The minimal clones above the permutations

H. Machida [1] M. Pinsker [2]

[1]Mathematics
Hitotsubashi University
Tokyo, Japan

[2]Algebra
Vienna University of Technology
Wien, Austria

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Outline

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The clown lattice

X … infinite base set of size $\aleph_\alpha$.
$\mathcal{O}^{(n)} = X^{X^n} = \{f : X^n \to X\}$ … n-ary functions on X.
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Fact

X infinite $\to |Cl(X)| = 2^{2^{\aleph_\alpha}}$.

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$\mathcal{S}\ldots$ permutations of X.

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Can we determine the interval $[\mathcal{S}, \mathcal{O}]$ of $Cl(X)$?
How about $[\mathcal{S}, \mathcal{O}^{(1)}]$?

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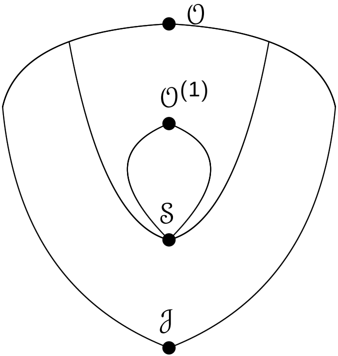

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Can we find all dual atoms (all atoms) of $[\mathcal{S}, \mathcal{O}]$?
How about $[\mathcal{S}, \mathcal{O}^{(1)}]$?

Maximal clones above $\mathcal{S}$

- $|X| = \aleph_\alpha$ regular $\rightarrow$ the maximal clones in $[\mathcal{S}, \mathcal{O}] \setminus [\mathcal{O}^{(1)}, \mathcal{O}]$ are known (Heindorf 2002, P. 2004). Their number is $|\alpha| + \aleph_0$.

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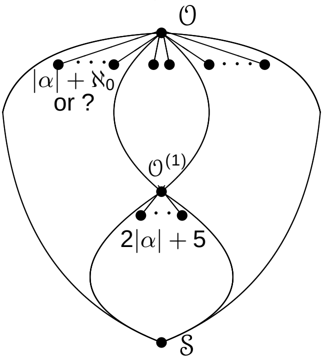

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Definition

$f \in \mathcal{O}^{(1)}$ is $\mathcal{S}$-minimal iff it generates a minimal clone (monoid) above $\mathcal{S}$.

$\mathcal{S}$-minimal functions

Definition

Let $f \in \mathcal{O}^{(1)}$. For all $0 \leq \xi \leq |X|$ set
$s_f(\xi) = |\{y \in X : |f^{-1}[y]| = \xi\}|$.
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$\mathrm{supp}(s_f) = \{1 \leq \xi \leq |X| : s_f(\xi) > 0\}$ ... support of s_f.
$\mathrm{supp}'(s_f) = \{1 \leq \xi \leq |X| : \xi \cdot s_f(\xi) > s_f(0)\}$ ... strong support.

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- $\mu_f = \min \text{supp}(s_f)$.
- $\varepsilon_f = \sup \text{supp}(s_f)$.
- $\lambda'_f = \sup\{\xi \in \text{supp}'(s_f) : \xi \leq s_f(0)\}$ (if non-void).
- $\chi_f = \min\{1 \leq \xi \leq |X| : \exists\zeta \in \text{supp}(s_f) : s_f(\geq \xi) \leq \zeta\}$.

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Theorem

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$f \in \mathcal{O}^{(1)} \setminus \mathcal{S}$ nonconstant $\rightarrow f$ is $\mathcal{S}$-minimal iff all of the following:

1. $\mu_f = 1$ *or* μ_f *is infinite.*
2. *If* $\mu_f = 1$, *then* $s_f(0)$ *is infinite or zero.*
3. $s_f(\mu_f) = |X|$.
4. $s_f(> \mu_f) < |X|$.
5. $s_f \restriction_{\mathrm{supp}'(s_f)}$ *is strictly decreasing.*
6. $n \notin \mathrm{supp}'(s_f)$ *for all* $1 < n < \aleph_0$.
7. $\varepsilon_f = 1$ *or* ε_f *is infinite.*
8. $\forall \xi \leq \chi_f$ *singular or finite* $s_f(\geq \xi) = \min\{s_f(\geq \zeta) : \zeta < \xi\}$.
9. *If* $\varepsilon_f \leq s_f(0)$, *then* $s_f(\geq \chi_f)$ *is finite.*
10. *If* $\varepsilon_f > s_f(0)$, *then* $s_f(\varepsilon_f)$ *is infinite.*
11. *If* $\varepsilon_f > s_f(0)$, *then* $s_f(\xi) = 0$ *for all* $\lambda'_f < \xi \leq s_f(0)$.

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Theorem

Let f, g be $\mathcal{S}$-minimal and non-constant.
Then $\langle \{f\} \cup \mathcal{S} \rangle = \langle \{g\} \cup \mathcal{S} \rangle$ iff all of the following:

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2. $s_g(0) = s_f(0)$.
3. $s_g \restriction_{\mathrm{supp}'(s_g)} = s_f \restriction_{\mathrm{supp}'(s_f)}$.
4. $\chi_g = \chi_f$.
5. $s_g(\geq \xi) = s_f(\geq \xi)$ *for all* $\xi < \chi_f$.
6. $\varepsilon_g = \varepsilon_f$.
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6. $\varepsilon_g = \varepsilon_f$.
7. $s_g(\varepsilon_g) = 0$ *iff* $s_f(\varepsilon_f) = 0$.

Corollary

$|X| = \aleph_\alpha$. The number of clones (monoids) minimal in $[\mathcal{S}, \mathcal{O}]$ (in $[\mathcal{S}, \mathcal{O}^{(1)}]$) is $|\alpha| + \aleph_0$.

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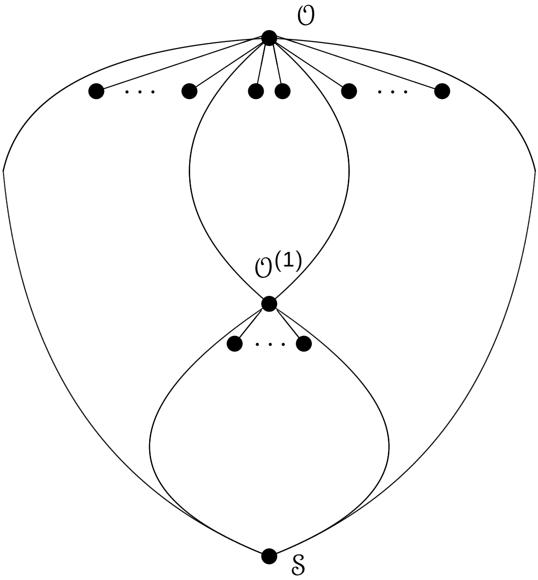

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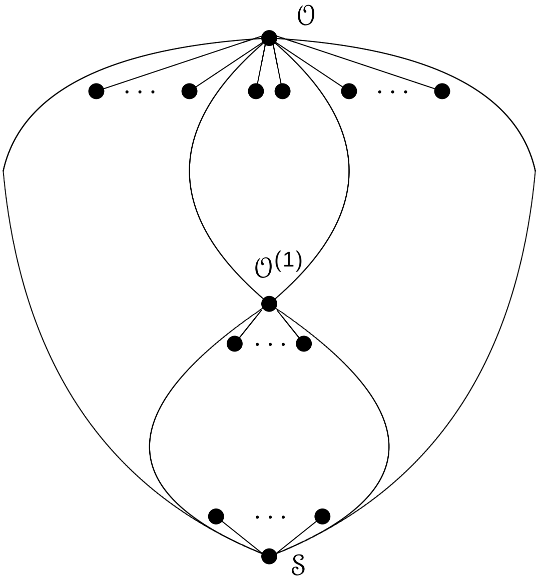

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