Algebraic lattices are complete sublattices of the clone lattice over an infinite set

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Outline

The clone lattice

$X \ldots$ base set.

$\mathcal{O}^{(n)} = X^{X^n} = \{f : X^n \to X\} \ldots$ *n*-ary functions on X.

$\mathcal{O} = \bigcup_{n \geq 1} \mathcal{O}^{(n)} \ldots$ finitary operations on X.

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Post's theorem

$|X| = 2 \to Cl(X)$ completely known ($|Cl(X)| = \aleph_0$).

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- we have (despite considerable effort) so far failed to do so.

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$|X| \geq 4 \rightarrow Cl(X)$ does not satisfy any non-trivial quasi-identity.

Monoidal intervals

For any monoid $\mathcal{G} \subseteq \mathcal{O}^{(1)}$,

$$\mathcal{I}_{\mathcal{G}} = \{\mathcal{C} \in \mathit{Cl}(X) : \mathcal{C} \cap \mathcal{O}^{(1)} = \mathcal{G}\}$$

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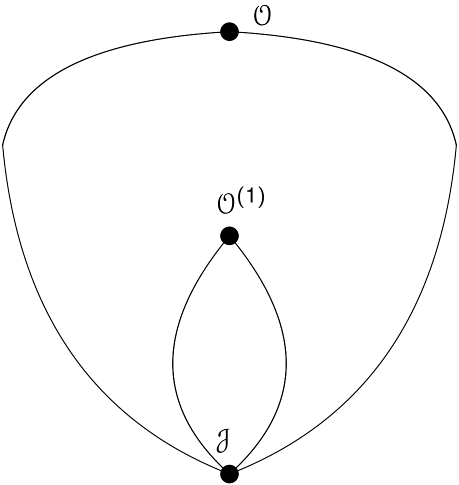
Theorem (P. 2005)

X infinite, $\mathcal{L}$ completely distributive algebraic with at most $2^{|X|}$ compact elements $\rightarrow$
$1 + \mathcal{L}$ is a monoidal interval of $Cl(X)$.

Monoidal intervals



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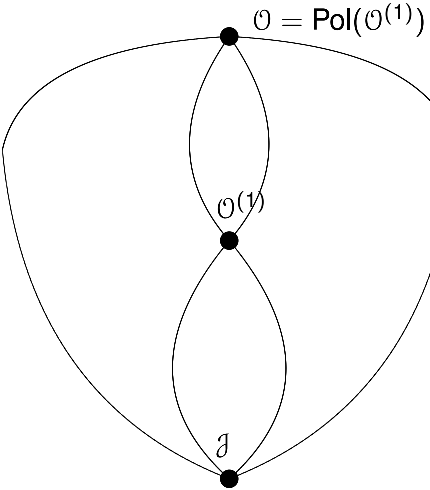

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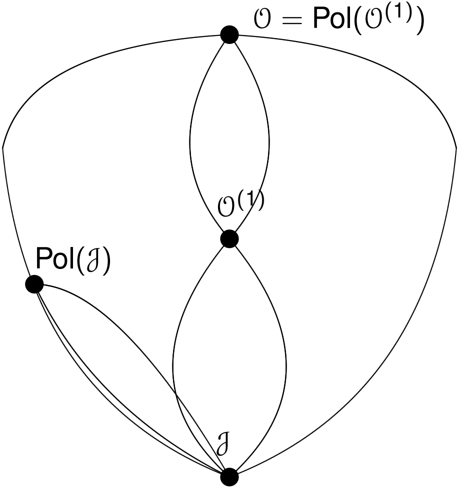

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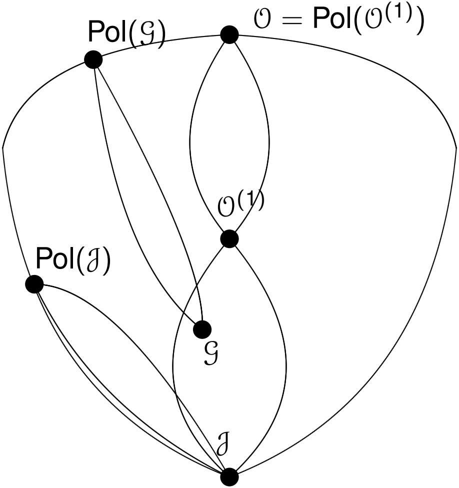

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- X infinite $\rightarrow$ $Cl(X)$ has $2^{|X|} = |\mathcal{O}|$ compact clones.

Theorem (P. 2006)

X infinite $\rightarrow$ Every algebraic lattice with at most $2^{|X|}$ compact elements is a complete sublattice of $Cl(X)$.

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Problem

X infinite. Is every algebraic lattice with at most $2^{|X|}$ compact elements an *interval* of $Cl(X)$? Even a *monoidal interval*?