Schaefer's theorem for graphs

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Outline

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- The graph satisfiability problem Graph-SAT(Ψ)

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- Schaefer's theorem for graphs

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- The graph satisfiability problem Graph-SAT(Ψ)

- Graph-SAT $\rightarrow$ CSPs of reducts of the random graph
- The algebraic approach: From reducts to polymorphisms
- Ramsey theory: Patterns in polymorphisms

- Schaefer's theorem for graphs

The graph satisfiability problem

Graph-SAT(Ψ)

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Computational problem: Boolean-SAT(Ψ)

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- A set W of propositional variables, and
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Theorem (Schaefer '78)

Boolean-SAT(Ψ) is either in P or NP-complete, for all Ψ.

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(Imagine: edge relation of an undirected graph.)

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Graph-SAT: Examples

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$$\begin{aligned}\psi_1(x,y,z) :=&(E(x,y) \wedge \neg E(y,z) \wedge \neg E(x,z)) \\ &\vee (\neg E(x,y) \wedge E(y,z) \wedge \neg E(x,z)) \\ &\vee (\neg E(x,y) \wedge \neg E(y,z) \wedge E(x,z)) \ .\end{aligned}$$

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Graph-SAT(Ψ_2) is in P.

Graph-SAT

$\rightarrow$

Constraint Satisfaction Problems

of reducts of the random graph

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For a graph formula $\psi(x_1, \ldots, x_n)$, define a relation

$$R_\psi := \{(a_1, \ldots, a_n) \in V^n : \psi(a_1, \ldots, a_n)\}.$$

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Γ_Ψ is a *reduct of* the random graph, i.e.,
a structure with a first-order definition in G.

Graph-SAT as Constraint Satisfaction Problem

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An instance

- $W = \{w_1, \ldots, w_m\}$
- $\phi_1, \ldots, \phi_n$

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whether or not a given *primitive positive sentence* holds in Γ_Ψ

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So Graph-SAT(Ψ) and CSP(Γ_Ψ) are one and the same problem.

The algebraic approach:

From reducts to polymorphisms

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set $\Gamma \leq_{pp} \Delta$ iff every relation of Γ has a pp-definition from Δ.

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A function $f : \Gamma^n \to \Gamma$ is a *polymorphism* of Γ iff
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Theorem (Bodirsky, Nešetřil). $\Gamma \leq_{pp} \Delta$ iff $\mathrm{Pol}(\Delta) \subseteq \mathrm{Pol}(\Gamma)$.

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Strategy:

(i) Prove hardness for certain relations

(ii) Prove that all reducts which do not pp-define any of these relations are tractable.

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Reducts of (ii) have polymorphisms violating the relations of (i).

Ramsey theory:

Patterns in polymorphisms

Canonical functions

A function $f : G \to G$ is *canonical* iff
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Canonical functions are finite objects!

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- Either Γ pp-defines one out of 4 hard relations, and CSP(Γ) is NP-complete,
- or CSP(Γ) is tractable.

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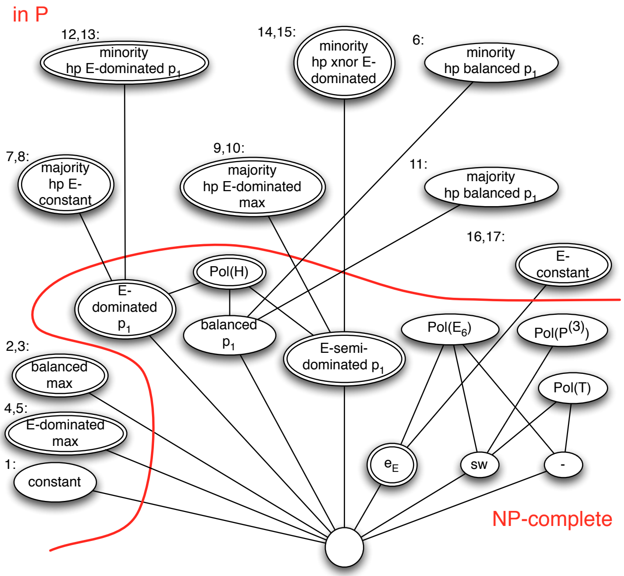
in P
12,13:
minority hp E-dominated p_1
14,15:
minority hp xnor E-dominated
6:
minority hp balanced p_1
7,8:
majority hp E-constant
9,10:
majority hp E-dominated max
11:
majority hp balanced p_1
16,17:
E-constant
Pol(H)
E-dominated p_1
balanced p_1
E-semi-dominated p_1
Pol(E_6)
Pol($P^{(3)}$)
Pol(T)
2,3:
balanced max
4,5:
E-dominated max
1:
constant
e_E
sw
-
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The Meta-Problem of Graph-SAT(Ψ) is decidable.