IDEAL CLONES: SOLUTION TO A PROBLEM OF CZÉDLI AND HEINDORF

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ABSTRACT. Given an infinite set X and an ideal I of subsets of X, the set of all finitary operations on X which map all (powers of) I-small sets to I-small sets is a clone. In [CH01], G. Czédli and L. Heindorf asked whether or not for two particular ideals I and J on a countably infinite set X, the corresponding ideal clones were a cover in the lattice of clones. We give an affirmative answer to this question.

1. CLONES AND IDEALS

Let X be an infinite set and denote the set of all n-ary operations on X by $\mathscr{O}^{(n)}$. Then $\mathscr{O} := \bigcup_{n \geq 1} \mathscr{O}^{(n)}$ is the set of all finitary operations on X. A subset $\mathscr{C}$ of $\mathscr{O}$ is called a *clone* iff it contains all projections, i.e. for all $1 \leq k \leq n$ the function $\pi_k^n \in \mathscr{O}^{(n)}$ satisfying $\pi_k^n(x_1, \ldots, x_n) = x_k$, and is closed under composition. The set of all clones on X, ordered by set-theoretical inclusion, forms a complete algebraic lattice $\mathrm{Cl}(X)$. The structure of this lattice has been subject to much investigation, many results of which are summarized in the recent survey [GP].

One such result, from [Ros76], states that there exist as many dual atoms ("*precomplete clones*") in $\mathrm{Cl}(X)$ as there are clones (that is, $2^{2^{|X|}}$), suggesting that it is impossible to describe all of them (as opposed to the clone lattice on finite X, where the dual atoms are finite in number and explicitly known [Ros70]). Much more recently, a new and short proof of this fact was given in [GS02]. It was observed that given an ideal I of subsets of X, one can associate with it a clone $\mathscr{C}_I$ consisting of those operations $f \in \mathscr{O}$ which satisfy $f[A^n] \in I$ for all $A \in I$. The authors then showed that prime ideals correspond to precomplete clones, and that moreover the clones induced by distinct prime ideals differ, implying that there exist as many precomplete clones as prime ideals on X; the latter are known to amount to $2^{2^{|X|}}$.

The study of clones that arise in this way from ideals was pursued in [CH01], for countably infinite X. The authors concentrated on the question of which ideals induce precomplete clones, and obtained a criterion for precompleteness. In the same paper, three open problems were posed, and we provide the solution to their second problem in this article.

We mention that in the article [BGHP] which is still in preparation, a theory of clones which arise from ideals is being developed. In particular, that paper contains a solution to the first problem from [CH01], which asked whether every ideal clone could be extended to a precomplete ideal clone. Its

third question, which asks whether every ideal clone is covered by another ideal clone, is still unsolved.

2. THE QUESTION AND ITS ANSWER

The problem from [CH01] we are going to solve concerns two particular ideals I, J on $X = \omega \times \omega$: We call sets of the form $\omega \times \{n\}$, where $n \in \omega$, *lines*, and sets of the form $\{n\} \times \omega$ *rows*. J is the ideal of those subsets of X whose intersection with every line is finite. The *width* of a subset Y of X is defined by $\sup\{|Y \cap (\omega \times \{n\})| : n \in \omega\}$. I is the ideal of those sets which have a finite width. Clearly, $I \subseteq J$.

It is easy to see that distinct proper ideals containing all finite subsets of the base set yield distinct clones, so $\mathscr{C}_I \neq \mathscr{C}_J$. In general, the mapping $K \mapsto \mathscr{C}_K$ is not monotone; in this case, however, we have $\mathscr{C}_I \subsetneq \mathscr{C}_J$. This was shown in [CH01] in order to provide the first example of an ideal (namely, I) which does not induce a precomplete clone. (It also follows easily from the more recent results in [BGHP], since J is the *regularization* of I.)

The second problem posed in [CH01] was

Problem. *Is $\mathscr{C}_J$ a cover of $\mathscr{C}_I$ in $\mathrm{Cl}(\omega \times \omega)$? That is, is the interval $(\mathscr{C}_I, \mathscr{C}_J)$ of $\mathrm{Cl}(\omega \times \omega)$ empty?*

We will now prove that the answer is

Answer. *Yes.*

We remark that our original motivation for working on this problem was the fact that a negative answer would have yielded a negative answer to the more general third problem from [CH01], asking whether every ideal clone is covered by an ideal clone. This is because one can show, and quite easily so with the methods from [BGHP], that if $\mathscr{C}_I$ has a cover which is an ideal clone, then this cover must be equal to $\mathscr{C}_J$.

3. THE PROOF

In order to prove that $\mathscr{C}_J$ covers $\mathscr{C}_I$, we must show that if we are given any $g \in \mathscr{C}_J$ and any $f \in \mathscr{C}_J \setminus \mathscr{C}_I$, then g can be written as a term over the set $\{f\} \cup \mathscr{C}_I$. We divide our proof into two parts: In the first part, we show that the m-ary function g can be decomposed as $g = g'(h_1, \ldots, h_m)$, where the h_i are m-ary operations in $\mathscr{C}_I$, and the m-ary operation $g' \in \mathscr{C}_J$ is what we will call *hereditarily thrifty*; this will be achieved in Corollary 16. In the second part, we show that every hereditarily thrifty operation in $\mathscr{C}_J$, so in particular g', can be written as a term over $\{f\} \cup \mathscr{C}_I$ (Lemma 19).

Before we go into Part 1, we fix some notation and give the definition of a hereditarily thrifty function.

From now on, we write $X = \omega \times \omega$. For $d \in X$, we write $d = (d^x | d^y)$ and refer to d^x as the *x-coordinate* and to d^y as the *y-coordinate* of d. We set $M = \{1, \ldots, m\}$ for all $m \geq 1$. An m-ary function on X maps the M-tuples of X^M into X.

For notational reasons we will often consider partial functions from X^M into a set Y, that is, functions into Y whose domain is a subset of X^M. We refer by $\mathrm{dom}(p)$ to the domain and by $\mathrm{ran}(p)$ to the range of a partial function p.

Definition 1. Let $\mathscr{C}$ be a clone. For a partial function p we write $p \in \mathscr{C}$ iff there is a total function $p' \in \mathscr{C}$ extending p.

If $p : X^M \to X^M$ is a partial function, then we write $p \in \mathscr{C}$ iff each of the k component functions $\pi_k^m \circ p$ of p is in $\mathscr{C}$.

Definition 2. For any partial function $p : X^M \to X$, let $\bar{p}$ be defined as the extension of p to X^M obtained by setting $\bar{p}(x) = (0|0)$ for all $x \in X^M \setminus \mathrm{dom}(p)$.

The following lemma justifies the use of partial functions in our proof, since we can always extend them to total functions respecting I.

Lemma 3. *For all partial functions $p : X^M \to X$ we have $p \in \mathscr{C}_I$ iff $\bar{p} \in \mathscr{C}_I$.*

Proof. Obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Definition 4. For every $m \geq 1$ and every $k \in \omega$, we define a subset $B_k^M \subseteq X^M$ to consist of all tuples of X^M which have at least one component whose y-coordinate is less than k. Formally, if for all $i \in M$ and all M-tuples $u \in X^M$ we write u_i for the i-th component of u,

$$B_k^M := \{u \in X^M : \exists i \in M \, ((u_i)^y < k)\}.$$

Definition 5. We call a subset of X^M *bounded* iff it is contained in B_k^M for some $k \in \omega$. A partial function $p : X^M \to Y$ is *thrifty* iff $p^{-1}[d]$ is bounded for all $d \in Y$.

Observe that when we talk about subsets of X having *finite width*, we want to restrict the possible *x-coordinates* for each y to a small set (of *size* $< k$); for a 1-dimensional *bounded* subset of X, on the other hand, we restrict the possible *y-coordinates* for each x to a small set (with *maximum* $< k$).

We will now give the definition of a hereditarily thrifty function. Loosely speaking, a function is hereditarily thrifty iff it is thrifty and whenever we fix some of its arguments to fixed values in X, then the function of smaller arity obtained this way is thrifty as well. The formal definition is as follows:

Definition 6. Let M be a finite index set, and let $S \subseteq M$. Set $T := M \setminus S$. An S-tuple is a total function from S to X. The M-tuples are exactly the unions of S-tuples with T-tuples. (Since $X^S \times X^T$ is naturally isomorphic with X^M through the map $(c, z) \mapsto c \cup z$, we may occasionally identify $X^S \times X^T$ with X^M, or the tuple $c \cup z$ with the ordered pair (c, z).)

For any partial function $p : X^M \to Y$, we write $p_{\cup c}$ for the function $p_{\cup c} : X^T \to Y$ defined by $p_{\cup c}(z) = p(z \cup c)$.

Definition 7. Let $p : X^M \to Y$ be a partial function. We call p *hereditarily thrifty* iff $p_{\cup c}$ is thrifty for all $S \subseteq M$ and all tuples $c \in X^S$.

Part 1: Decomposing $\mathscr{C}_J$–functions into hereditarily thrifty functions and $\mathscr{C}_I$–functions. In this part, we show that given any function $g : X^M \to Y$, we can find a function $h : X^M \to X^M$, $h \in \mathscr{C}_I$, and a hereditarily thrifty $g' \subseteq g$ such that $g = g' \circ h$ (Corollary 16). Clearly, if $Y = X$ and $g \in \mathscr{C}_J$, then also $g' \in \mathscr{C}_J$; thus, we can decompose the $\mathscr{C}_J$–function g into

the composition of a hereditarily thrifty $\mathscr{C}_J$–function g' with a $\mathscr{C}_I$–function h.

We first state some trivial facts about the composition of functions. In order to avoid confusion with the components of elements of X, we use the symbol $\mathbb{N}$ (rather than ω) to denote the index set of countable families of functions.

Fact 8. *Assume that f_n, $n \in \mathbb{N}$ are functions with pairwise disjoint domains, and similarly g_n, $n \in \mathbb{N}$. Then:*

- *$\bigcup_n f_n$ and $\bigcup_n g_n$ are functions.*
- *The functions $f_n \circ g_n$ (for $n \in \mathbb{N}$) have pairwise disjoint domains, and hence their union is a function.*
- *$\bigcup_n \left(f_n \circ g_n \right) \subseteq \left(\bigcup_n f_n \right) \circ \left(\bigcup_n g_n \right)$.*

Fact 9. *If $g \subseteq g' \circ h'$, then there is a function $h \subseteq h'$ such that $g = g' \circ h$.*

Definition 10. We call a partial function $p : X^M \to Y$ *wasteful* iff for every $y \in \mathrm{ran}(p)$ the set $p^{-1}[y]$ is unbounded.

Lemma 11. *For every partial function $p : X^M \to Y$ we can find disjoint sets $W_p, T_p \subseteq \mathrm{dom}(p)$ such that $\mathrm{dom}(p) = W_p \cup T_p$, and $p{\restriction}W_p$ is wasteful and $p{\restriction}T_p$ is thrifty.*

Proof. Easy.   □

Lemma 12. *Let p_1, p_2 be partial functions on X^M with disjoint domains. Then $p_1 \cup p_2 \in \mathscr{C}_I$ iff $p_1 \in \mathscr{C}_I$ and $p_2 \in \mathscr{C}_I$. Moreover, $p_1 \cup p_2$ is thrifty iff both p_1 and p_2 are thrifty.*

Proof. This is clear since I is closed under finite unions.   □

Recall that we defined the width of a subset $A \subseteq X$ as $\sup\{|A \cap (\omega \times \{n\})| : n \in \omega\}$. We extend this definition to subsets of X^M.

Definition 13. The *width* of a set $A \subseteq X^M$ is the maximum of the widths of its projections onto the components of M.

Lemma 14. *Let $g_n : X^M \to Y$, $n \in \mathbb{N}$, be wasteful partial functions. Then we can find*

- *a set $A \subseteq X^M$ of width 1*
- *a family of injective functions $g'_n \subseteq g_n$, with A being the disjoint union of the sets $\mathrm{dom}(g'_n)$*
- *partial functions $h_n : X^M \to A$*

such that $g_n = g'_n \circ h_n$.

Note that the g'_n are thrifty (as they are injective), and that the h_n are in $\mathscr{C}_I$ (as A has width 1).

Proof. For all $n \in \mathbb{N}$ and every $c \in \mathrm{ran}(g_n)$, pick a tuple $u^{c,n} \in X^M$ in such a way that for distinct $(d,i),(e,j) \in Y \times \mathbb{N}$, the components of the corresponding tuples $u^{d,i}, u^{e,j}$ have no y-coordinates in common; this is possible as $g_n^{-1}[c]$ is unbounded for all $n \in \mathbb{N}$ and all $c \in \mathrm{ran}(g_n)$. Set A_n to consist of all tuples chosen for g_n, and let $A = \bigcup_n A_n$. Let g'_n be the restriction of

g_n to A_n. Clearly, A has width 1 and the g'_n are injective on their respective domains A_n. The existence of the h_n is then trivial. $\qquad\square$

The core of Part 1 of our proof is the following lemma.

Lemma 15. *Let $g : X^M \to Y$ be a partial function and fix a set $S \subseteq M$. Then we can write g as $g = g' \circ h$, where $g' \subseteq g$, $g'_{\cup c}$ is thrifty for each $c \in X^S$, and $h : X^M \to X^M$ is in $\mathscr{C}_I$.*

Before we prove the lemma, we observe that it implies all we need in this section.

Corollary 16. *Let $g : X^M \to Y$ be a partial function. Then we can write g as $g = g' \circ h$, where $g' \subseteq g$, g' is hereditarily thrifty, and h is in $\mathscr{C}_I$.*

Proof of the corollary. We enumerate all subsets S of M as $S_1, \ldots, S_k$. Applying Lemma 15 repeatedly, we inductively define a sequence $g = g^0 \supseteq g^1 \supseteq \cdots \supseteq g^k$ such that:

- $g^i_{\cup c}$ is thrifty for each $c \in X^{S_i}$
- $g^{i-1} = g^i \circ h^i$ for some function $h^i : X^M \to X^M$ in $\mathscr{C}_I$.

We have $g = g^0 = g^1 \circ h^1 = g^2 \circ h^2 \circ h^1 = \cdots = g^k \circ (h^k \circ \cdots \circ h^1)$. As $g' := g^k \subseteq g^i$ for all i, we see that g' is hereditarily thrifty. Clearly, $h := h^k \circ \cdots \circ h^1$ is in $\mathscr{C}_I$. $\qquad\square$

Definition 17. Let $S \subseteq M$, $T := M \setminus S$, and $c \in X^S$.

- For any set $A \subseteq X^T$ we write $c * A$ for the set $\{c \cup z : z \in A\} \subseteq X^M$.
- For any partial function $g : X^T \to Y$ we write $c * g$ for the function with domain $c * \mathrm{dom}(g)$ defined by $(c * g)(c \cup z) = g(z)$.
- If $Y = X^T$, then we write $c\#g$ for the partial function from X^M to X^M mapping each $c \cup z \in c * \mathrm{dom}(g)$ to $c \cup g(z)$.

The following lemma shows how to calculate with the operators just defined; we leave its straightforward verification to the reader.

Lemma 18. *Let M, S, T, and c as in the preceding definition.*

(1) *For all partial functions $f : X^T \to Y$ and all partial $g : X^T \to X^T$, $c * (f \circ g) = (c * f) \circ (c\#g)$.*

(2) *For every partial function $g : X^T \to Y$, $(c * g)_{\cup c} = g$.*

(3) *For every partial function $g : X^M \to Y$, $g = \bigcup_{c \in X^S} c * (g_{\cup c})$.*
   Moreover: Whenever $(g_c : c \in X^S)$ is a family of partial functions from X^T to Y with $g = \bigcup_{c \in X^S} c * g_c$, then we must have $g_c = g_{\cup c}$ for all $c \in X^S$.

We are now ready to finish Part 1 and provide the proof of its main lemma.

Proof of Lemma 15. Using Lemma 18, we split g into its "components" $g_{\cup c} : X^T \to Y$, and use Lemma 14 to deal with those functions; that is, we write $g = \bigcup_{c \in X^S} c * g_{\cup c}$. By Lemma 11, each function $g_{\cup c}$ can be written as $t_c \cup w_c$, where t_c is thrifty, w_c is wasteful, and t_c and w_c have disjoint domains.

By Lemma 14, we can find a set A of width 1, injective functions $w'_c \subseteq w_c$ whose domains are disjoint subsets of A, and partial functions h_c with

$\operatorname{ran}(h_c) \subseteq A$, such that $w_c = w'_c \circ h_c$, for all $c \in X^S$. We write t_c as the composition $t_c = t_c \circ i_c$, where i_c is the identity function on $\operatorname{dom}(t_c) \subseteq X^T$.

Now the domains of i_c and h_c, as well as those of t_c and w'_c, are disjoint, so Fact 8 implies that for all $c \in X^S$,

$$g_{\cup c} = t_c \cup w_c = (t_c \circ i_c) \cup (w'_c \circ h_c) \subseteq (t_c \cup w'_c) \circ (i_c \cup h_c).$$

Therefore, by Lemma 18,

$$c * g_{\cup c} \subseteq c * [(t_c \cup w'_c) \circ (i_c \cup h_c)] = (c * (t_c \cup w'_c)) \circ (c\#(i_c \cup h_c)).$$

This, together with Fact 8, allows us to calculate:

$$g = \bigcup_{c \in X^S} c * g_{\cup c} \subseteq \bigcup_{c \in X^S} (c * (t_c \cup w'_c)) \circ (c\#(i_c \cup h_c)) \subseteq$$

$$\subseteq \left(\bigcup_{c \in X^S} (c * (t_c \cup w'_c)) \right) \circ \left(\bigcup_{c \in X^S} c\#(i_c \cup h_c) \right) =: g' \circ h'.$$

Now for all $c \in X^S$, $g'_{\cup c} = t_c \cup w'_c$ is thrifty, since t_c is thrifty and since w'_c is injective (and hence thrifty), and by Lemma 12. Since by Fact 9 there is $h \subseteq h'$ such that $g = g' \circ h$, it remains to see that h' is in $\mathscr{C}_I$. A quick check of the definitions shows that h' is the union of $\bigcup_{c \in X^S} c\#i_c$ with $\bigcup_{c \in X^S} c\#h_c$; by Lemma 12, it suffices to check that each of these unions is in $\mathscr{C}_I$. All components of the first union are projections, thus certainly in $\mathscr{C}_I$. For the second union, those components of with index in S are just projections as well; the components with index in T, on the other hand, have range in a projection of A, and hence are elements of $\mathscr{C}_I$ as well. □

Part 2: Generating hereditarily thrifty functions. In this section, we fix an arbitrary $f \in \mathscr{C}_J \setminus \mathscr{C}_I$ and prove the following:

Lemma 19. *Let $q : X^M \to X$ be a hereditarily thrifty partial function in $\mathscr{C}_J$. Then q can be written as a term of f and partial functions in $\mathscr{C}_I$.*

Thus, given an arbitrary $g \in \mathscr{C}_J$, we can use Corollary 16 to decompose it as $g = q \circ h$, and by Lemma 19, we can write q as a term of f and partial $\mathscr{C}_I$–functions. By Lemma 3, we can extend each of the partial $\mathscr{C}_I$–functions in this term to total functions in $\mathscr{C}_I$, which yields a representation of g as a term over $\{f\} \cup \mathscr{C}_I$, finishing our proof.

In order to prove the lemma, we first show that we can assume f to be unary.

Lemma 20. *$\{f\} \cup \mathscr{C}_I$ generates a unary operation in $\mathscr{C}_J \setminus \mathscr{C}_I$.*

Proof. For a set of unary operations $\mathscr{M} \subseteq \mathscr{O}^{(1)}$, set

$$\operatorname{Pol}(\mathscr{M}) := \{g \in \mathscr{O} : g(g_1, \ldots, g_n) \in \mathscr{M} \text{ for all } g_1, \ldots, g_n \in \mathscr{M}\}.$$

It is a fact (see [BGHP]) and easy to see that for any ideal K, $\mathscr{C}_K = \operatorname{Pol}(\mathscr{C}_K \cap \mathscr{O}^{(1)})$. Hence, since $f \notin \mathscr{C}_I$, there exist unary $g_1, \ldots, g_n \in \mathscr{C}_I$ such that $f(g_1, \ldots, g_n) \notin \mathscr{C}_I$. As $f, g_1, \ldots, g_n$ are elements of $\mathscr{C}_J$, so is $f(g_1, \ldots, g_n)$. Hence, the latter operation witnesses our assertion. □

Referring to the preceding lemma, we assume without loss of generality that f is unary itself. Hence, there is a set $A \subseteq X$ of width 1 that f maps onto a set in $J \setminus I$. Since every function which maps every line $\omega \times \{n\}$ into itself is an element of $\mathscr{C}_I$, we can simplify notation by assuming that all elements of A have x-coordinate 0.

Fix an injective map $(n, k) \mapsto n \oplus k$ from $\{(n, k) : k < n \in \omega\}$ into a co-infinite subset of ω (for example $n \oplus k = n^2 + k$). By further permuting of lines and rows we may without loss of generality assume that

$$f((0|n \oplus j)) = (j|n) \tag{*}$$

for all $j < n < \omega$.

For any finite index set $M = \{1, \ldots, m\}$, define

$$P^*(M) = \{(S, j) : S \subseteq M, j \in M \setminus S\}.$$

We will show that if $q : X^M \to X$ is hereditarily thrifty in and in $\mathscr{C}_J$, then there are partial $\mathscr{C}_I$–functions $Q : X^M \times X^{P^*(M)} \to X$ and $h^{S,j} : X^M \to X$, $(S, j) \in P^*(M)$, such that

$$q(u) = Q\bigg(u, \big(f(h^{S,j}(u)) : (S, j) \in P^*(M)\big)\bigg).$$

Clearly, this is sufficient for the proof of Lemma 19. We start by defining the $h^{S,j}$; to do this, we need the following lemma.

Lemma 21. *Let $t : X^M \to X$ be a partial function which is thrifty and in $\mathscr{C}_J$. Then $t^{-1}[\omega \times \{n\}]$ is bounded for all $n \in \omega$.*

Proof. Suppose this is not the case, and write $t^{-1}[\omega \times \{n\}] = \{c^0, c^1, \ldots\}$. Since this set is unbounded, we can find an infinite $U \subseteq \mathbb{N}$ such that the sequences $((c^i_j)^y : i \in U)$ are injective for all $j \in M$ (where c^i_j is the j-th component of the M-tuple c^i). Now given some value $d \in \omega \times \{n\}$, only finitely many of the points $\{c^i : i \in U\}$ can be mapped to d, as t is thrifty. Hence, the set $\{c^i : i \in U\} \in I^M$ is mapped by t to an infinite subset of $\omega \times \{n\}$ and hence to a set outside J, a contradiction. $\square$

Lemma 21 allows us to make the following definition.

Definition 22. For any thrifty partial $\mathscr{C}_J$–function $t : X^M \to X$ define a function $K_t : \omega \to \omega$ by

$$K_t(n) = \min\{k : t^{-1}(\omega \times \{n\}) \subseteq B_k^M\}.$$

Definition 23. Let $q : X^M \to X$ be a hereditarily thrifty partial $\mathscr{C}_J$–function, and $(S, j) \in P^*(M)$. Write $T := M \setminus S$; then each tuple $u \in X^M$ can be written as $u = c \cup z$, $c \in X^S$, $z \in X^T$. Write z_j for the j-th component of any such z.

We define $h^{S,j} : X^M \to X$ as follows. For any $c \in X^S$, $z \in X^T$ we let

$$h^{S,j}(c \cup z) = \begin{cases} \big(0|K_{q_{\cup c}}(q(u)^y) \oplus z_j^y\big) & \text{if } z_j^y < K_{q_{\cup c}}(q(u)^y) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Observe that since the range of $h^{S,j}$ has width 1, it is an element of $\mathscr{C}_I$. We turn to the definition of Q.

Definition 24. For any $q : X^M \to Y$, we define a partial function $Q : X^M \times X^{P^*(M)} \to Y$: For any $u = (u_i : i \in M)$ and any $v = (v_{S,j} : (S,j) \in P^*(M))$ we let

$$Q(u,v) = \begin{cases} q(u), & \text{if } v_{S,j} = f(h^{S,j}(u)) \text{ for all } (S,j) \in P^*(M) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Clearly, $q(u) = Q\Big(u, \big(f(h^{S,j}(u)) : (S,j) \in P^*(M)\big)\Big)$. Therefore, to obtain a proof of Lemma 19, it suffices to show that if q in $\mathscr{C}_J$, then $Q \in \mathscr{C}_I$. This will be the direct consequence of the following final lemma.

Lemma 25. *Assume that $(A_i : i \in M)$ and $(A_{S,j} : (S,j) \in P^*(M))$ are sets of width 1. Let $A \subseteq X^M \times X^{P^*(M)}$ be the product of those sets. Then $Q[A]$ is a set of width $\leq m!$.*

Observe that if $E \in I$, then it has finite width, so it can be written as a finite union $E = \bigcup_{i \in N} E_i$ of sets of width 1. Set $R = M \times P^*(M)$. Now $E^R = \bigcup_{r \in N^R} \prod_{j \in R} E_{r(j)}$. Since every of the products in the big union is mapped by Q to a set of width $\leq m!$, the whole union is mapped to a set of finite width. Hence, $Q \in \mathscr{C}_I$.

Proof of Lemma 25. We show:

> For all $n \in \omega$, and all permutations $\pi : M \to M$ there is at most one tuple $(u,v) \in A$ with $Q(u,v) \in \omega \times \{n\}$ such that $u^y_{\pi(1)} \leq \cdots \leq u^y_{\pi(m)}$.

This clearly implies the assertion. For notational simplicity (but without loss of generality, since the definition of Q did not refer to any order of the indices) we will prove this only for the special case where π is the identity; that is, we show that there is at most one tuple $(u,v) \in A$ with $u^y_1 \leq \cdots \leq u^y_m$ and $Q(u,v) \in \omega \times \{n\}$.

By assumption, all factors $B \subseteq X$ of the product set A have width 1, meaning that their intersection with every line contains at most one element; by replacing these factors by supersets (still of width 1), we may assume that these intersections are never empty. For every factor B, we write $B\langle n \rangle$ for the unique k with $(k|n) \in B$.

We now define inductively tuples $c^0, \ldots, c^m$ such that $c^j \in X^{\{1,\ldots,j\}}$ for all $j \leq m$ and such that for all $1 \leq j \leq m$, c^j is the extension of c^{j-1} by one coordinate.

Let $c^0 \in X^\emptyset$ be the empty tuple. To continue, set

$$k_1 := K_q(n), \quad b_1 := A_{\emptyset,1}\langle k_1 \rangle, \quad a_1 := A_1\langle b_1 \rangle,$$

and let c^1 be the tuple mapping 1 to $(a_1|b_1)$. For $j \geq 1$, having already defined the tuple $c^{j-1} \in X^{\{1,\ldots,j-1\}}$, we set

$$k_j := K_{q_{\cup c^{j-1}}}(n), \quad b_j := A_{\{1,\ldots,j-1\},j}\langle k_j \rangle, \quad a_j := A_j\langle b_j \rangle,$$

and let c^j be the tuple extending c^{j-1} which maps j to $(a_j|b_j)$. After m steps, we arrive at $c^m = ((a_1|b_1), \ldots, (a_m|b_m))$.

We claim that for all $(u,v) \in A$ with $u^y_1 \leq \cdots \leq u^y_m$, if $Q(u,v) \in \omega \times \{n\}$ then $u_j = (a_j|b_j)$ for all $1 \leq j \leq m$. This will finish the proof since if these

hypotheses uniquely determine u, then they also determine v, as is obvious from the definition of Q.

To see the truth of our final claim, take any tuple (u, v) satisfying the hypotheses; so $Q(u, v) = q(u) \in \omega \times \{n\}$.

We start by showing $u_1 = (a_1|b_1)$. Since $q^{-1}[\omega \times \{n\}] \subseteq B^M_{k_1}$, there is some $i \in M$ such that $u^y_i < k_1$; by our assumption $u^y_1 \leq \cdots \leq u^y_m$, this implies $u^y_1 < k_1$. Therefore, the definition of $h^{\emptyset,1}$ yields $h^{\emptyset,1}(u) = (0|K_{q_{\cup\emptyset}}(q(u)^y) \oplus u^y_1) = (0|K_q(n) \oplus u^y_1) = (0|k_1 \oplus u^y_1)$. Consequently, by our assumption $(*)$ on f, $f(h^{\emptyset,1}(u)) = (u^y_1|k_1)$. Since $Q(u, v)$ is defined, we must have $v_{\emptyset,1} = f(h^{\emptyset,1}(u)) = (u^y_1|k_1)$. But $v_{\emptyset,1} \in A_{\emptyset,1}$, hence $u^y_1 = A_{\emptyset,1}\langle k_1 \rangle$. As $u_1 = (u^x_1|u^y_1) \in A_1$ we must also have $u^x_1 = A_1\langle u^y_1 \rangle$. Hence, $u_1 = (a_1|b_1)$.

We now show that $u_2 = (a_2|b_2)$. To this end, we consider the function $q_{\cup c^1}$, a function from $X^{\{2,\ldots,m\}}$ into X. We have $q_{\cup c^1}(u_2, \ldots, u_n) \in \omega \times \{n\}$. Therefore, $q^{-1}_{\cup c^1}[\omega \times \{n\}] \subseteq B^{\{2,\ldots,m\}}_{k_2}$, so we must have $u^y_2 < k_2$. As above we conclude (in this order):

- $h^{\{1\},2}(u) = (0|k_2 \oplus u^y_2)$
- $f(h^{\{1\},2}(u)) = (u^y_2|k_2)$
- $v_{\{1\},2} = f(h^{\{1\},2}(u)) = (u^y_2|k_2) \in A_{\{1\},2}$
- $u^y_2 = A_{\{1\},2}\langle k_2 \rangle$
- $u^x_2 = A_2\langle u^y_2 \rangle$
- So $u_2 = (u^x_2|u^y_2) = (A_2\langle u^y_2 \rangle|A_{\{1\},2}\langle k_2 \rangle) = (a_2|b_2)$.

Continuing inductively in this fashion, one sees that indeed, $u_j = (a_j|b_j)$ for all $1 \leq j \leq m$. $\qquad\square$

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