{ P, Q \} \) is a basis of the lattice \( \Phi(\mathbb{Z}[\alpha]) \) by (1.4), there exist positive integers \( m_1, m_2 \) such that

\[
B_{r_1}(0) \subset \bigcup_{n_1 \in \{-m_1, \ldots, m_1\}} \bigcup_{n_2 \in \{-m_2, \ldots, m_2\}} (T + n_1 P + n_2 Q) \subset B_{r_2}(0).
\]

The second inclusion follows again from the compactness of \( T \), \( r_2 > 0 \) is simply chosen large enough.

Let now \( \varepsilon > 0 \), \( n \geq n_0 \) such that \( \text{diam}(\psi_w(B_{r_2}(0))) < \varepsilon \) for every \( w \) of length \( |w| = n \), and let \( w \in p_n(F, \circ) = G_n \) such that \( x \in \psi_w(T) \). Then using (6.1) and (6.2), the following inclusions hold:

\[
x \in \psi_w(T) \subset \psi_w(B_{r_1}(0)) \subset \psi_w \left( \bigcup_{n_1 \in \{-m_1, \ldots, m_1\}} \bigcup_{n_2 \in \{-m_2, \ldots, m_2\}} (T + n_1 P + n_2 Q) \right) \subset \psi_w(B_{r_2}(0)).
\]

\[
= \bigcup_{n_1 \in \{-m_1, \ldots, m_1\}} \bigcup_{n_2 \in \{-m_2, \ldots, m_2\}} \psi_w(T + n_1 P + n_2 Q)
\]

Our aim is to find a \( y \in \partial T \) in the union above. Since \( x \) and \( y \) will then both belong to \( \psi_w(B_{r_2}(0)) \), which has diameter less than \( \varepsilon \), we will be done. Note that \( \psi_w(B_{r_1}(0)) \) is a neighborhood of \( x \), a point of \( \partial \mathcal{M} \), hence this neighborhood has nonempty intersection with the complement of \( \mathcal{M} \) in \( \mathbb{R}^2 \).

Remember that \( w \) is a walk of \( p_n(F, \circ) \). Now we make the following assumption:

- Each of the following additions is admissible,
- Each of the following additions yields a walk that is contained in \( p(F, \circ) \).

\[
W_1 := \{ \Phi_{n_1} \circ \Phi_{n_2}(w) \mid 0 \leq n_1 \leq m_1; 0 \leq n_2 \leq m_2 \},
\]

\[
W_2 := \{ \Phi_{n_1}(-p) \circ \Phi_{n_2}(w) \mid 0 \leq n_1 \leq m_1; 0 \leq n_2 \leq m_2 \},
\]

\[
W_3 := \{ \Phi_{n_1}(p) \circ \Phi_{n_2}(-q)(w) \mid 0 \leq n_1 \leq m_1; 0 \leq n_2 \leq m_2 \},
\]

\[
W_4 := \{ \Phi_{n_1}(-p) \circ \Phi_{n_2}(q)(w) \mid 0 \leq n_1 \leq m_1; 0 \leq n_2 \leq m_2 \}.
\]

Set \( W := W_1 \cup W_2 \cup W_3 \cup W_4 \). With a slight abuse of notation we may write

\[
W := \{ \Phi_{n_1} \circ \Phi_{n_2}(w) \mid -m_1 \leq n_1 \leq m_1; -m_2 \leq n_2 \leq m_2 \}.
\]
By assumption all walks of $W$ are contained in $G_n = p_n(F, \circ)$. Thus
\[
\psi_w(B_{r_1}(0)) \subset \bigcup_{n_1, n_2} \psi_w(T + n_1 P + n_2 Q)
\]
\[
= \bigcup_{n_1, n_2} \psi_{\Phi_{n_1} \circ \Phi_{n_2} Q}(w)(T) \quad \text{(by Remark 3.9.1)}
\]
\[
\subset \mathcal{M}(G_n)
\]
\[
\subset \mathcal{M},
\]
which contradicts the fact that $\psi_w(B_{r_1}(0))$ contains points of the complement $\mathbb{R}^2 \setminus \mathcal{M}$. So our assumption is wrong.

Therefore one of the following alternatives must hold:

- at least one of the additions in $W$ is not admissible or
- at least one element $w_0 \in W$ does not belong to $G_n$.

In view of Proposition 4.1 we conclude that at least one $w_0 \in W$ does not belong to $G_n = p_n(F, \circ)$. Proposition 4.1 also shows that all additions are admissible for each element of $G_n$, i.e., starting from a word in $G_n$, each addition $\Psi_{\pm P}, \Psi_{\pm Q}$ leads to a word in $p_n(F)$. Thus, starting at $w$, by a sequence of admissible additions we can reach a word $w_0 \in W$ which belongs to $p_n(F) \setminus p_n(F, \circ)$. Let us write
\[
w_0 = \Phi_{n_1} P \circ \Phi_{n_2} Q(w) \in p_n(F) \setminus p_n(F, \circ)
\]
for some $n_1 \in \{-m_1, \ldots, m_1\}$, $n_2 \in \{-m_2, \ldots, m_2\}$. (Note that by Remark 3.9.1
\[
\psi_w(T + n_1 P + n_2 Q) = \psi_{w_0}(T)
\]
because we have a sequence of admissible additions from $w$ to $w_0$.) For this choice of $w_0$ we have

- $\psi_{w_0}(T) \subset \psi_w(B_{r_2}(0))$ by assumption,
- $\psi_{w_0}(T) \cap \partial T \neq \emptyset$ by Proposition 6.1.

This implies that $\psi_w(B_{r_2}(0))$ contains $x$ as well as some point of $\partial T$, hence $\text{dist}(x, \partial T) < \varepsilon$.

Consequently, in both cases for every $\varepsilon > 0$, $\text{dist}(x, \partial T) < \varepsilon$, thus $x \in \partial T$, and this holds for every $x \in \partial M$, hence, $\partial M \subset \partial T$.

\section{Generalized fundamental inequality and consequences}

This section is devoted to a generalization of the fundamental inequality found in [1]. This will lead to the construction of an arcwise connected skeleton inside the interior of the tile $T$ together with some of its neighbors. We denote by $S$ the set of elements of $\mathbb{Q}[\alpha]$ with integer part zero with respect to the basis $\mathcal{N} = \{0, 1, 2, 3, 4\}$:
\[
S := \left\{ \sum_{i=1}^{l} \Phi(\alpha^{-i} a_i), \ l \in \mathbb{N}, (a_i)_{1 \leq i \leq l} \in \mathcal{N}^l \right\}.
\]
Consequently we have $\overline{\Phi(S)} = T$.

\begin{remark}
We recall that the tile $T$ is symmetric with respect to the point $\Phi(c) := \Phi\left(\frac{4}{2(\alpha - 1)}\right)$ (see [1, Lemma 3.2]).
\end{remark}

\begin{proposition}[Generalized fundamental inequality]
There is an $\varepsilon > 0$ such that for any $x \in S + 2\alpha$ we have
\[
\mathfrak{M}(x) > \mathfrak{M}(c) + \varepsilon.
\]
\end{proposition}
Proof. This follows from the minoration
\[
\Im \left( \sum_{i=1}^{l} a_i \alpha^{-i} \right) \geq - \left| \sum_{i=1}^{l} a_i \alpha^{-i} \right| \geq - \frac{4}{|\alpha - 1|},
\]
so that for \( x \in S + 2\alpha \) we have \( \Im(x) > 0 > I(c) = \frac{-1}{5} \). \( \square \)

Corollary 7.3. Let \( \gamma \in \mathbb{Z}[\alpha] \) and put \( \gamma = u + v\alpha \) with \( u, v \in \mathbb{Z} \). Then there exists a constant \( \varepsilon > 0 \) such that for any \( x \in S \),
\[
\begin{cases}
\Im(x) + \Im(\gamma) > \Im(c) + \varepsilon & \text{if } v \geq 2 \\
\Im(x) + \Im(\gamma) < \Im(c) - \varepsilon & \text{if } v \leq -2.
\end{cases}
\]

Proof. This is proved in the same way as in [1, Lemma 4.3]. \( \square \)

We will now construct a generalized version of the skeleton constructed in [1] where the tiles were disk-like. To this matter set the backbone
\[
\mathcal{L} := \{ \Phi(c) + w\Phi(1) \mid w \in [0, 4] \}.
\]
Furthermore, let
\[
V_0 := \bigcup_{S \in \mathcal{B}_0} (T + S).
\]
Then the following lemma holds.

Lemma 7.4. We have \( \mathcal{L} \subset \text{Int}(AV_0) \).

Proof. Note that
\[
\text{Int}(AV_0) = \mathbb{R}^2 \setminus \bigcup_{x \in \mathbb{Z}^2 \setminus \mathcal{B}_0} (A(T + x)) = \mathbb{R}^2 \setminus \left( \bigcup_{u \in \mathbb{Z}, |v| \geq 2} (T + \Phi(u + v\alpha)) \cup \bigcup_{v \geq 1, u \leq -1 \text{ or } u \geq 10} (T + \Phi(u + v\alpha)) \cup \bigcup_{v = 0, u \leq -6 \text{ or } u \geq 10} (T + \Phi(u + v\alpha)) \cup \bigcup_{v = -1, u \leq -6 \text{ or } u \geq 5} (T + \Phi(u + v\alpha)) \right).
\]
Thus we have to show that \((T + \Phi(u + v\alpha)) \cap \mathcal{L} = \emptyset\) for all constellations \((u, v)\) occurring in the above unions.

Let first \( \gamma = u + v\alpha \) with \( u \in \mathbb{Z} \) and \(|v| \geq 2\). Then Corollary 7.3 and the fact that \( \Phi(S) = T \) imply that \((T + \Phi(\gamma)) \cap \mathcal{L} = \emptyset\).

For the pairs of the shape \((u, 0)\), \( u \leq -6 \text{ or } u \geq 10 \) we see that \((T + \Phi(\gamma)) \cap \mathcal{L} = \emptyset\) in exactly the same way as in [1, Lemma 4.4].

If now \( v = -1 \), suppose first that \( u \leq -6 \). If \( x \in \alpha^{-1}(S - \alpha^2) + u \), then \( \alpha x \in S + 5 + (u + 4)\alpha \). From \( u + 4 \leq -2 \) and by Corollary 7.3, we can write \( \Im(\alpha x) < \Im(c) - \varepsilon \), and consequently:
\[
\forall j \in \{0, \ldots, 4\}, \ \exists(\alpha x + j) = \exists(\alpha(x + \alpha^{-1}j)) < \exists(c) - \varepsilon,
\]
thus
\[
\forall j \in \{0, \ldots, 4\}, \forall x \in \alpha^{-1}(S - \alpha^2 + j) + u, \ \exists(\alpha x) < \exists(c) - \varepsilon.
\]
From $\Phi(S) = T$, we conclude that this inequality also holds if we replace $S$ by $T$, and from the set equation of $T$ we obtain, taking the union over $j \in \{0, \ldots, 4\}$,

$$\forall \Phi(x) \in T + \Phi(u - \alpha), \exists(\alpha x) \leq \exists(c) - \varepsilon.$$ 

On the other hand, for an element of $L$, i.e., $\Phi(x) = \Phi(c) + w\Phi(1), w \in [0, 4]$, we have

$$\exists(\alpha x) = w + \exists(\alpha c) \geq \exists(c) = -1/5 = \exists(c),$$

which implies that $(T + \Phi(u - \alpha)) \cap L = \emptyset$ for all $u \leq -6$.

Suppose secondly that $u \geq 5$. If again $x \in \alpha^{-1}(S - \alpha^{2}) + u$, then $\alpha x \in S + 5 + (u + 4)\alpha$. From $u \geq 5$ and by Corollary 7.3, we have $\exists(\alpha x) > \exists(c) + \varepsilon + 4\exists(\alpha)$, and consequently, by a similar reasoning as above, we obtain

$$\forall \Phi(x) \in T + \Phi(u - \alpha), \exists(\alpha x) \geq \exists(c) + \varepsilon + 4.$$ 

On the other hand, for $\Phi(x) \in L$, we have $\exists(\alpha x) \leq \exists(c) + 4$, thus again $(T + \Phi(u - \alpha)) \cap L = \emptyset$ for all $u \geq 5$.

The remaining case $v = 1$ is treated likewise, thus the proof is complete. \hfill \Box

Composing small pieces of backbones, let us define the $n$-skeleton by

$$K_{n} = \bigcup_{m=1}^{n} \bigg( \bigcup_{(a_{1}, a_{2}, \ldots, a_{m})} \sum_{i=1}^{m} A^{-i} \Phi(a_{i}) + A^{-m-1}(L) \bigg).$$

**Lemma 7.5.** $K_{n}$ is arcwise connected and $K_{n} \subset \text{Int}(V_{0})$.

**Proof.** The arcwise connectivity can be shown as in [1, Lemma 4.6].

For the second part, note that for every $(a_{1}, \ldots, a_{m})$, by Lemma 7.4 we have for the small pieces

$$\sum_{i=1}^{m} A^{-i} \Phi(a_{i}) + A^{-m}L \subset \sum_{i=1}^{m} A^{-i} \Phi(a_{i}) + \text{Int}(A^{-m}V_{0}).$$

Now remember that using the automaton $B_{0}$ one can compute for $S \in B_{0}$ that

$$\sum_{i=1}^{m} A^{-i} \Phi(a_{i}) + A^{-m}S = A^{-m} \left( S + \sum_{i=0}^{m-1} A^{-i} \Phi(a_{m-i}) \right)$$

$$= A^{-m} \left( A^{m} S' + \sum_{i=0}^{m-1} A^{-i} \Phi(a'_{m-i}) \right) \quad \text{(by (3.2))}$$

$$= \sum_{i=1}^{m} A^{-i} \Phi(a'_{i}) + S'$$

with $S' \in B_{0}$ and $a'_{i} \in N$. From this we can conclude that

$$\sum_{i=1}^{m} A^{-i} \Phi(a_{i}) + A^{-m}V_{0} \subset V_{0},$$

hence the union of the backbones remains in $\text{Int}(V_{0})$. \hfill \Box

**Remark 7.6.** Note that the middle points $\sum_{i=1}^{m} A^{-i} \Phi(a_{i})$ of the union of all backbones are dense in $T$.

8. **The Component of Int($T$) Containing 0**

We are almost ready to prove Theorem 2.4 concerning the description of $C_{0}$. We will first construct an arc from an arbitrary point in $\text{Int}(M)$ to zero entirely contained in $\text{Int}(M)$.

**Lemma 8.1.** If $x \in \text{Int}(M)$, then there is an $n \geq 3$ such that $x \in M(G_{n})$. 

Proposition 8.3. Then there is an arc \( (3.1), (3.2) \) and Remark 3.2, been shown in a more general context in Wang [25]. Let \( 2B \)
\[ S \]
Then the point \( S \)
Thus to each point \( A \)
Now choose \( M \)
x
Since \( S \)
\( v, (1 + S) \) and Remark 3.2, this remains true after applying the homeomorphism \( v, T \).
We show that \( T \) belongs to the triple intersection \( M \)
\[ \text{diam}(\psi_{w_n}(T)) < \varepsilon, \]
\[ \psi_{w_n}(T) \cap \partial T \neq \emptyset \] (by Lemma 6.1).
Thus for every \( \varepsilon > 0, \) dist \((x, \partial T) < \varepsilon, \) hence \( x \in \partial T, \) since \( \partial T \) is a closed set. \( \square \)

Lemma 8.2. Let \( n \geq 3, S \in B \) and \( v_1, v_2 \in G_n \) such that \( v_2 = \Psi_S(v_1). \) Then \( \psi_{v_1}(T) \cap \psi_{v_2}(T) \) contains points of \( \text{Int}(T). \)

Proof. Since both subpieces \( \psi_{v_1}(T) \) and \( \psi_{v_2}(T) \) are subsets of \( T, \) points of their intersection that are not in \( \text{Int}(T) \) must be in \( \partial T. \) We show that there are at most countably many such points, whereas \( \psi_{v_1}(T) \cap \psi_{v_2}(T) \) is uncountable.

The uncountability of \( \psi_{v_1}(T) \cap \psi_{v_2}(T) \) follows from the fact that \( v_2 = \Psi_S(v_1), \) hence \( \psi_{v_2}(T) = \psi_{v_1}(T + S), \) with \( S \in B \) (see Remark 3.9.1). Since \( T \cap (T + S) \) has uncountably many points for \( S \in B \) (see [1, Section 9]), this remains true after applying the homeomorphism \( \psi_{v_1}. \)

On the other side, a point \( x \) of \( \psi_{v_1}(T) \cap \psi_{v_2}(T) \) which lies on \( \partial T \) also belongs to a translate \( T + S' \) of \( T \) with \( S' \in B, \) by the boundary equation (3.3). This translate is the union of the subpieces \( \psi_{v_1}(T) + S' \) with \( |w| = |v_1| = n. \) Let us write \( v_1 =: (a_1, \ldots, a_n) \) and \( w =: (b_1, \ldots, b_n). \) Then the point \( A^nx \) belongs to the triple intersection
\[ \left( T + \sum_{i=0}^{n-1} A^i\Phi(a_{n-i}) \right) \cap \left( T + S + \sum_{i=0}^{n-1} A^i\Phi(a_{n-i}) \right) \cap \left( T + A^nS' + \sum_{i=0}^{n-1} A^i\Phi(b_{n-i}) \right), \]
or, equivalently, the point \( A^nx - \sum_{i=0}^{n-1} A^i(a_{n-i}) \) belongs to the triple intersection
\[ T \cap (S + T) \cap (S'' + T) =: V(S, S''), \]
with \( S'' := A^nS' + \sum_{i=0}^{n-1} A^i\Phi(b_{n-i}) - \sum_{i=0}^{n-1} A^i\Phi(a_{n-i}). \) Note that \( S'' \notin \{\emptyset, S\}. \) Indeed, using (3.1), (3.2) and Remark 3.2, \( S'' = \emptyset \) as well as \( S'' = S \) would imply \( S' = \emptyset. \)
Thus to each point \( x \) of \( \psi_{v_1}(T) \cap \psi_{v_2}(T) \cap \partial T \) corresponds exactly one point of \( V_3 \), the set of all triple points of \( T \) (i.e., where \( T \) meets with two other translates). Since \( V_3 \) is at most countable (see [1, Theorem 10.1]), there are at most countably many points in \( \psi_{v_1}(T) \cap \psi_{v_2}(T) \cap \partial T. \) Together with the first part of this proof this means that \( \psi_{v_1}(T) \cap \psi_{v_2}(T) \cap \text{Int}(T) \) is not empty. \( \square \)

Proposition 8.3. Let \( x \in \text{Int}(M) \) and an \( n \geq 3 \) given by Lemma 8.1, i.e., such that \( x \in \mathcal{M}(G_n). \) Then there is an arc \( p \) from \( x \) to an element \( y \) of \( \mathcal{M}(\mathcal{G}(G_{n-1})) \) with \( p \subset \text{Int}(T). \)

Proof. In this proof we will often use the fact that \( T \) is the closure of its interior. This has been shown in a more general context in Wang [25]. Let \( w \in G_n \) such that \( x \in \psi_w(T). \) By Proposition 5.2 there exists a finite chain of walks \( v_1, \ldots, v_m \in G_{n+1} \) with the following properties:
\[ v_1 = w \& d \text{ for some } d \in \{0, \ldots, 4\}, \]
\[ \Psi_{S_j}(v_i) = v_{i+1} \text{ for some } S_j \in B \quad (1 \leq i \leq m - 1), \]
\[ v_m|_{n-1} \in G_{n-1}. \]
Now choose \( x_1 \in \text{Int}(\psi_{v_1}(T)) \) arbitrary and set \( y := x_m. \) Note that \( \psi_{v_m}(T) \subset \psi_{v_m|_{n-1}}(T) \subset \mathcal{M}(G_{n-1}), \) thus \( y \) has the required property. First we shall construct an arc \( p_1 \subset \text{Int}(T) \) from \( x \) to \( x_1. \) Without loss of generality, one can suppose that \( x \in \psi_{v_1}(T) \) (see Remark 5.3.1).

Since \( x \in \text{Int}(M) \) there exists an \( \varepsilon > 0 \) such that \( B_\varepsilon(x) \subset \text{Int}(M). \) Thus there is a \( z_1 \in B_\varepsilon(x) \cap \text{Int}(\psi_{v_1}(T)). \) Connect \( x \) with \( z_1 \) by a straight line segment \( \ell_1. \) Obviously, \( \ell_1 \subset \text{Int}(M). \)
Now \(x_1, z_1 \in \text{Int}(\psi_{v_1}(T))\). Thus there exists \(\varepsilon_2 > 0\) such that

\[
B_{\varepsilon_2}(z_1) \subset \text{Int}(\psi_{v_1}(T)),
\]

\[
B_{\varepsilon_2}(x_1) \subset \text{Int}(\psi_{v_1}(T)).
\]

By Remark 7.6 at the end of the previous section there exists a \(j \in \mathbb{N}\) such that \(\psi_{v_1}(K_j)\) contains points \(z_2, z_3\) with

\[
z_2 \in B_{\varepsilon_2}(z_1),
\]

\[
z_3 \in B_{\varepsilon_2}(x_1).
\]

Now connect \(z_1\) with \(z_2\) by the line segment \(\ell_2\) and connect \(z_3\) with \(x_1\) by the line segment \(\ell_3\). Both of these line segments are obviously contained in \(\text{Int}(T)\). Since \(K_j\) is arcwise connected by Lemma 7.5 there exists an arc \(q_1 \in \psi_{v_1}(K_j)\) connecting \(z_2\) with \(z_3\). We have to show that \(q_1 \subset \text{Int}(T)\).

What we know from Lemma 7.5 is that

\[
q_1 \subset \text{Int}(\psi_{v_1}(V_0)) = \text{Int} \left( \bigcup_{S \in B_0} \psi_{v_1}(T + S) \right) = \text{Int} \left( \bigcup_{S \in B_0} \psi_{\Psi_S(v_1)}(T) \right).
\]

We used here Remark 3.9.1. Indeed, since \(v_1 \in G_{n+1}\), all the additions \(\Psi_S(v_1)\) with \(S \in B\) are admissible by Proposition 4.1. So for all \(S \in B_0\), \(\psi_{v_1}(T + S)\) is contained in \(T\), because it is a subpiece of level \(n+1\) of \(T\). This implies that \(\text{Int}(\psi_{v_1}(V_0)) \subset \text{Int}(T)\). Thus \(q_1 \subset \text{Int}(T)\). Summing up we have constructed an arc \(p_1 := \ell_1 q_1 \ell_3 \) from \(x\) to \(x_1\) which is contained in \(\text{Int}(T)\).

In the next step we construct an arc \(p_{i+1}\) from \(x_i\) to \(x_{i+1}\), still inside \(\text{Int}(T)\). Because \(\Psi_S(v_1) = v_{i+1}\) for some \(S_i \in B\), Lemma 8.2 implies the existence of a \(z_1 \in \psi_{v_1}(T) \cap \psi_{v_{i+1}}(T)\) which is contained in \(\text{Int}(T)\). Thus there exists an \(\varepsilon_1 > 0\) with \(B_{\varepsilon_1}(z_1) \subset \text{Int}(T)\). Furthermore,

\[
z_1 \in \psi_{v_1}(T) \implies \exists z_2 \in B_{\varepsilon_1}(z_1) \cap \text{Int}(\psi_{v_1}(T)),
\]

\[
z_1 \in \psi_{v_{i+1}}(T) \implies \exists z_3 \in B_{\varepsilon_1}(z_1) \cap \text{Int}(\psi_{v_{i+1}}(T)).
\]

Now connect \(z_2\) with \(z_1\) by the line segment \(\ell_1\) and connect \(z_1\) with \(z_3\) by the line segment \(\ell_2\). Both of these line segments are obviously contained in \(\text{Int}(T)\).

As above we can now construct using the \(n\)-skeletons an arc \(q_1 \subset \text{Int}(T)\) connecting \(x_i\) with \(z_2\) and an arc \(q_2 \subset \text{Int}(T)\) connecting \(z_3\) with \(x_{i+1}\). The arc \(p_{i+1} := q_1 \ell_1 q_2 \) now connects \(x_i\) with \(x_{i+1}\).

Setting \(p := p_1 \ldots p_m\) we have a path connecting \(x\) with \(y\) lying entirely in the interior of \(T\). \(\square\)

**Proposition 8.4.** Let \(x\) be a point of \(\text{Int}(M)\). Then there is an arc connecting \(x\) to 0.

**Proof.** By Lemma 8.1 there is an \(n \geq 3\) such that \(x \in M(G_n)\). By applying Proposition 8.3 \(n - 3\) times we can construct an arc inside \(\text{Int}(T)\) from \(x\) to some \(y \in \text{Int}(M(G_3))\). Since \(M(G_3) = \psi_{(0,0,0)}(T) = \psi_{(1,4,4)}(T)\) with \((F;1,4,4) = \Psi_{-1}(F;0,0,0)\) (see Remark 5.1), an arc \(q''\) from \(y\) to 0 in \(\psi_{(0,0,0)}(T)\) inside \(\text{Int}(T)\) can be constructed in the same way as the arcs \(p_i\) in the proof of Proposition 8.3, using the \(n\)-skeletons.

Now \(q = q' q''\) does the job. \(\square\)

We obtain directly from the above proposition the following result.

**Corollary 8.5.** The set \(\text{Int}(M)\) is a subset of the interior component of \(T\) containing 0.

The reverse inclusion is the purpose of the next proposition.

**Proposition 8.6.** The component of \(\text{Int}(T)\) containing 0 is a subset of \(\text{Int}(M)\).
Proof. Let \( y \) be a point in \( \text{Int}(T) \) such that there is an arc \( p : [0, 1] \rightarrow \text{Int}(T) \) connecting 0 and \( y \). Suppose that \( y \) does not belong to \( \text{Int}(M) \). Since \( \partial M \subset \partial T \) (see Proposition 6.3), \( y \) does not belong to \( M \). Let
\[
t_0 := \inf \{ t \in [0, 1] \mid p(t) \notin M \}.
\]
Then \( t_0 \in (0, 1) \) and \( p(t_0) \in \partial M \), because every neighborhood of this point encounters \( M \) as well as its complement. Since again \( \partial M \subset \partial T \), we obtain that \( p(t_0) \in \partial T \), a contradiction to the definition of \( p \).

\[\text{Lemma 8.7.} \quad \text{The set } M \text{ is the closure of its interior, i.e., } \overline{\text{Int}(M)} = M.\]

\[\text{Proof.} \quad \text{Let } x \in M \text{ and } \varepsilon > 0 \text{ be arbitrary. Let } n \geq 3 \text{ be large enough such that } \text{diam}(\psi_v(T)) < \varepsilon \text{ for each } w \in p_n(F). \text{ There exists a walk } v \in p_n(F) \text{ such that } x \in \psi_v(T). \text{ It can easily be read off from the graph } G \text{ that each } v \in p_n(F) \text{ can be extended to a walk } v' = v \& (b_1, b_2) \in G_{n+2} = p_{n+2}(F, o). \text{ Thus}
\]
\[
\psi_{v'}(T) \subset M \implies \text{Int}(\psi_{v'}(T)) \subset \text{Int}(M)
\]
and
\[
\psi_{v'}(T) \subset \psi_v(T).
\]
Select \( y \in \text{Int}(\psi_{v'}(T)) \). Then the above inclusions imply that \( y \in \text{Int}(M) \) and \( \text{dist}(x, y) < \varepsilon \).

Since \( \varepsilon \) can be arbitrarily small we have that \( x \in \overline{\text{Int}(M)} \). \[\Box\]

\[\text{Proof of Theorem 2.4.} \quad \text{The first part of this theorem follows from Corollary 8.5 and Proposition 8.6. The second part is given by Lemma 8.7.} \quad \Box\]

9. Concluding remarks

We have shown in this paper how to obtain the closure \( M \) of the interior component of \( T \) that contains the point \( \Phi(0) \). This component is depicted in Figure It is of natural interest to wonder how the closure of the interior component of \( T \) containing a given point \( x \) could be computed. For the other “big” components (see Figure 4), \( i.e., \) containing \( \Phi(1), \Phi(2) \) and \( \Phi(3) \), the description is similar: it suffices to replace the edges at the top of \( G \) in a way that can be seen in Figure 5.

![Figure 4. Tile associated to the base \(-2 + \sqrt{-1}\) with “big” interior components.](image)

![Figure 5. Top edges in \( G \) for the closure of the component containing \( \Phi(d) \), \( d = 0, 1, 2, 3 \).](image)
Thus the closure of these components are simply images of \( M \) by translations.

For other “smaller components”, we conjecture that there closure is also similar to \( M \), the similitude may be given by a pre-graph that would be connected to \( \mathcal{G} \) via the state \( F \).

References