Euler's constant and averages of fractional parts

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1 Introduction and statement of results

Euler's constant is defined by $\gamma := \lim_{y\to\infty}\left(\sum_{n\le y} 1/n - \log y\right) = 0.5772156649\ldots$. There is a huge amount of literature on this famous mathematical constant among which we just refer to the book [2] and the references therein. Like π or e, Euler's constant γ appears in many number-theoretic identities and relations; see, e.g., [5]. One of these is a result from de la Vallée Poussin [4] from the year 1898. He showed that if one divides an integer m by all integers less than it, then the average of the fractional parts of these fractions tends to $1-\gamma$ when m approaches infinity. This result is mentioned (without any proof), in an equivalent form, in [2, Chapter 12.3], and it can also be found in [5]. The same result holds true when the divisors are only those in an arithmetic progression. Using the notation $\{x\} = x - \lfloor x \rfloor$ for the fractional part of x, de la Vallée Poussin's [4] result can be stated in the following form:

Theorem 0 (de la Vallée Poussin, 1898) *For all reals $x \ge 1$ and integers $a \ne 0$ and b we have*

$$\sum_{\substack{d\le x\\ d\equiv b \pmod a}} \left\{\frac{x}{d}\right\} = \frac{x}{a}(1-\gamma) + O(\sqrt{x}),$$

where $\sum_{d\le x}$ means summation over all integers d such that $1 \le d \le x$.

It should be remarked that a similar result holds true if the divisors are only the primes. More precisely, in [4] it was also shown that, as $x\to\infty$, we have

$$\sum_{p\le x}\left\{\frac{x}{p}\right\} = \frac{x}{\log x}(1-\gamma) + o\left(\frac{x}{\log x}\right), \tag{1}$$

where the sum is over all primes p such that $p \le x$.

In this note we present an easy proof of Theorem 0 and we show a new formula of the same kind, but where the divisors are only allowed to be a fixed power of integers (see Theorem 1).

For the statement of the new result we will need a generalization of Euler's constant: for reals $\alpha > 0$ we define $\gamma_\alpha := \lim_{x\to\infty}\left(\sum_{n\le x} 1/n^\alpha - \int_1^x 1/t^\alpha \,\mathrm{d}t\right)$ (note that the limit exists since $x \mapsto 1/x^\alpha$ is a decreasing function). In this setting, Euler's constant can be obtained by choosing $\alpha = 1$. With this notation we can state our second theorem:

Theorem 1 *For integers $a > 1$ and reals $x \ge 1$ we have*

$$\sum_{k\le \sqrt[a]{x}} \left\{\frac{x}{k^a}\right\} = \sqrt[a]{x}\left(1-\gamma_{1/a}\right) + O\left(x^{\frac{1}{a+1}}\right).$$

2 The proof of Theorems 0 and 1

For the proof of the two theorems we study sums of the form

$$\sum_{d \le x} f(d) \left\{ \frac{x}{d} \right\} \tag{2}$$

for real $x \ge 1$, where f is an arithmetic function. For example if $f(d) = 1$ if $d \equiv b \pmod{a}$ and 0 otherwise, then we obtain the sum in Theorem 0, and if $f(d) = 1$ if $d = k^a$ for some integer k and 0 otherwise, then we obtain the sum in Theorem 1.

We need some notation and lemmas: for arithmetic functions $f, g : \mathbb{N} \to \mathbb{C}$ the convolution $f * g$ is defined by $f * g(n) := \sum_{d|n} f(d) g(n/d)$, where the sum is over all integers d such that $1 \le d \le n$ and d is a divisor of n. Let $\mathbf{1} : \mathbb{N} \to \mathbb{N}$ be defined by $\mathbf{1}(n) = 1$.

We start with the following simple observation:

Lemma 1 *For any real $x \ge 1$ we have*

$$\sum_{d \le x} f(d) \left\{ \frac{x}{d} \right\} = x \sum_{d \le x} \frac{f(d)}{d} - \sum_{n \le x} f * \mathbf{1}(n).$$

Proof. Convolution of the arithmetic function f with $\mathbf{1}$ yields

$$\sum_{n \le x} f * \mathbf{1}(n) \;=\; \sum_{n \le x} \sum_{d|n} f(d) \mathbf{1}\left(\frac{n}{d}\right) = \sum_{d \le x} f(d) \sum_{\substack{n \le x \\ d|n}} 1 = \sum_{d \le x} f(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

Hence we have

$$\sum_{d \le x} f(d) \left\{ \frac{x}{d} \right\} = \sum_{d \le x} f(d) \left(\frac{x}{d} - \left\lfloor \frac{x}{d} \right\rfloor \right) = x \sum_{d \le x} \frac{f(d)}{d} - \sum_{n \le x} f * \mathbf{1}(n)$$

as desired. □

Furthermore, we will often need some elementary and well-known asymptotic formulas which we collect in the following lemma.

Lemma 2 *(a) For $x \ge 1$ we have $\sum_{n \le x} 1/n^s = x^{1-s}/(1-s) + \zeta(s) + O(x^{-s})$ for $s > 0$, $s \ne 1$. Here $\zeta(s)$ denotes the Riemann zeta function which is defined by $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$ if $s > 1$ and by $\zeta(s) := \lim_{x \to \infty} \left(\sum_{n \le x} 1/n^s - x^{1-s}/(1-s) \right)$ if $0 < s < 1$.*

(b) For $v \ge 0$ and $x \ge 1$ we have $\sum_{n \le x} 1/(n+v) = \log(x+v) + C(v) + O(1/x)$, where $C(v)$ only depends on v. (For $v = 0$ we have $C(0) = \gamma$.)

Proof. A proof for (a) can be found in [1, Theorem 3.2] and a proof for (b) can be found in [3, Kapitel VI, §4, 3.]. □

Now we present the proof of Theorem 0.

Proof of Theorem 0. We assume that $0 \le b < a$. Let $f(d) = 1$ if $n \equiv b \pmod a$ and 0 otherwise. From Lemma 1 we obtain

$$\sum_{\substack{d \le x \\ d \equiv b \pmod a}} \left\{ \frac{x}{d} \right\} = x \sum_{\substack{d \le x \\ d \equiv b \pmod a}} \frac{1}{d} - \sum_{n \le x} f * \mathbf{1}(n). \tag{3}$$

Using Lemma 2(b) and the notation $t_b := 1/b$ if $b \ne 0$ and $t_0 := 0$ we have

$$\sum_{\substack{d \le x \\ d \equiv b \pmod a}} \frac{1}{d} = t_b + \frac{1}{a} \sum_{k \le \frac{x-b}{a}} \frac{1}{k + \frac{b}{a}} = \frac{1}{a} \log\left(\frac{x}{a}\right) + t_b + \frac{C(b/a)}{a} + O\left(\frac{1}{x}\right). \tag{4}$$

For the evaluation of the last sum in (3) we use *Dirichlets hyperbola method* (see, e.g., [1, Theorem 3.17]) which states that for arithmetic functions $f, g : \mathbb{N} \to \mathbb{C}$ and with $F(x) := \sum_{n \le x} f(n)$ and $G(x) := \sum_{n \le x} g(n)$ for $1 \le y \le x$ we have

$$\sum_{n \le x} f * g(n) = \sum_{n \le y} g(n) F\left(\frac{x}{n}\right) + \sum_{m \le x/y} f(m) G\left(\frac{x}{m}\right) - F\left(\frac{x}{y}\right) G(y). \tag{5}$$

Therefore and with $y = \sqrt{x}$ we obtain

$$\begin{aligned}
\sum_{n \le x} f * \mathbf{1}(n) &= \sum_{n \le \sqrt{x}} \sum_{m \le x/n} f(m) + \sum_{m \le \sqrt{x}} f(m) \left\lfloor \frac{x}{m} \right\rfloor - \left(\sum_{n \le \sqrt{x}} f(n) \right) \left(\sum_{n \le \sqrt{x}} 1 \right) \\
&= \sum_{n \le \sqrt{x}} \sum_{\substack{m \le x/n \\ m \equiv b \pmod a}} 1 + \sum_{\substack{m \le \sqrt{x} \\ m \equiv b \pmod a}} \left\lfloor \frac{x}{m} \right\rfloor - \lfloor \sqrt{x} \rfloor \sum_{\substack{n \le \sqrt{x} \\ n \equiv b \pmod a}} 1 \\
&= \sum_{n \le \sqrt{x}} \left(\frac{x}{an} + O(1) \right) + \sum_{\substack{m \le \sqrt{x} \\ m \equiv b \pmod a}} \frac{x}{m} + O(\sqrt{x}) \\
&\quad -(\sqrt{x} + O(1)) \left(\frac{\sqrt{x}}{a} + O(1) \right) \\
&= \frac{x}{a} \left(\log \sqrt{x} + \gamma \right) + x \left(\frac{1}{a} \log\left(\frac{\sqrt{x}}{a}\right) + t_b + \frac{C(b/a)}{a} \right) - \frac{x}{a} + O(\sqrt{x}).
\end{aligned}$$

Inserting this result and (4) into (3) yields

$$\sum_{\substack{d \le x \\ d \equiv b \pmod a}} \left\{ \frac{x}{d} \right\} = \frac{x}{a}(1 - \gamma) + O(\sqrt{x}),$$

as desired. $\square$

The same method can be applied to give a proof of Theorem 1.

Proof of Theorem 1. Using Lemma 1 with $f(d) = 1$ if $d = k^a$ for some $k \in \mathbb{N}$ and 0 otherwise we obtain

$$\sum_{k \le \sqrt[a]{x}} \left\{ \frac{x}{k^a} \right\} = x \sum_{d \le \sqrt[a]{x}} \frac{1}{d^a} - \sum_{n \le x} f * \mathbf{1}(n). \tag{6}$$

Then for $1 \le y \le x$, Dirichlets hyperbola method (5), the fact that $\lfloor a \rfloor = a + O(1)$, and Lemma 2(a) (in this order) give

$$\begin{aligned}
\sum_{n \le x} f * \mathbf{1}(n) &= \sum_{n \le y} \left\lfloor \sqrt[a]{\frac{x}{n}} \right\rfloor + \sum_{m \le \sqrt[a]{x/y}} \left\lfloor \frac{x}{m^a} \right\rfloor - \left\lfloor \sqrt[a]{\frac{x}{y}} \right\rfloor \lfloor y \rfloor \\
&= \sum_{n \le y} \sqrt[a]{\frac{x}{n}} + \sum_{m \le \sqrt[a]{x/y}} \frac{x}{m^a} - y \sqrt[a]{\frac{x}{y}} + O\left(\max\left\{ y, \sqrt[a]{\frac{x}{y}} \right\} \right) \\
&= \sqrt[a]{x} \left(\frac{y^{1-1/a}}{1-1/a} + \zeta\left(\frac{1}{a}\right) + O\left(\frac{1}{\sqrt[a]{y}}\right) \right) \\
&\quad + x \left(\frac{\left(\sqrt[a]{\frac{y}{x}}\right)^{1-a}}{1-a} + \zeta(a) + O\left(\frac{y}{x}\right) \right) - y \sqrt[a]{\frac{x}{y}} + O\left(\max\left\{ y, \sqrt[a]{\frac{x}{y}} \right\} \right) \\
&= \sqrt[a]{x} \zeta\left(\frac{1}{a}\right) + x\zeta(a) + O\left(\max\left\{ y, \sqrt[a]{\frac{x}{y}} \right\} \right).
\end{aligned}$$

Choosing $y = x^{\frac{1}{a+1}}$ we therefore obtain

$$\sum_{n \le x} f * \mathbf{1}(n) = \sqrt[a]{x} \zeta\left(\frac{1}{a}\right) + x\zeta(a) + O\left(x^{\frac{1}{a+1}}\right). \tag{7}$$

Furthermore, by Lemma 2(a) we have

$$x \sum_{d \le \sqrt[a]{x}} \frac{1}{d^a} = \frac{\sqrt[a]{x}}{1-a} + x\zeta(a) + O(1). \tag{8}$$

Inserting (7) and (8) into (6) yields

$$\sum_{k \le \sqrt[a]{x}} \left\{ \frac{x}{k^a} \right\} = \sqrt[a]{x} \left(\frac{1}{1-a} - \zeta\left(\frac{1}{a}\right) \right) + O\left(x^{\frac{1}{a+1}}\right).$$

Finally we use

$$\frac{1}{1-a} - \zeta\left(\frac{1}{a}\right) = \frac{1}{1-a} - \lim_{x \to \infty} \left(\sum_{n \le x} \frac{1}{n^{1/a}} - \int_1^x \frac{\mathrm{d}t}{t^{1/a}} \right) + \frac{1}{1-1/a} = 1 - \gamma_{1/a}$$

to obtain the desired result. □

3 Concluding remarks

In this note we have presented a short proof of a century-old result due to de la Vallée Poussin. The technique used allows us to prove further similar results such as Theorem 1. Motivated by the presented relations one may state the following open question:

For which functions f is it true that $\sum_{d\le x} f(d)\,\{x/d\} \sim c_f \sum_{d\le x} f(d)$ with some proportionality constant c_f? Here $g(x)\sim h(x)$ means that $\lim_{x\to\infty} g(x)/h(x)=1$.

For example, with our approach it can be easily shown that for reals $\alpha>0$ we have $\sum_{d\le x} d^{\alpha}\,\{x/d\} \sim x^{\alpha+1}(1-\gamma_{\alpha+1})/(\alpha+1)$.

Furthermore, we remark that (1) remains true if we sum over all divisors that are prime powers instead of primes. More precisely, we have

$$\sum_{p^{\nu}\le x}\left\{\frac{x}{p^{\nu}}\right\} \sim \frac{x}{\log x}(1-\gamma), \tag{9}$$

where the sum is over all $(p,\nu)\in\mathbb{P}\times\mathbb{N}$ such that $p^{\nu}\le x$.

This can be seen as follows: splitting the sum over all prime powers $p^{\nu}\le x$ into a sum over all primes $p\le x$ and a sum over all prime powers $p^{\nu}\le x$ with $\nu\ge 2$ we obtain

$$\sum_{p^{\nu}\le x}\left\{\frac{x}{p^{\nu}}\right\} = \sum_{p\le x}\left\{\frac{x}{p}\right\} + \sum_{\substack{p^{\nu}\le x\\ \nu\ge 2}}\left\{\frac{x}{p^{\nu}}\right\} = \frac{x}{\log x}(1-\gamma) + o\left(\frac{x}{\log x}\right) + O\left(\sum_{\substack{p^{\nu}\le x\\ \nu\ge 2}} 1\right),$$

where for the second equality we used (1) and the fact that $0\le\{x\}\le 1$. Now we have

$$\sum_{\substack{p^{\nu}\le x\\ \nu\ge 2}} 1 \le \sum_{\substack{p^{\nu}\le x\\ \nu\ge 2}} \frac{\log p}{\log 2} = \sum_{p^{\nu}\le x}\frac{\log p}{\log 2} - \sum_{p\le x}\frac{\log p}{\log 2} = O(x^{1/2}(\log x)^2) = o\left(\frac{x}{\log x}\right)$$

by [1, Theorem 4.1] and the result (9) follows. □

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