Minimal functions on the random graph

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Problem

Classify the reducts of Γ.

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Theorem (Junker, Ziegler '08)

$(\mathbb{Q}; <, 0)$ has 116 reducts, up to f.o.-interdefinability.

Thomas' conjecture

Conjecture (Thomas '91)

Let Γ be homogeneous in a finite language.

Then Γ has finitely many reducts up to f.o.-interdefinability.

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Clone. . . set of finitary operations which contains all projections and which is closed under composition

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5. The full symmetric group S_V.

How to find all reducts up to . . .-interdefinability?

Climb up the lattice!

Canonical functions

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Definition

$f : \Gamma \to \Gamma$ is *canonical* iff
for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type in Γ
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sw_c is canonical for $(V; E, c)$.

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Definition

A class $\mathcal{C}$ of structures of the same signature is called a *Ramsey class* iff
for all $H, P \in \mathcal{C}$ there is N in $\mathcal{C}$ such that $N \to (H)^P$.

Patterns in functions on Ramsey structures

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Refining this idea, one can show:

If Γ is a reduct of an ordered Ramsey structure,
then every non-trivial function *generates*
a non-trivial function which is canonical
with respect to $(\Gamma, c_1, \ldots, c_n)$ for constants $c_1, \ldots, c_n$.

The minimal monoids on the random graph

Theorem (Thomas '96)

Let $f : V \to V$, $f \notin \mathrm{Aut}(G)$.
Then f generates one of the following:

- A constant operation
- An injection that deletes all edges
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Corollary. All reducts of the random graph are model-complete.

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Application. Constraint Satisfaction in theoretical computer science.

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- Every closed supermonoid of $\mathrm{Aut}(\Gamma)$ contains a minimal closed supermonoid of $\mathrm{Aut}(\Gamma)$.
- There are finitely many minimal closed clones containing $\mathrm{Aut}(\Gamma)$. (Arity bound: $|S_2(\Gamma)|$.)
- Every closed clone above $\mathrm{Aut}(\Gamma)$ contains a minimal one.

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Same for existential positive / existential.

Most important problem

Does Thomas' conjecture hold for Ramsey structures?