Reducts of homogeneous structures with the Ramsey property

Michael Pinsker

ÉLM Université Denis-Diderot Paris 7

Algebra TU Wien

Einstein Institute of Mathematics Jerusalem

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Outline

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Problem

Classify the reducts of Γ.

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Example (Thomas 1996)

The homogeneous k-graph has $2^k + 1$ reducts, up to f.o.-interdefinability.

Thomas' conjecture

Conjecture (Thomas 1991)

Γ has always finitely many reducts up to f.o. interdefinability.

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Clone. . . set of finitary operations which contains all projections and which is closed under composition

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- The full symmetric group S_V.

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Conclusion: Every finite graph has a copy in G on which f is canonical.

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Turning everything into edges (e_E), or
turning everything into non-edges (e_N), or
behaving like $-$, or
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The structure $(V; E, c, d)$ has similar Ramsey properties as $(V; E)$:

The subsets of elements of the same type contain the Random graph or have just one element.

An improved version of Thomas' theorem

Theorem (Thomas 1996)

Let $f : V \to V$, $f \notin \mathrm{Aut}(G)$.
Then f generates one of the following:

- A constant operation
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We thus know the *minimal closed monoids* containing $\mathrm{Aut}(G)$.

Ramsey classes

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Definition

A class $\mathcal{C}$ of structures of the same signature is called a *Ramsey class* iff

for all $H, P \in \mathcal{C}$ there is N in $\mathcal{C}$ such that $N \to (H)^P$.

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$f : \Gamma \to \Gamma$ is *canonical* iff
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Observation. Let H be a finite structure in the age of Γ.
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We would like to fix $c_1, \ldots, c_n$ witnessing $f \notin \mathrm{Aut}(\Gamma)$,
and have canonical behavior on $(\Gamma, c_1, \ldots, c_n)$.

Adding constants to Ramsey classes

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If Γ is Ramsey, is $(\Gamma, c_1, \ldots, c_n)$ still Ramsey?

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Theorem (Kechris, Pestov, Todorcevic 2005)

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Corollary

If Γ is ordered Ramsey, then so is $(\Gamma, c_1, \ldots, c_n)$.

Minimal monoid-reducts of Ramsey structures

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$, then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$ which behaves like f on $\{c_1, \ldots, c_n\}$.

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Note: There are infinitely many closed supermonoids of $\mathrm{Aut}(\Gamma)$.

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We might lose surjectivity.

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Project: Refine the method so that back and forth is possible.

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There are 14 minimal closed clones containing the automorphism group of the random graph.

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Theorem (Bodirsky, P. 2010)

If Γ is ordered Ramsey, then there are finitely many minimal closed clones containing Aut(Γ). (Arity bound: $|S_2(\Gamma)|$.)

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Can a clone containing the automorphism group of an ordered Ramsey structure Γ have infinitely many superclones?

Open problems II

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Determine the reducts of the random partial order.

Danke