MONOIDAL INTERVALS OF CLONES ON INFINITE SETS

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Abstract. Let X be an infinite set of cardinality κ. We show that if $\mathfrak{L}$ is an algebraic and dually algebraic distributive lattice with at most 2^κ completely join irreducibles, then there exists a monoidal interval in the clone lattice on X which is isomorphic to the lattice $\mathbf{1}+\mathfrak{L}$ obtained by adding a new smallest element to $\mathfrak{L}$. In particular, we find that if $\mathfrak{L}$ is any chain which is an algebraic lattice, and if $\mathfrak{L}$ does not have more than 2^κ completely join irreducibles, then $\mathbf{1}+\mathfrak{L}$ appears as a monoidal interval; also, if $\lambda \leq 2^\kappa$, then the power set of λ with an additional smallest element is a monoidal interval. Concerning cardinalities of monoidal intervals these results imply that there are monoidal intervals of all cardinalities not greater than 2^κ, as well as monoidal intervals of cardinality 2^λ, for all $\lambda \leq 2^\kappa$.

1. The problem

Let X be a set of cardinality κ, and for all $n \geq 1$ denote the set of n-ary operations on X by $\mathscr{O}^{(n)}$. Then $\mathscr{O} = \bigcup_{n\geq 1} \mathscr{O}^{(n)}$ is the set of all finitary operations on X. A set of operations $\mathscr{C} \subseteq \mathscr{O}$ is called a *clone* iff it is closed under composition and contains all projections, that is, all functions of the form $\pi^n_k(x_1, \ldots, x_n) = x_k$ $(1 \leq k \leq n)$. The set of all clones on X equipped with the order of set-theoretical inclusion forms a complete algebraic lattice $\mathrm{Cl}(X)$ called the *clone lattice (on X)*. After this introductory section, we are going to work exclusively with an infinite base set X, in which case the cardinality of $\mathrm{Cl}(X)$ is 2^{2^κ}. For finite X with at least three elements we have $|\mathrm{Cl}(X)| = 2^{\aleph_0}$, and $|\mathrm{Cl}(X)| = \aleph_0$ if the base set has two elements. Only in the last case has the structure of the clone lattice been completely resolved [Pos41]. If X has at least three elements, then $\mathrm{Cl}(X)$ seems to be too large and complicated to be fully understood. In particular, it has been shown recently that for infinite X, $\mathrm{Cl}(X)$ contains all algebraic lattices which have at most 2^κ compact elements as complete sublattices [Pin]. We

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refer the reader to [GP] for a survey of clones on infinite sets, and to [Sze86] for clones on finite sets.

One approach to the problem of describing the clone lattice is to partition $\mathrm{Cl}(X)$ into so-called monoidal intervals. Let $\mathscr{G}$ be a submonoid of the monoid of unary operations $\mathscr{O}^{(1)}$. The set of all clones $\mathscr{C}$ with unary part $\mathscr{G}$ (that is, with $\mathscr{C}^{(1)} = \mathscr{G}$, where $\mathscr{C}^{(1)} = \mathscr{C} \cap \mathscr{O}^{(1)}$) forms an interval $\mathscr{I}_{\mathscr{G}}$ of the clone lattice; such intervals are referred to as *monoidal*. The smallest element of $\mathscr{I}_{\mathscr{G}}$ is obviously $\langle \mathscr{G} \rangle$, the clone generated by $\mathscr{G}$ which in this case consists of all essentially unary functions (i.e. functions depending on at most one variable) whose corresponding unary function is an element of $\mathscr{G}$. The largest element of $\mathscr{I}_{\mathscr{G}}$ is easily seen to be $\mathrm{Pol}(\mathscr{G})$, defined to contain precisely those functions $f \in \mathscr{O}$ for which $f(g_1, \ldots, g_{n_f}) \in \mathscr{G}$ whenever $g_1, \ldots, g_{n_f}$ are functions in $\mathscr{G}$. Functions with this property are called *polymorphisms* of $\mathscr{G}$.

We are interested in the structure of monoidal intervals, in particular in the cardinalities monoidal intervals can have; this question was first posed by Szendrei [Sze86]. For dividing the clone lattice into monoidal intervals allows us to approach smaller parts of it; in some sense, this procedure is "orthogonal" to the study of the lattice of monoids, since we fix a monoid and investigate the behaviour of functions of higher arity together with the monoid under consideration. A classification of monoidal intervals, together with insights on the monoid lattice, would help us to understand the clone lattice.

There is a deeper concept behind the partition of the clone lattice into monoidal intervals. If $\mathscr{C}, \mathscr{D} \subseteq \mathscr{O}$ are two distinct clones, then there exists $n \geq 1$ such that $\mathscr{C}^{(n)} \neq \mathscr{D}^{(n)}$, where $\mathscr{C}^{(n)} = \mathscr{C} \cap \mathscr{O}^{(n)}$. Moreover, if this is the case and $m \geq n$, then also $\mathscr{C}^{(m)} \neq \mathscr{D}^{(m)}$. Therefore, we can say that two clones are closer the later their n-ary parts start to differ. More precisely, the function

$$d(\mathscr{C}, \mathscr{D}) = \begin{cases} \frac{1}{2^{n-1}}, & \mathscr{C} \neq \mathscr{D} \wedge n = \min\{k : \mathscr{C}^{(k)} \neq \mathscr{D}^{(k)}\}, \\ 0, & \mathscr{C} = \mathscr{D} \end{cases}$$

defines a metric on the clone lattice, first introduced by Machida [Mac98]. Formulated in this metric, a monoidal interval is just an open sphere of radius 1 in the metric space $(\mathrm{Cl}(X), d)$. It also makes sense to consider refinements of this partition, for example open spheres of radius $\frac{1}{2}$, or equivalently sets of clones with identical binary parts; they are of the form $[\langle \mathscr{H} \rangle, \mathrm{Pol}(\mathscr{H})]$, where $\mathscr{H} \subseteq \mathscr{O}^{(2)}$ is a set of binary functions closed under composition and containing the two binary projections.

For a finite base set X it has been observed by Rosenberg and Sauer [RS] that all intervals are either at most countably infinite or of size continuum. We shall give a short argument proving this: On a finite base set, the clone lattice equipped with Machida's metric is homeomorphic to a closed subset of the Cantor space 2^ω. To see this, notice first that $\mathscr{O}$ is countably infinite, and let $(f_i)_{i \in \omega}$ be an enumeration of $\mathscr{O}$ with the property that for all $i < j$

the arity of f_i is not greater than the arity of f_j; this is possible, since $\mathscr{O}^{(n)}$ is finite for all $n \geq 1$. Now we can assign to every set of operations $\mathscr{C} \subseteq \mathscr{O}$ a sequence $s(\mathscr{C}) \in 2^\omega$ by defining $s(\mathscr{C})(i) = 1$, if $f_i \in \mathscr{C}$, and $s(\mathscr{C})(i) = 0$ otherwise. This gives a bijection from the power set $\mathscr{P}(\mathscr{O})$ of $\mathscr{O}$ onto 2^ω, and if we extend Machida's metric from the clone lattice to $\mathscr{P}(\mathscr{O})$ (with the same definition), this mapping is easily seen to be a homeomorphism. The set of sequences of 2^ω that correspond to clones is a closed subset of 2^ω. To see this, for $i \in \omega$ and $j \in 2$ let A_i^j be a set consisting of all $s \in 2^\omega$ with $s(i) = j$; the A_i^j form a clopen subbasis of the topology of 2^ω. Now the property that $\mathscr{C} \subseteq \mathscr{O}$ contains all projections is equivalent to $s(\mathscr{C})$ being an element of $\Lambda_1 = \bigcap\{A_i^1 : i \in \omega \text{ and } f_i \text{ is a projection}\}$. Moreover, that $\mathscr{C}$ is closed under composition can be stated in the language of sequences by saying that $s(\mathscr{C})$ is an element of

$$\Lambda_2 = \bigcap\{(A_{i_0}^0 \cup \ldots \cup A_{i_n}^0) \cup A_j^1 : f_j = f_{i_0}(f_{i_1}, \ldots, f_{i_n})\}.$$

Thus $\mathscr{C} \subseteq \mathscr{O}$ is a clone iff $s(\mathscr{C})$ is an element of $\Lambda = \Lambda_1 \cap \Lambda_2$, a closed set since both Λ_i are intersections of closed sets and hence closed themselves. Whence, $(\mathrm{Cl}(X), d)$ is indeed homeomorphic to a closed subset of 2^ω, which immediately yields the topological properties of the clone space proven in [Mac98].

Now if $\mathscr{C}_1 \subseteq \mathscr{C}_2 \subseteq \mathscr{O}$, then the interval $[\mathscr{C}_1, \mathscr{C}_2]$ in the power set of $\mathscr{O}$ corresponds to the interval $[s(\mathscr{C}_1), s(\mathscr{C}_2)]$ in 2^ω with the pointwise order, a closed set. Therefore it is a Polish (separable complete metric) space and satisfies the continuum hypothesis by the Cantor-Bendixon theorem ([Kec95, Theorem 6.4]). Also, if $\mathscr{C}_1$ and $\mathscr{C}_2$ are clones, then the interval $[\mathscr{C}_1, \mathscr{C}_2]$ in $\mathrm{Cl}(X)$ corresponds to $[s(\mathscr{C}_1), s(\mathscr{C}_2)] \cap \Lambda$ in 2^ω, again a closed set which as such is either countable or of size continuum. We conclude that all intervals of the clone lattice on a finite set satisfy CH. In particular, monoidal intervals can only be finite, countably infinite, or of size continuum.

The same argument does not work for infinite sets, and we shall prove that on a countably infinite set there exist monoidal intervals of all cardinalities between $\aleph_0$ and $2^{\aleph_0}$.

Of the possible sizes finite, $\aleph_0$, and $2^{\aleph_0}$ for monoidal intervals over a finite set with at least three elements, all possibilities occur: There must be a monoidal interval of size continuum, since there exist only finitely many monoids and $|\mathrm{Cl}(X)| = 2^{\aleph_0}$. Also some finite sizes appear, for example the interval corresponding to the monoid $\mathscr{O}^{(1)}$ is of size $|X|+1$ ([Bur67]), and we will see in this paper that if $|X| \geq 3$, then the group of all permutations on X is an example of a monoid whose monoidal interval has only one element (for infinite X, but the same proof works on finite sets with at least three elements). See [PS83], [Kro95], [Kro97] for more examples. However, for one fixed base set, only finitely many finite numbers appear as sizes of monoidal intervals, again because there exist only finitely many monoids. Krokhin

[Kro97] proved that there exist countably infinite monoidal intervals over a finite set.

Goldstern and Shelah [GS] showed that on a countably infinite base set, many monoids define a monoidal interval which is as large as the clone lattice ($2^{2^{\aleph_0}}$). Starting from this result, we investigated the question whether all monoidal intervals on infinite sets are that large, and found that the situation is much more diverse.

2. Results

For a lattice $\mathfrak{L}$, denote by $\mathbf{1} + \mathfrak{L}$ the lattice obtained by adding a new smallest element to $\mathfrak{L}$. We are going to prove the following

Theorem 1. *Let X be an infinite set of cardinality κ, and let $\mathfrak{L}$ be an algebraic and dually algebraic distributive lattice with at most 2^κ completely join irreducible elements. Then there is a monoidal interval in $\mathrm{Cl}(X)$ which is isomorphic to $\mathbf{1} + \mathfrak{L}$.*

Remark 2. The class of algebraic and dually algebraic distributive lattices is the class of completely distributive lattices, or equivalently the class of lattices of order ideals of partial orders (see e.g. [CD73, p.83] for the latter statement).

As an immediate consequence we obtain

Corollary 3. *Let X be an infinite set of cardinality κ, and let $\lambda \leq 2^\kappa$. Then there is a monoidal interval in $\mathrm{Cl}(X)$ which is isomorphic to $\mathbf{1}+\mathscr{P}(\lambda)$, where $\mathscr{P}(\lambda)$ is the power set of λ.*

Let $\mathfrak{L}$ be a chain which is complete as a lattice. Then $\mathfrak{L}$ is algebraic iff for all elements $p, q \in \mathfrak{L}$ with $p <_{\mathfrak{L}} q$ there exists a covering pair $u \prec_{\mathfrak{L}} v$ in $\mathfrak{L}$ (i.e. $u <_{\mathfrak{L}} v$ and the interval $[u, v]_{\mathfrak{L}}$ contains only u and v) such that $p \leq_{\mathfrak{L}} u \prec_{\mathfrak{L}} v \leq_{\mathfrak{L}} q$ (see for example [BS81, Chapter I, Section 4, Exercise 4]). Moreover, an element $p \in \mathfrak{L}$ is completely join irreducible iff it is the smallest element of $\mathfrak{L}$ or there exists $q \in \mathfrak{L}$ such that $q \prec_{\mathfrak{L}} p$.

Corollary 4. *Let X be an infinite set of cardinality κ, and let $\mathfrak{L}$ be any chain which is a complete algebraic lattice with at most 2^κ completely join irreducibles. Then there is a monoidal interval in $\mathrm{Cl}(X)$ which is isomorphic to $\mathbf{1} + \mathfrak{L}$.*

Corollary 5. *Let X be an infinite set of cardinality κ, and let $1 \leq \mu < (2^\kappa)^+$ be an ordinal (where $(2^\kappa)^+$ is the successor cardinal of 2^κ). Then there is a monoidal interval in $\mathrm{Cl}(X)$ which has the same order as μ.*

Remark 6. Clearly, all intervals of $\mathrm{Cl}(X)$ are algebraic lattices with at most 2^κ compact elements (since a clone is compact in an interval iff it is finitely generated over the smallest element of that interval). Hence, the chains exposed in Corollary 4 are all chains of the form $\mathbf{1} + \mathfrak{L}$, where $\mathfrak{L}$ is a chain with smallest element, which can occur as monoidal intervals. The ordinals of Corollary 5 are all ordinals that can appear as monoidal intervals, since all larger ordinals have more than 2^κ compact elements.

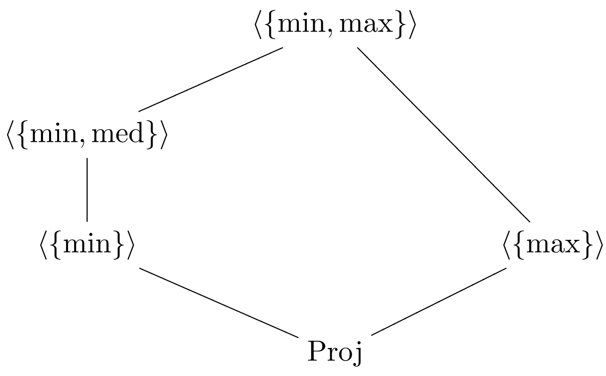
Corollary 7. *On infinite X of size κ, there exist at least monoidal intervals in $\mathrm{Cl}(X)$ of the following cardinalities:*

- *λ for all $\lambda \leq 2^\kappa$.*
- *2^λ for all $\lambda \leq 2^\kappa$.*

The monoidal intervals exposed in our theorem are completely distributive and therefore still quite special lattices. Therefore it is not surprising that they are not all monoidal intervals that can appear.

Proposition 8. *There exists a non-modular monoidal interval.*

Proof. Let $\mathfrak{B} = (X, \vee, \wedge, -, 0, 1)$ be a boolean algebra on X. Then the clone of $\mathfrak{B}$ is isomorphic to the clone of the 2-element Boolean algebra $\mathfrak{B}_2$; the latter clone is, of course, the clone of all operations on the 2-element set. In particular, the subclone lattice of $\mathfrak{B}$ is isomorphic to the lattice of clones on the 2-element set, which is known as Post's lattice and has been completely described [Pos41]. Also, in the isomorphism between the clone of $\mathfrak{B}$ and the operations on the 2-element set, the idempotent terms of $\mathfrak{B}$ correspond exactly to the idempotent operations on the 2-element set. Therefore, the interval of idempotent clones on the 2-element set is isomorphic to the interval of idempotent subclones of $\mathfrak{B}$; the latter lattice is a subinterval of the monoidal interval corresponding to the trivial monoid $\{\pi^1_1\}$. Since it is known that the interval of idempotent clones on the 2-element set is non-modular, we have that $\mathscr{I}_{\{\pi^1_1\}}$ on X is non-modular too. For a concrete example of the non-modularity of this monoidal interval, let X be linearly ordered, and write $\min(x_1, x_2)$ for the minimum function, $\mathrm{med}(x_1, x_2, x_3)$ for the median function, and $\max(x_1, x_2)$ for the maximum function with respect to that linear order. Denote by Proj the clone of projections. Then



is a sublattice of the monoidal interval corresponding to the trivial monoid $\{\pi^1_1\}$. That $\langle\{\min, \mathrm{med}\}\rangle \cap \langle\{\max\}\rangle = \mathrm{Proj}$ follows from [Pin04] but is also not difficult to verify. $\square$

The fact that monoidal intervals must be algebraic lattices with no more than 2^κ compact elements is the only restriction for them we know of. Therefore we pose the following problem.

Problem 9. *If $\mathfrak{L}$ is any algebraic lattice with at most 2^κ compact elements, is there a monoidal interval isomorphic to $\mathfrak{L}$?*

Concerning cardinalities our theorem leaves the following cases open:

Problem 10. *Are the cardinalities of Corollary 7 all possible sizes of monoidal intervals? That is, if $2^\kappa < \lambda < 2^{2^\kappa}$ and λ is not a cardinality of a power set, does there exist a monoidal interval of size λ?*

2.1. **Notation.** The smallest clone containing a set $\mathscr{F} \subseteq \mathscr{O}$ shall be denoted by $\langle \mathscr{F} \rangle$; moreover, we write $\mathscr{F}^*$ for the set of all functions which arise from functions of $\mathscr{F}$ by identification of variables, addition of fictitious variables, or permutation of variables. For $n \geq 1$ we denote the set of n-ary operations on X by $\mathscr{O}^{(n)}$; if $\mathscr{F} \subseteq \mathscr{O}$, then $\mathscr{F}^{(n)}$ will stand for $\mathscr{F} \cap \mathscr{O}^{(n)}$. We will see X equipped with a vector space structure; then we write $\operatorname{span}(S)$ for the subspace of X generated by a set of vectors $S \subseteq X$. We shall denote the zero vector of X by 0, and use the same symbol for the constant function with value 0. We write $\mathscr{L}$ for the set of linear functions on X. The sum $f+g$ of two linear functions f, g on X is defined pointwise, as is the binary function $f(x)+g(y)$ obtained by the sum of two unary functions of different variables. The range of a function $f \in \mathscr{O}$ is given the symbol $\operatorname{ran} f$. For a set Y we write $\mathscr{P}(Y)$ for the power set of Y.

3. Monoids of linear functions

Let $\mathfrak{P}$ be a partial order. The set of all order ideals (also called lower subsets) on $\mathfrak{P}$ with the operations of set-theoretical intersection and union is a complete algebraic lattice, a sublattice of the power set of $\mathfrak{P}$. Given an arbitrary $\mathfrak{P}$ with $|\mathfrak{P}| = \lambda \leq 2^\kappa$, we construct a monoid $\mathscr{M}$ such that $\mathscr{I}_\mathscr{M}$ is isomorphic to $\mathbf{1} + \mathfrak{L}$, where $\mathfrak{L}$ is the lattice of order ideals of $\mathfrak{P}$.

Equip X with a vector space structure of dimension κ over any field K of characteristic $\neq 2, 3$ and fix a basis B of X. Fix moreover three distinguished elements $a, b, c \in B$ and write $A = B \setminus \{a, b, c\}$.

Let $\mathscr{A} \subseteq \mathscr{P}(A)$ be any family of subsets of A of cardinality κ such that $|\mathscr{A}| = \lambda$, and such that $A_1 \nsubseteq A_2$ for all distinct $A_1, A_2 \in \mathscr{A}$. Such families exist; see the textbook [Jec03, Lemma 7.7] for a proof of this. We now define a family $\mathscr{I} \subseteq \mathscr{P}(A)$ to consist of all proper subsets of sets in $\mathscr{A}$, plus all finite subsets of A. Clearly, $\mathscr{I}$ is an (order) ideal in the partial order $\mathscr{P}(A)$ equipped with set-theoretical inclusion, and we call the sets in $\mathscr{I}$ *small.* This ideal has the property that if $\alpha \in \mathscr{O}^{(1)}$ maps A_1 bijectively onto A_2, where $A_1, A_2 \in \mathscr{A}$, and if $S \subseteq A_1$ is small, then $\alpha[S]$ is small; we will need this property throughout our proof. Observe also that the sets in $\mathscr{A}$ are not elements of $\mathscr{I}$, but their nontrivial intersections are. We index the family $\mathscr{A}$ by the elements of $\mathfrak{P}$: $\mathscr{A} = (A_p)_{p \in \mathfrak{P}}$.

The monoid $\mathscr{M}$ we are going to construct will be one of linear functions on the vector space X; recall that we denote the set of linear functions on X by $\mathscr{L}$. We shall sometimes speak of the *support* of a linear function f, by which we mean the subset of A of those basis vectors which f does not send to 0. The monoid $\mathscr{M}$ will be the union of seven classes of functions, plus the identity and the zero function. Three classes, namely $\mathscr{N}$, $\mathscr{N}'$ and $\mathscr{N}''$, do "almost nothing", in the sense that they have small support; $\mathscr{N}$ essentially guarantees that the polymorphisms $\mathrm{Pol}(\mathscr{M})$ of the monoid $\mathscr{M}$ are sums of linear functions, and $\mathscr{N}'$ and $\mathscr{N}''$ consist of auxiliary functions necessary for the monoid to be closed under composition. The class Φ represents the elements of the partial order $\mathfrak{P}$, the class Ψ its order. Finally, the classes $\mathscr{S}_\Phi$ and $\mathscr{S}_{\mathscr{N}'}$ ensure that there exist nontrivial polymorphisms of the monoid, and that they correspond to elements of the partial order.

We start with the set $\mathscr{N}$ of those linear functions $n \in \mathscr{L}$ which satisfy the following conditions:

- $n(a) = a$
- $n(b) = 0$
- $n(c) = c$
- n has small support.

Next we add the set $\mathscr{N}' \subseteq \mathscr{L}$ consisting of all linear functions n' for which:

- $n'(a) = 0$
- $n'(b) = 0$
- $n'(c) = b$
- n' has small support
- $\operatorname{ran} n' \subseteq \operatorname{span}(\{b\})$.

The class $\mathscr{N}''$ contains all $n'' \in \mathscr{L}$ with

- $n''(a) = a$
- $n''(b) = 0$
- $n''(c) = 0$
- n'' has small support
- $\operatorname{ran} n'' \subseteq \operatorname{span}(\{a\})$.

Observe that all functions f in these three classes have small support, and that the range of any of the functions of $\mathscr{N}'$ and $\mathscr{N}''$ is only a one-dimensional subspace of X.

Now we define for all $p \in \mathfrak{P}$ a function $\phi_p \in \mathscr{L}$ by setting

- $\phi_p(a) = 0$
- $\phi_p(b) = 0$
- $\phi_p(c) = b$
- $\phi_p(d) = b$ for all $d \in A_p$
- $\phi_p(d) = 0$ for all other $d \in B$.

So ϕ_p is essentially the characteristic function of A_p. Observe that $\operatorname{ran} \phi_p \subseteq \operatorname{span}(\{b\})$. We write $\Phi = \{\phi_p : p \in \mathfrak{P}\}$.

We fix for all $p, q \in \mathfrak{P}$ with $q \leq_{\mathfrak{P}} p$ a function $\psi_{p,q} \in \mathscr{L}$ such that

- $\psi_{p,q}$ maps A_q bijectively onto A_p
- $\psi_{p,q}(a) = a$
- $\psi_{p,q}(b) = 0$
- $\psi_{p,q}(c) = c$
- $\psi_{p,q}(d) = 0$ for all other $d \in B$
- If $q \leq_{\mathfrak{P}} r \leq_{\mathfrak{P}} p$, then $\psi_{p,r} \circ \psi_{r,q} = \psi_{p,q}$.

This is possible: Let Y be a set of cardinality κ and choose for all $p \in \mathfrak{P}$ a bijection μ_p mapping A_p onto Y. Then setting $\psi_{p,q}(d) = \mu_p^{-1} \circ \mu_q(d)$ for all $d \in A_q$, $\psi_{p,q}(a) = a$, $\psi_{p,q}(c) = c$, and $\psi_{p,q}(d) = 0$ for all remaining $d \in B$ yields the required functions. We set $\Psi = \{\psi_{p,q} : p, q \in \mathfrak{P}, q \leq_{\mathfrak{P}} p\}$. The idea behind $\psi_{p,q}$ is that it "translates" the function ϕ_p of Φ into the function ϕ_q, and that such a translation function exists only if $q \leq_{\mathfrak{P}} p$. More precisely we have

Lemma 11. *Let $\phi_r \in \Phi$ and $\psi_{p,q} \in \Psi$. If $r = p$, then $\phi_r \circ \psi_{p,q} = \phi_q$; otherwise, $\phi_r \circ \psi_{p,q} \in \mathscr{N}'$.*

Proof. Assume first that $r = p$. Then in the composite $\phi_r \circ \psi_{p,q}$, first $\psi_{p,q}$ maps A_q onto A_p, and all other vectors of A to 0, and then ϕ_r sends $A_r = A_p$ to b, so that the composite indeed sends A_q to b and all other vectors of A to 0, as does ϕ_q; one easily checks that also the extra conditions on $a, b, c \in B$ are satisfied. If on the other hand $r \neq p$, then the only basis vectors in A which $\phi_r \circ \psi_{p,q}$ does not send to zero are those in $\psi_{p,q}^{-1}[A_r \cap A_p]$, a small set since $\psi_{p,q}$ is one-one on its support and by the properties of the family $\mathscr{A}$. Moreover, $\operatorname{ran}(\phi_r \circ \psi_{p,q}) \subseteq \operatorname{ran} \phi_r \subseteq \operatorname{span}(\{b\})$. Hence, since also the respective additional conditions on $a, b, c \in B$ are satisfied we have $\phi_r \circ \psi_{p,q} \in \mathscr{N}'$.

□

The remaining functions to be added to our monoid are those of the form $\phi_p + n''$, where $\phi_p \in \Phi$ and $n'' \in \mathscr{N}''$, the set of which we denote by $\mathscr{S}_\Phi$, and all functions of the form $n' + n''$, where $n' \in \mathscr{N}'$ and $n'' \in \mathscr{N}''$; this set we call $\mathscr{S}_{\mathscr{N}'}$. The elements f of $\mathscr{S}_\Phi$ and $\mathscr{S}_{\mathscr{N}'}$ both satisfy

- $f(a) = a$
- $f(b) = 0$
- $f(c) = b$.

We set $\mathscr{M} = \mathscr{N} \cup \mathscr{N}' \cup \mathscr{N}'' \cup \Phi \cup \Psi \cup \mathscr{S}_\Phi \cup \mathscr{S}_{\mathscr{N}'} \cup \{0, \pi_1^1\}$. Observe the following properties which hold for all $f \in \mathscr{M}$ except the identity π_1^1 and which will be useful:

- $f(a) \in \{0, a\}$
- $f(b) = 0$
- $f(c) \in \{0, b, c\}$.

Lemma 12. *$\mathscr{M}$ is a monoid.*

Proof. The following table describes the composition of the different classes of functions in $\mathscr{M}\setminus\{0,\pi_1^1\}$. Here, the meaning of $\mathscr{X}\circ\mathscr{Y}=\mathscr{Z}$ is: Whenever $f\in\mathscr{X}$ and $g\in\mathscr{Y}$, then $f\circ g\in\mathscr{Z}$. For the sake of a smaller table, we did not include the trivial composition with the operations π_1^1 and 0.

$\circ$	$\mathscr{N}$	$\mathscr{N}'$	$\mathscr{N}''$	Φ	Ψ	$\mathscr{S}_\Phi$	$\mathscr{S}_{\mathscr{N}'}$
$\mathscr{N}$	$\mathscr{N}$	$\{0\}$	$\mathscr{N}''$	$\{0\}$	$\mathscr{N}$	$\mathscr{N}''$	$\mathscr{N}''$
$\mathscr{N}'$	$\mathscr{N}'$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathscr{N}'$	$\{0\}$	$\{0\}$
$\mathscr{N}''$	$\mathscr{N}''$	$\{0\}$	$\mathscr{N}''$	$\{0\}$	$\mathscr{N}''$	$\mathscr{N}''$	$\mathscr{N}''$
Φ	$\mathscr{N}'$	$\{0\}$	$\{0\}$	$\{0\}$	$\Phi\cup\mathscr{N}'$	$\{0\}$	$\{0\}$
Ψ	$\mathscr{N}$	$\{0\}$	$\mathscr{N}''$	$\{0\}$	$\Psi\cup\mathscr{N}$	$\mathscr{N}''$	$\mathscr{N}''$
$\mathscr{S}_\Phi$	$\mathscr{S}_{\mathscr{N}'}$	$\{0\}$	$\mathscr{N}''$	$\{0\}$	$\mathscr{S}_\Phi\cup\mathscr{S}_{\mathscr{N}'}$	$\mathscr{N}''$	$\mathscr{N}''$
$\mathscr{S}_{\mathscr{N}'}$	$\mathscr{S}_{\mathscr{N}'}$	$\{0\}$	$\mathscr{N}''$	$\{0\}$	$\mathscr{S}_{\mathscr{N}'}$	$\mathscr{N}''$	$\mathscr{N}''$

We check the fields of the table. The fact that $\operatorname{ran} n'\subseteq\operatorname{span}(\{b\})$ for all $n'\in\mathscr{N}'$ and $f(b)=0$ for all $f\in\mathscr{M}$ yields the $\mathscr{N}'$-column; in the same way we get the Φ-column.
If $g=\phi_p+n''\in\mathscr{S}_\Phi$ and $f\in\mathscr{M}$, then $f\circ g=f\circ\phi_p+f\circ n''=f\circ n''$, so the $\mathscr{S}_\Phi$-column is equal to the $\mathscr{N}''$-column, and the same holds for the $\mathscr{S}_{\mathscr{N}'}$-column.
We turn to the $\mathscr{N}$- and $\mathscr{N}''$-columns. The $\mathscr{S}_\Phi$- and the $\mathscr{S}_{\mathscr{N}'}$-row are the sum of the Φ- and the $\mathscr{N}'$-row with the $\mathscr{N}''$-row, respectively, since $(f+g)\circ h=(f\circ h)+(g\circ h)$ for all $f,g,h\in\mathscr{O}^{(1)}$. For the other rows of those columns, note that if $f,g\in\mathscr{L}$ and g has small support, then also $f\circ g$ has small support. It is left to the reader to check the conditions on $a,b,c\in B$ and on the range for the composites.
It remains to verify the Ψ-column. For the first row, observe that since all $n\in\mathscr{N}$ have small support and since $\psi_{p,q}^{-1}[S]$ is small for all small $S\subseteq A$ and all $\psi_{p,q}\in\Psi$ by the properties of $\mathscr{A}$, any composition $n\circ\psi_{p,q}$ will have small support. Thus, together with the readily checked fact that the extra conditions on $a,b,c\in B$ are satisfied we get that $n\circ\psi_{p,q}\in\mathscr{N}$. The same argument yields the $\mathscr{N}'$- and $\mathscr{N}''$-rows.
The Φ-row is a consequence of Lemma 11. Similarly to the proof of that lemma, we show that $\psi_{p,s}\circ\psi_{t,q}$ is an element of $\mathscr{N}$ unless $s=t$, in which case it is $\psi_{p,q}$ by construction. To see this, assume $s\neq t$; then $\psi_{t,q}$ takes A_q to A_t, but $\psi_{p,s}$ has support A_s; therefore, the composite $\psi_{p,s}\circ\psi_{t,q}$ has support $\psi_{t,q}^{-1}[A_t\cap A_s]$, a small set since $\psi_{t,q}$ is injective on its support and by the properties of the family $\mathscr{A}$. The conditions on a,b,c for the composite to be in $\mathscr{N}$ are left to the reader, and we are done with the Ψ-row.
The $\mathscr{S}_\Phi$- and $\mathscr{S}_{\mathscr{N}'}$-rows are the sums of the $\mathscr{N}''$-row with the Φ-row and the $\mathscr{N}'$-row respectively, by the definitions of $\mathscr{S}_\Phi$ and $\mathscr{S}_{\mathscr{N}'}$. □

Recall that if $\mathscr{F}\subseteq\mathscr{O}$, then $\mathscr{F}^*$ consists of all functions which arise from functions of $\mathscr{F}$ by identification of variables, adding of fictitious variables, as well as by permutation of variables. Functions in $\mathscr{F}^*$ are called *polymers* of functions in $\mathscr{F}$. Set

$$\mathscr{V} = \{n'(x) + n''(y) : n' \in \mathscr{N}', n'' \in \mathscr{N}''\}.$$

Moreover, define for all $I \subseteq \mathfrak{P}$ sets of functions

$$\mathscr{D}_I = \{\phi_p(x) + n''(y) : p \in I, n'' \in \mathscr{N}''\}$$

and

$$\mathscr{C}_I = (\mathscr{M} \cup \mathscr{V} \cup \mathscr{D}_I)^*.$$

In these definitions, the variables x and y have no particular order, despite the alphabetical order one might associate with them; so for example, a function in $\mathscr{V}$ can be of the form $n'(x_2)+n''(x_1)$, with $n' \in \mathscr{N}'$ and $n'' \in \mathscr{N}''$, where x_1 is the first and x_2 the second variable of the function. This technical statement is necessary for the proof of the following lemma.

Lemma 13. *Let $I \subseteq \mathfrak{P}$ be an order ideal. Then $\mathscr{C}_I$ is a clone in $\mathscr{I}_{\mathscr{M}}$.*

Proof. We first show that $\mathscr{C}_I^{(1)} = \mathscr{M}$. To see this, observe that by its definition the unary functions in $\mathscr{C}_I$ are exactly $\mathscr{M}$ and those functions which arise when one identifies the two variables of a function in $\mathscr{V} \cup \mathscr{D}_I$. If $f \in \mathscr{V} \cup \mathscr{D}_I$, then $f = n'(x) + n''(y)$ or $f = \phi_p(x) + n''(y)$. Identifying its variables, we obtain a function of $\mathscr{S}_{\mathscr{N}'}$ in the first and of $\mathscr{S}_\Phi$ in the second case, and in either case an element of $\mathscr{M}$. Therefore, the unary part of $\mathscr{C}_I$ is exactly $\mathscr{M}$ and $\mathscr{C}_I$, if a clone, is indeed an element of $\mathscr{I}_{\mathscr{M}}$.
$\mathscr{C}_I$ contains $\pi_1^1 \in \mathscr{M}$ and therefore all projections, as it is by definition closed under the addition of fictitious variables.
We prove that $\mathscr{C}_I$ is closed under composition. To do this it suffices to prove that if $f(x_1, \ldots, x_n), g(y_1, \ldots, y_m) \in \mathscr{C}_I$, then the $(n+m-1)$-ary operation $f(x_1, \ldots, x_{i-1}, g(y_1, \ldots, y_m), x_{i+1}, \ldots, x_n) \in \mathscr{C}_I$, for all $1 \leq i \leq n$. Moreover, since $\mathscr{C}_I$ is closed under the addition of fictitious variables, we may assume that f, g depend on all of their variables, so by the definition of $\mathscr{C}_I$ they are at most binary, and therefore $f, g \in \mathscr{M} \cup \mathscr{V} \cup \mathscr{D}_I$. There is nothing to show if either f or g is the identity, so we consider only the case where $f, g \neq \pi_1^1$. Also, since $\mathscr{C}_I$ is by definition closed under identification of variables, we may assume that y_i and x_j are different variables, for all $1 \leq i \leq m$ and $1 \leq j \leq n$.
Let first $f \in \mathscr{M}$. If we substitute any $g \in \mathscr{M}$ for the only variable of f, then we stay in $\mathscr{M} \subseteq \mathscr{C}_I$ since $\mathscr{M}$ is a monoid by Lemma 12. If g is binary and of the form $m'(x) + m''(y) \in \mathscr{V}$, then by the multiplication table in the proof of Lemma 12 we have $f(m'(x)+m''(y)) = f(m'(x)) + f(m''(y)) = f(m''(y))$. Therefore, the binary operation $f(g(x,y))$ does not depend on its first variable, and $f(g(x,y)) \in \mathscr{M}^* \subseteq \mathscr{C}_I$, since the unary function $f \circ m'' \in \mathscr{M}$ as $\mathscr{M}$ is a monoid by Lemma 12. Similarly, if $g = \phi_p(x)+m''(y) \in \mathscr{D}_I$ we get $f(g(x,y)) = f(\phi_p(x)+m''(y)) = f(\phi_p(x))+f(m''(y)) = f(m''(y)) \in \mathscr{M}^*$.
We proceed with the case where f is binary, so $f \in \mathscr{V} \cup \mathscr{D}_I$. Assume $f = n'(x) + n''(y) \in \mathscr{V}$, and that we substitute a unary $g(z) \in \mathscr{M}$ for x. By

the multiplication table in the proof of Lemma 12, $n' \circ g \in \mathscr{N}' \cup \{0\}$; hence, $f(g(z), y)$ is a function of the form $m'(z) + n''(y) \in \mathscr{V}$ if $n' \circ g \in \mathscr{N}'$, and the essentially unary function $n''(y) \in \mathscr{M}^*$ if $n' \circ g = 0$. If we substitute a unary $g(z) \in \mathscr{M}$ for y, then $n'' \circ g \in \mathscr{N}'' \cup \{0\}$, so that again we stay in $\mathscr{V} \cup \mathscr{M}^*$. So say that $f = \phi_p(x) + n''(y) \in \mathscr{D}_I$, and that we substitute a unary $g(z) \in \mathscr{M}$ for x. From the multiplication table in the proof of Lemma 12 we know that $\phi_p \circ g \in \mathscr{N}' \cup \Phi \cup \{0\}$. If $\phi_p \circ g$ vanishes, then we obtain an essentially unary function in $(\mathscr{N}'')^* \subseteq \mathscr{M}^*$ for $f(g(z), y)$. If $\phi_p \circ g \in \mathscr{N}'$, then the sum with $n''(y)$ is in $\mathscr{V}$. The interesting case is the one where $\phi_p \circ g \in \Phi$; from the proof of Lemma 12 we know that this can only happen if g equals some $\psi_{s,t} \in \Psi$. Moreover, from Lemma 11 we infer that the composition is only in Φ if $s = p$, and then we have $\phi_p \circ \psi_{p,t} = \phi_t$. Hence in this case, $f(g(z), y) = \phi_t(z) + n''(y) \in \mathscr{D}_I$ since $t \leq p \in I$. To finish the case where we substitute a unary function for a variable of a binary function, let $f = \phi_p(x) + n''(y)$ and substitute $g(z) \in \mathscr{M}$ for y. Then, since $n'' \circ g \in \mathscr{N}'' \cup \{0\}$, the result will either be of the form $\phi_p(x) + m''(z)$ and thus in $\mathscr{D}_I$, or just $\phi_p(x) \in \mathscr{M}^*$ in case $n'' \circ g$ vanishes.
We now substitute binary functions $g(v, w) \in \mathscr{V} \cup \mathscr{D}_I$ into one variable of a binary $f(x, y) \in \mathscr{V} \cup \mathscr{D}_I$, thereby obtaining a ternary operation. Let $g(v, w) = m'(v) + m''(w) \in \mathscr{V}$. Since $h \circ m' = 0$ for all $h \in \mathscr{M} \setminus \{\pi_1^1\}$, and since $f(x, y)$ is of the form $f_1(x) + f_2(y)$ for some $f_1, f_2 \in \mathscr{M} \setminus \{\pi_1^1\}$, and since all involved functions are linear, m' will vanish in any substitution with g. Therefore substituting g is the same as substituting only an essentially unary function, which we already discussed. So let $g(v, w) = \phi_q(v) + m''(w)$. Then again, $h \circ \phi_q = 0$ for all $h \in \mathscr{M} \setminus \{\pi_1^1\}$, so substitution of g is equivalent to substituting only $m''(y)$ and we are done.

□

We now prove that $\langle \mathscr{M} \rangle$ and the $\mathscr{C}_I$ are the only clones in $\mathscr{I}_{\mathscr{M}}$.

Lemma 14. *Let $\mathscr{G}$ be a monoid of linear functions on the vector space X which contains the constant function 0, and let $k \geq 1$ be a natural number. If for any finite sequence of vectors $d_1, \ldots, d_k \in X$ there exist $e_1, \ldots, e_k \in X$ and $h_1, \ldots, h_k \in \mathscr{G}$ such that $h_j(e_j) = d_j$ and $h_j(e_i) = 0$ for all $1 \leq i, j \leq k$ with $i \neq j$, then all functions in $\mathrm{Pol}(\mathscr{G})^{(k)}$ are of the form $g_1(x_1) + \ldots + g_k(x_k)$, with $g_1, \ldots, g_k \in \mathscr{G}$.*

Proof. Let $F(x_1, \ldots, x_k) \in \mathrm{Pol}(\mathscr{G})^{(k)}$. Since $0 \in \mathscr{G}$, the functions $g_j(x_j) = F(0, \ldots, 0, x_j, 0, \ldots, 0)$ are elements of $\mathscr{G}$ for all $1 \leq j \leq k$. We claim $F(d_1, \ldots, d_k) = g_1(d_1) + \ldots + g_k(d_k)$ for all $d_1, \ldots, d_k \in X$. To see this, let $e_1, \ldots, e_k \in X$ and $h_1, \ldots, h_k \in \mathscr{G}$ be provided by the assumption of the lemma. Then $h(x) = F(h_1(x), \ldots, h_k(x))$ is an element of $\mathscr{G}$; therefore it is linear. Hence,

$$\begin{aligned} h(e_1+\ldots+e_k) &= h(e_1)+\ldots+h(e_k) \\ &= F(h_1(e_1),\ldots,h_k(e_1))+\ldots+F(h_1(e_k),\ldots,h_k(e_k)) \\ &= F(d_1,0,\ldots,0)+\ldots+F(0,\ldots,0,d_k) \\ &= g_1(d_1)+\ldots+g_k(d_k). \end{aligned}$$

On the other hand,

$$\begin{aligned} h(e_1+\ldots+e_k) &= F(h_1(e_1+\ldots+e_k),\ldots,h_k(e_1+\ldots+e_k)) \\ &= F(h_1(e_1)+\ldots+h_1(e_k),\ldots,h_k(e_1)+\ldots+h_k(e_k)) \\ &= F(d_1,\ldots,d_k). \end{aligned}$$

This proves the lemma. □

Lemma 15. *Let $\mathscr{G}$ be a monoid of linear functions on the vector space X which contains 0. If $\mathscr{G}$ contains $\mathscr{N}$, then the condition of the preceding lemma is satisfied for all $k \geq 1$.*

Proof. Given $d_1,\ldots,d_k \in X$ we choose any distinct $e_1,\ldots,e_k \in A$. Now for $1 \leq j \leq k$ we define $h_j \in \mathscr{N}$ to map e_j to d_j, a to a, c to c, and all remaining basis vectors to 0. □

Lemma 16. *Let $f,g \in \mathscr{M}$ be nonconstant. If $f+g \in \mathscr{M}$, then $f \in \mathscr{N}' \cup \Phi$ and $g \in \mathscr{N}''$ (or the other way round).*

Proof. Observe where the nonconstant functions of $\mathscr{M}$ map $a, c \in B$:

	a	c
$\{\pi_1^1\}$	a	c
$\mathscr{N}$	a	c
$\mathscr{N}'$	0	b
$\mathscr{N}''$	a	0
Φ	0	b
Ψ	a	c
$\mathscr{S}_\Phi$	a	b
$\mathscr{S}_{\mathscr{N}'}$	a	b

All functions $f \in \mathscr{M}$ satisfy $f(a) \in \{a,0\}$ and $f(c) \in \{b,c,0\}$. Hence, if $f+g \in \mathscr{M}$, then $(f+g)(a) = f(a)+g(a) \in \{a,0\}$ and $(f+g)(c) = f(c)+g(c) \in \{b,c,0\}$. Since the field K has characteristic $\neq 2$ we have that $a+a, b+b, c+c, b+c \notin \{0,a,b,c\}$. Thus it can be seen from the table that if $f(a)+g(a) \in \{a,0\}$, then at least one of the functions must map a to 0 and thereby be an element of $\mathscr{N}' \cup \Phi$. From the condition $f(c)+g(c) \in \{b,c,0\}$ we infer that either f or g must map c to 0 and hence belong to $\mathscr{N}''$. This proves the lemma. □

Lemma 17. *Let $f,g,h \in \mathscr{M}$ be nonconstant. Then $f+g+h \notin \mathscr{M}$.*

Proof. Since K has characteristic $\neq 2, 3$ we have that no sum of two or three elements of $\{a, b, c\}$ is an element of $\{0, a, b, c\}$. If $f + g + h \in \mathscr{M}$, then $f(a) + g(a) + h(a) \in \{a, 0\}$. This implies that at least two of the three functions have to map a to 0 and therefore belong to $\mathscr{N}' \cup \Phi$. Also, $f(c) + g(c) + h(c) \in \{b, c, 0\}$, from which we conclude that at least two functions must map c to 0 and thus be elements of $\mathscr{N}''$. So one function would have to be both in $\mathscr{N}' \cup \Phi$ and in $\mathscr{N}''$ which is impossible. Hence, $f + g + h \notin \mathscr{M}$. □

Lemma 18. $\mathrm{Pol}(\mathscr{M}) = \mathscr{C}_{\mathfrak{P}}$. *In particular, all functions in* $\mathrm{Pol}(\mathscr{M})$ *depend on at most two variables.*

Proof. Since $\mathscr{C}_{\mathfrak{P}}$ is a clone with unary part $\mathscr{M}$ by Lemma 13, we have that $\mathscr{C}_{\mathfrak{P}} \subseteq \mathrm{Pol}(\mathscr{M})$. To see the other inclusion, let $F(x_1, \ldots, x_k) \in \mathrm{Pol}(\mathscr{M})^{(k)}$. Then by Lemmas 14 and 15, $F(x_1, \ldots, x_k) = f_1(x_1) + \ldots + f_k(x_k)$, with $f_i \in \mathscr{M}$, $1 \leq i \leq k$. We show $F \in \mathscr{C}_{\mathfrak{P}}$; since clones are closed under the addition of fictitious variables, we may assume that F depends on all of its variables, i.e. f_i is nonconstant for all $1 \leq i \leq k$. If $k = 1$, then $F \in \mathscr{M}$, so $F \in \mathscr{C}_{\mathfrak{P}}$. If $k = 2$, then since $F(x, x) = (f_1 + f_2)(x)$ has to be an element of $\mathscr{M}$, Lemma 16 implies that $F \in \mathscr{V} \cup \mathscr{D}_I \subseteq \mathscr{C}_{\mathfrak{P}}$. To conclude, observe that $k \geq 3$ cannot occur by Lemma 17, since $F(x, x, x, 0, \ldots, 0) = f_1(x) + f_2(x) + f_3(x)$ must be an element of $\mathscr{M}$ if $F \in \mathrm{Pol}(\mathscr{M})$. □

Lemma 19. *Let* $\mathscr{C}$ *be a clone containing* $\mathscr{M}$ *and any function of* $\mathscr{V}$. *Then* $\mathscr{C}$ *contains* $\mathscr{V}$.

Proof. Let $n'(x) + n''(y) \in \mathscr{V} \cap \mathscr{C}$, where $n' \in \mathscr{N}'$ and $n'' \in \mathscr{N}''$, and let $m'(x) + m''(y)$ with $m' \in \mathscr{N}'$ and $m'' \in \mathscr{N}''$ be an (up to permutation of variables) arbitrary function in $\mathscr{V}$. Since $\mathrm{ran}\, m' = \mathrm{ran}\, n' = \mathrm{span}(\{b\})$, there is $n_1 \in \mathscr{L}$ with $m' = n' \circ n_1$. This n_1 can be chosen to satisfy $n_1(a) = a$, $n_1(b) = 0$, and $n_1(c) = c$; also, since m' has small support, we can choose n_1 to have small support too. Then $n_1 \in \mathscr{N} \subseteq \mathscr{M} \subseteq \mathscr{C}$. Similarly, there is $n_2 \in \mathscr{N}$ such that $m'' = n'' \circ n_2$. Hence, $m'(x) + m''(y) = n'(n_1(x)) + n''(n_2(y)) \in \mathscr{C}$. □

Lemma 20. *Let* $\mathscr{C}$ *be a clone containing* $\mathscr{M}$ *and any function of* $\mathscr{D}_{\mathfrak{P}}$. *Then* $\mathscr{C}$ *contains* $\mathscr{V}$.

Proof. Let $\phi_p(x) + n''(y) \in \mathscr{C} \cap \mathscr{D}_{\mathfrak{P}}$, where $\phi_p \in \Phi$ and $n'' \in \mathscr{N}''$. Taking any $n \in \mathscr{N}$ we set $n' = \phi_p \circ n \in \mathscr{N}'$. Then $\mathscr{C}$ contains $n'(x) + n''(y) \in \mathscr{V}$ and hence all functions of $\mathscr{V}$ by the preceding lemma. □

Lemma 21. *Let* $\mathscr{C}$ *be a clone containing* $\mathscr{M}$ *and a function* $\phi_p(x) + n''(y) \in \mathscr{D}_{\mathfrak{P}}$, *where* $\phi_p \in \Phi$ *and* $n'' \in \mathscr{N}''$. *If* $q \leq_{\mathfrak{P}} p$ *and* $m'' \in \mathscr{N}''$, *then* $\mathscr{C}$ *contains the function* $\phi_q(x) + m''(y)$.

Proof. As discussed in the proof of Lemma 19, there is $n \in \mathscr{N}$ such that $m'' = n'' \circ n$. Therefore $\mathscr{C}$ contains $\phi_p(\psi_{p,q}(x)) + n''(n(y)) = \phi_q(x) + m''(y)$. □

Proposition 22. *If $\mathscr{C} \in \mathscr{I}_{\mathscr{M}}$ is a clone, then $\mathscr{C} = \mathscr{M}^* = \langle \mathscr{M} \rangle$, or $\mathscr{C} = \mathscr{C}_I$, where $I \subseteq \mathfrak{P}$ is an order ideal on $\mathfrak{P}$.*

Proof. Let $\mathscr{C} \neq \langle \mathscr{M} \rangle$, that is, $\mathscr{C}$ contains an essentially binary function. Set $I = \{p \in \mathfrak{P} : \exists n'' \in \mathscr{N}'' \, (\phi_p(x) + n''(y) \in \mathscr{C})\}$. By Lemma 21, I is an order ideal of $\mathfrak{P}$. We claim $\mathscr{C} = \mathscr{C}_I$. Being elements of $\mathscr{I}_{\mathscr{M}}$, both $\mathscr{C}$ and $\mathscr{C}_I$ have $\mathscr{M}$ as their unary part. Let $f(x,y) \in \mathscr{C}^{(2)}$ be essentially binary, i.e. depending on both of its variables; then $f(x,y) \in \mathscr{V} \cup \mathscr{D}_{\mathfrak{P}}$ by Lemma 18. If $f \in \mathscr{V}$, then $f \in \mathscr{C}_I$ by definition of $\mathscr{C}_I$. If $f \in \mathscr{D}_{\mathfrak{P}}$, then it is of the form $\phi_p(x) + n''(y)$, where $p \in \mathfrak{P}$ and $n'' \in \mathscr{N}''$. But then $p \in I$ by definition of I and so $f \in \mathscr{C}_I$. Hence, $\mathscr{C}^{(2)} \subseteq \mathscr{C}_I^{(2)}$. Because $\mathscr{C}$ contains a binary function from $\mathscr{V} \cup \mathscr{D}_{\mathfrak{P}}$, Lemmas 19 and 20 imply $\mathscr{C}^{(2)} \supseteq \mathscr{V}$. Also, $\phi_q(x) + m''(y) \in \mathscr{C}^{(2)}$ for all $q \in I$ and all $m'' \in \mathscr{N}''$ by Lemma 21, so that we have $\mathscr{C}^{(2)} \supseteq \mathscr{C}_I^{(2)}$ and thus $\mathscr{C}^{(2)} = \mathscr{C}_I^{(2)}$. Lemma 18 implies that clones in $\mathscr{I}_{\mathscr{M}}$ are uniquely determined by their binary parts, so that we conclude $\mathscr{C} = \mathscr{C}_I$. □

Proposition 23. *Let X be an infinite set of size κ. Moreover, let $\mathfrak{P}$ be any partial order of size at most 2^κ, and denote by $\mathfrak{L}$ the lattice of order ideals on $\mathfrak{P}$. Then for the monoid $\mathscr{M}$ on X constructed in this section, the monoidal interval $\mathscr{I}_{\mathscr{M}}$ is isomorphic to $\mathbf{1} + \mathfrak{L}$.*

Proof. The mapping $\sigma : \mathbf{1} + \mathfrak{L} \to \mathscr{I}_{\mathscr{M}}$ taking an order ideal $I \in \mathfrak{L}$ to $\mathscr{C}_I$, as well as the smallest element of $\mathbf{1} + \mathfrak{L}$ to $\langle \mathscr{M} \rangle$, is obviously isotone and injective. By the preceding proposition it is also surjective. Since the inverse σ^{-1} is clearly isotone as well, we conclude that σ is a lattice isomorphism. □

Proposition 24. *Let X be an infinite set of size κ. If $\mathfrak{P}$ is any partial order with smallest element which has cardinality at most 2^κ, and if $\mathfrak{L}$ is the lattice of order ideals on $\mathfrak{P}$, then there exists a monoidal interval in the clone lattice over X which is isomorphic to $\mathfrak{L}$.*

Proof. Given a partial order $\mathfrak{P}$ with smallest element, we consider the partial order $\mathfrak{P}'$ obtained from $\mathfrak{P}$ by taking away the smallest element. By the preceding proposition, we can construct a monoid $\mathscr{M}$ such that $\mathscr{I}_{\mathscr{M}}$ is isomorphic to $\mathbf{1} + \mathfrak{L}'$, where $\mathfrak{L}'$ is the lattice of order ideals on $\mathfrak{P}'$. Now it is enough to observe that $\mathbf{1} + \mathfrak{L}'$ is isomorphic to the lattice $\mathfrak{L}$ of order ideals on $\mathfrak{P}$. □

Proof of Theorem 1. Let $\mathfrak{L}$ be an algebraic and dually algebraic distributive lattice. Then $\mathfrak{L}$ is isomorphic to the lattice of order ideals of the partial order of completely join irreducibles of $\mathfrak{L}$ (with the induced order), see the textbook [CD73, p.82-83]; therefore we can refer to Proposition 23. □

Proof of Corollary 3. The completely join irreducibles of $\mathscr{P}(\lambda)$ are exactly the singleton sets, so there are exactly $\lambda \leq 2^\kappa$ of them and we can refer to Theorem 1. □

Proof of Corollary 4. $\mathfrak{L}$ is completely distributive, so this is a direct consequence of Theorem 1. ☐

Definition 25. A monoid $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ is called *collapsing* iff its monoidal interval has only one element, i.e. $\langle \mathscr{G} \rangle = \mathrm{Pol}(\mathscr{G})$.

Denote by $\mathscr{S}$ the monoid of all permutations of X.

Proposition 26. *$\mathscr{S}$ is collapsing.*

Proof. Let $f \in \mathrm{Pol}(\mathscr{S}) \cap \mathscr{O}^{(2)}$. Then $\gamma(x) = f(x,x)$ is a permutation. Now let $a, b \in X$ be distinct. There exists $c \in X$ with $\gamma(c) = f(a,b)$. If $c \notin \{a,b\}$, then we can find $\alpha, \beta \in \mathscr{S}$ with $\alpha(a) = a$, $\alpha(b) = c$, $\beta(a) = b$, and $\beta(b) = c$. But then $f(\alpha,\beta)(a) = f(a,b) = f(c,c) = f(\alpha,\beta)(b)$, so $f(\alpha,\beta)(x)$ is not a permutation. Thus, $c \in \{a,b\}$, so we have shown that $f(x,y) \in \{f(x,x), f(y,y)\}$ for all $x, y \in X$.
Next we claim that for all distinct $a, b \in X$, if $f(a,b) = f(a,a)$, then $f(b,a) = f(b,b)$. To see this, consider any permutation α which has a cycle (ab). Then $f(a,\alpha(a)) = f(a,b) = f(a,a)$, so $f(b,\alpha(b)) = f(b,a)$ has to be different from $f(a,a)$, because otherwise the function $f(x,\alpha(x))$ is not injective. Therefore, $f(b,a) = f(b,b)$.
Assume without loss that $f(a,b) = f(a,a)$, for some distinct $a, b \in X$. We first claim that $f(a,c) = f(a,a)$ for all $c \in X$. For assume not; then $f(a,c) = f(c,c)$, and therefore $f(c,a) = f(a,a)$. Let $\beta \in \mathscr{S}$ map a to b and c to a. Then $f(a,\beta(a)) = f(a,b) = f(a,a)$, but also $f(c,\beta(c)) = f(c,a) = f(a,a)$, a contradiction since f preserves $\mathscr{S}$. Hence, $f(a,c) = f(a,a)$ for all $c \in X$. Now if $f(\tilde{a},\tilde{b}) \neq f(\tilde{a},\tilde{a})$ for some $\tilde{a}, \tilde{b} \in X$, then $\tilde{a} \neq a$ by the observation we just made, and $f(\tilde{a},\tilde{b}) = f(\tilde{b},\tilde{b})$ and so $f(\tilde{b},\tilde{a}) = f(\tilde{a},\tilde{a})$; thus, $\tilde{b} \neq a$. Therefore, the conditions $f(\tilde{a},\tilde{b}) = f(\tilde{b},\tilde{b})$ but $f(a,\tilde{b}) = f(a,a) \neq f(\tilde{b},\tilde{b})$ lead to a similar contradiction as before. Hence, $f(x,y) = f(x,x)$ for all $x, y \in X$, and we have shown that f depends on at most one variable. Since $f \in \mathrm{Pol}(\mathscr{S}) \cap \mathscr{O}^{(2)}$ was arbitrary, all binary functions of $\mathrm{Pol}(\mathscr{S})$ are essentially unary. By a result of Grabowski [Gra97], this implies that $\mathscr{S}$ is collapsing. (The mentioned result was proved for finite base sets of at least three elements but the same proof works on infinite sets.) ☐

Proof of Corollary 5. The preceding proposition gives us the ordinal 1. For larger ordinals, we can refer to Corollary 4. ☐

Proof of Corollary 7. This is the direct consequence of Corollaries 3 and 5. ☐

References

[BS81] S. Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.

[Bur67] G. A. Burle. Classes of k-valued logic which contain all functions of a single variable. *Diskret. Analiz, Novosibirsk*, 10:3–7, 1967. Russian.

[CD73] P. Crawley and R. P. Dilworth. *Algebraic theory of lattices.* Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

[GP] M. Goldstern and M. Pinsker. A survey of clones on infinite sets. Preprint available from `http://arxiv.org/math.RA/0701030`.

[Gra97] J.-U. Grabowski. Binary operations suffice to test collapsing of monoidal intervals. *Algebra universalis*, 38:92–95, 1997.

[GS] M. Goldstern and S. Shelah. Very many clones above the unary clone. Preprint.

[Jec03] T. Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[Kec95] A. S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kro95] A. A. Krokhin. Monoidal intervals in clone lattices. *Algebra i Logika*, 34(3):288–310, 1995.

[Kro97] A. A. Krokhin. On clones, transformation monoids, and associative rings. *Algebra universalis*, 37:527–540, 1997.

[Mac98] H. Machida. The clone space as a metric space. *Acta Applicandae Mathematicae*, 52:297–304, 1998.

[Pin] M. Pinsker. Algebraic lattices are complete sublattices of the clone lattice on an infinite set. Preprint available from `http://arxiv.org/math.RA/0605411`.

[Pin04] M. Pinsker. Clones containing all almost unary functions. *Algebra universalis*, 51:235–255, 2004.

[Pos41] E. L. Post. *The Two-Valued Iterative Systems of Mathematical Logic*. Annals of Mathematics Studies, no. 5. Princeton University Press, Princeton, N. J., 1941.

[PS83] P. P. Pálfy and Á. Szendrei. Unary polynomials in algebras. II. In *Contributions to general algebra, 2 (Klagenfurt, 1982)*, pages 273–290. Hölder-Pichler-Tempsky, Vienna, 1983.

[RS] I. G. Rosenberg and N. Sauer. Interval cardinality in the lattice of clones. Unpublished.

[Sze86] Á. Szendrei. *Clones in universal algebra*, volume 99 of *Séminaire de Mathématiques Supérieures*. Presses de l'Université de Montréal, Montreal, QC, 1986.

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