Exercises Course A

(1) Prove that $R(3,3) = 6$.

(2) Prove that $\alpha(G)\chi(G) \geq |G|$.

(3) Let $G = (V, E)$ be a graph with n vertices and e edges. Then G contains a bipartite subgraph with at least $e/2$ edges.
(Hint: Consider a random set T of vertices and the number of edges that connect T and $V \setminus T$.)

(4) Prove that
$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d_v + 1},$$
where d_v denote the degree of $v \in V(G)$.
(Hint: Choose a random total order on $V(G)$ and consider the random number of vertices for which all neighbors are larger.)

(5) Prove that there is an absolute constant $c > 0$ with the following property:
Let A be an n by n matrix with pairwise distinct entries. Then there is a permuation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.
(Hint: Fix k and compute the expected number of increasing subsequences of length k.)

Exercises Course A (2)

(6) Let $(Y_\alpha)_{\alpha\in A}$ be a system of independent random variables. Further let I be an index set and for every $i \in I$ let A_i be a subset of A. Now suppose that X_i is a function of the variables $(Y_\alpha)_{\alpha\in A_i}$.
Prove that the graph with vertex set I and edge set $E = \{(i,j) : A_i \cap A_j \neq \emptyset\}$ is a dependency graph for the family $(X_i)_{i\in I}$.

(7) Suppose that $Y_1, Y_2, \ldots$ is a sequence of random variables with zero means $\mathbb{E}\, Y_j = 0$ and bounded third moments $\mathbb{E}\,|Y_j|^3 < \infty$, that are k-independent, that is, whenever A, B are two subsets of the postive integers with $\min\{|i-j| : i \in A,\ j \in B\} > k$ then the subsystems $(Y_i, i \in A)$ and $(Y_j, j \in B)$ are independent.
Set $S_n = Y_1 + \ldots + Y_n$ and $\sigma_n^2 := \mathbb{V} S_n$. Prove that if

$$\lim_{n\to\infty} \frac{1}{\sigma_n^3} \sum_{j=1}^{n} \mathbb{E}\,|Y_j|^3 = 0$$

then $S_n = Y_1 + \ldots + Y_n$ satisfies a central limit theorem.

(8) Let $A = \{a_1, a_2, \ldots, a_m\}$ be a finite alphabet with probability distribution $p_1, p_2, \ldots, p_m$, that is, $\mathbb{P}\{a_j\} = p_j$ and $p_1 + \cdots + p_m = 1$. Let $x_1x_2x_3 \ldots x_n$ a random sequence of length n over the alphabet A where the letter x_j are chosen independently (according to the above probability distribution $p_1, p_2, \ldots, p_m$). Further, let $B = b_1b_2 \ldots b_k \in A^k$ a given block of size k (over A) and let X_n denote the (random) number of occurences of B as a (consecutive) subblock of $x_1x_2x_3 \ldots x_n$.
Prove that X_n satisfies a central limit theorem. Compute, $\mathbb{E}\, X_n$ and $\mathbb{V} X_n$, too.