Structure in mappings on the random graph

Michael Pinsker

jointly with Manuel Bodirsky (LIX Paris)

LMNO
Université de Caen
France

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Outline

The random graph and homogeneous structures

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Why is it random?

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Examples: Homogeneous K_n-free graph, dense linear order.

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How many classes of n-tuples are there?

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Do all homogeneous structures have oligomorphic automorphism groups?

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All homogeneous structures in a finite language
have oligomorphic automorphism groups.

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Is every closed group the automorphism group of a structure?

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What can we say about structures with comparable automorphism groups?

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Theorem

Let Γ be homogeneous.
Let Δ be any structure.
Then Δ has a first-order definition in Γ iff Aut(Δ) *contains* Aut(Γ).

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Tell us all the groups containing Aut(G)!

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Theorem (Thomas '91 / Bodirsky & P '09)

There are exactly 5 closed groups containing Aut(G).

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We want more examples!

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Other examples

There are 5 groups containing Aut(($\mathbb{Q}, <$)) (Cameron '76).
There are 2 groups containing the homogeneous K_n-free graph (Thomas '91 / P '09).

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Is the number of groups containing the automorphisms of a homogeneous structure always finite?

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Conjecture (Thomas '91)

The number of groups containing the automorphisms of a homogeneous structure is always finite.

Non-permutations on G

Why did you reprove a 20-years old theorem?

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Similarly, closed clones $\supseteq \mathrm{Aut}(G)$ are the polymorphism clones of reducts of G.

Definability

Groups: First-order interdefinability.
Monoids: Existential positive interdefinability.
Clones: Primitive positive interdefinability.

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Corollary

Let Γ be a reduct of the random graph.
Then Γ is a reduct of $(V; =)$, or its endomorphisms are generated by its automorphisms.

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What does this have to do with mappings on G?

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Theorem (Bodirsky & P '09)

All reducts of the random graph are model-complete.

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Given h, p, can we choose n large enough such that $n \longrightarrow (h)^p$ holds?

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Theorem (Ramsey's theorem)

For all p, h there exists n such that $n \to (h)^p$.

Ramsey classes

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Let N, H, P be graphs.

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means:

For all partitions of the copies of P in N into *good* and *bad*
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Definition

A class $\mathcal{C}$ of structures of the same signature is called a *Ramsey class* iff
for all $H, P \in \mathcal{C}$ there is N in $\mathcal{C}$ such that $N \to (H)^P$.

A variant of Thomas' conjecture

Theorem (Nešetřil-Rödl)

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Theorem (Nešetřil-Rödl)

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Conjecture

Any homogeneous structure **whose set of finite induced substructures (+ order) is a Ramsey class** has only finitely many reducts.

Enjoy your coffee break!

Where can we find your paper?

Manuel Bodirsky and Michael Pinsker,

All reducts of the random graph are model-complete,

available from arXiv.