Lattices of subgroups of the symmetric group

Michael Pinsker (Paris 7)

joint work with

Saharon Shelah (Jerusalem)

AAA 84 Dresden 2012

Outline

Outline

- **Part I**

  Lattices of: Groups - Monoids - Clones

Outline

- **Part I**

  Lattices of: Groups - Monoids - Clones

- **Part II**

  (Locally) closed groups - monoids - clones

Three lattices

Let X be an infinite set.

Three lattices

Let X be an infinite set.

$\mathrm{Gr}(X)$ . . . lattice of all permutation groups on X.
(order = containment, meet = intersection)

Three lattices

Let X be an infinite set.

Gr(X) . . . lattice of all permutation groups on X.
(order = containment, meet = intersection)

Alternatively: Gr(X) subgroup lattice of the symmetric group Sym(X).

Three lattices

Let X be an infinite set.

Gr(X) . . . lattice of all permutation groups on X.
(order = containment, meet = intersection)

Alternatively: Gr(X) subgroup lattice of the symmetric group Sym(X).

Mo(X) . . . lattice of all transformation monoids on X.
(order = containment, meet = intersection)

Three lattices

Let X be an infinite set.

Gr(X) . . . lattice of all permutation groups on X.
(order = containment, meet = intersection)

Alternatively: Gr(X) subgroup lattice of the symmetric group Sym(X).

Mo(X) . . . lattice of all transformation monoids on X.
(order = containment, meet = intersection)

Alternatively: Mo(X) submonoid lattice of the monoid X^X.

Three lattices

Let X be an infinite set.

Gr(X) ... lattice of all permutation groups on X.
(order = containment, meet = intersection)

Alternatively: Gr(X) subgroup lattice of the symmetric group Sym(X).

Mo(X) ... lattice of all transformation monoids on X.
(order = containment, meet = intersection)

Alternatively: Mo(X) submonoid lattice of the monoid X^X.

Cl(X) ... lattice of all (concrete) clones on X.
(order = containment, meet = intersection)

Three lattices

Let X be an infinite set.

Gr(X) . . . lattice of all permutation groups on X.
(order = containment, meet = intersection)

Alternatively: Gr(X) subgroup lattice of the symmetric group Sym(X).

Mo(X) . . . lattice of all transformation monoids on X.
(order = containment, meet = intersection)

Alternatively: Mo(X) submonoid lattice of the monoid X^X.

Cl(X) . . . lattice of all (concrete) clones on X.
(order = containment, meet = intersection)

Alternatively: Cl(X) subclone lattice of the *full clone* $\mathcal{O} := \bigcup_n X^{X^n}$.

Lattice worries

Lattice worries

- What do these lattices look like?

Lattice worries

- What do these lattices look like? (lattices: Gr(X), Mo(X), Cl(X))

Lattice worries

- What do these lattices look like? (lattices: $\mathrm{Gr}(X)$, $\mathrm{Mo}(X)$, $\mathrm{Cl}(X)$)
- Are they ugly?

Lattice worries

- What do these lattices look like? (lattices: $\mathrm{Gr}(X)$, $\mathrm{Mo}(X)$, $\mathrm{Cl}(X)$)
- Are they ugly?
- What do they contain?

Lattice worries

- What do these lattices look like? (lattices: Gr(X), Mo(X), Cl(X))
- Are they ugly?
- What do they contain?

Is this vegetarian?

Lattice worries

- What do these lattices look like? (lattices: Gr(X), Mo(X), Cl(X))
- Are they ugly?
- What do they contain?

Is this vegetarian?

Lattice worries

- What do these lattices look like? (lattices: $\mathrm{Gr}(X)$, $\mathrm{Mo}(X)$, $\mathrm{Cl}(X)$)
- Are they ugly?
- What do they contain?

Is this vegetarian?

Comparing the lattices

$\mathrm{Gr}(X)$ is a sublattice of $\mathrm{Mo}(X)$ is a sublattice of $\mathrm{Cl}(X)$.

Comparing the lattices

$\mathrm{Gr}(X)$ is a sublattice of $\mathrm{Mo}(X)$ is a sublattice of $\mathrm{Cl}(X)$.

Problem

Converse true?

Size of the lattices

Observations.

Size of the lattices

Observations.

- $|\operatorname{Sym}(X)| = |X^X| = |\mathcal{O}| = 2^{|X|}$.

Size of the lattices

Observations.

- $|\operatorname{Sym}(X)| = |X^X| = |\mathcal{O}| = 2^{|X|}$.
- $|\operatorname{Gr}(X)| = |\operatorname{Mo}(X)| = |\operatorname{Cl}(X)| = 2^{2^{|X|}}$.

Size of the lattices

Observations.

- $|\operatorname{Sym}(X)| = |X^X| = |\mathcal{O}| = 2^{|X|}$.
- $|\operatorname{Gr}(X)| = |\operatorname{Mo}(X)| = |\operatorname{Cl}(X)| = 2^{2^{|X|}}$.

But: number of *finitely generated* groups / monoids / clones: $2^{|X|}$.

Size of the lattices

Observations.

- $|\mathrm{Sym}(X)| = |X^X| = |\mathcal{O}| = 2^{|X|}$.
- $|\mathrm{Gr}(X)| = |\mathrm{Mo}(X)| = |\mathrm{Cl}(X)| = 2^{2^{|X|}}$.

But: number of *finitely generated* groups / monoids / clones: $2^{|X|}$.

$\mathcal{G}$ finitely generated $\leftrightarrow$ $\mathcal{G}$ is *compact*, i.e.,
whenever $\mathcal{G} \leq \bigvee_{i \in I} \mathcal{G}_i$, then also $\mathcal{G} \leq \bigvee_{i \in J} \mathcal{G}_i$ for some $J \subseteq I$ finite.

Size of the lattices

Observations.

- $|\operatorname{Sym}(X)| = |X^X| = |\mathcal{O}| = 2^{|X|}$.
- $|\operatorname{Gr}(X)| = |\operatorname{Mo}(X)| = |\operatorname{Cl}(X)| = 2^{2^{|X|}}$.

But: number of *finitely generated* groups / monoids / clones: $2^{|X|}$.

$\mathcal{G}$ finitely generated $\leftrightarrow$ $\mathcal{G}$ is *compact*, i.e.,
whenever $\mathcal{G} \leq \bigvee_{i \in I} \mathcal{G}_i$, then also $\mathcal{G} \leq \bigvee_{i \in J} \mathcal{G}_i$ for some $J \subseteq I$ finite.

Every group (monoid, clone) is the join of compact elements.

So $\operatorname{Gr}(X)$, $\operatorname{Mo}(X)$, $\operatorname{Cl}(X)$ are *algebraic*.

Size of the lattices

Observations.

- $|\operatorname{Sym}(X)| = |X^X| = |\mathcal{O}| = 2^{|X|}$.
- $|\operatorname{Gr}(X)| = |\operatorname{Mo}(X)| = |\operatorname{Cl}(X)| = 2^{2^{|X|}}$.

But: number of *finitely generated* groups / monoids / clones: $2^{|X|}$.

$\mathcal{G}$ finitely generated $\leftrightarrow$ $\mathcal{G}$ is *compact*, i.e.,
whenever $\mathcal{G} \leq \bigvee_{i \in I} \mathcal{G}_i$, then also $\mathcal{G} \leq \bigvee_{i \in J} \mathcal{G}_i$ for some $J \subseteq I$ finite.

Every group (monoid, clone) is the join of compact elements.

So $\operatorname{Gr}(X)$, $\operatorname{Mo}(X)$, $\operatorname{Cl}(X)$ are *algebraic*.

Fact: A *complete sublattice* of an algebraic lattice is algebraic and cannot have more compact elements.

Non-vegetarian lattices

Non-vegetarian lattices

Theorem (MP '06)

$\mathrm{Cl}(X)$ is *universal*, i.e., every algebraic lattice with at most $2^{|X|}$ compact elements is a complete sublattice of $\mathrm{Cl}(X)$.

Non-vegetarian lattices

Theorem (MP '06)

Cl(X) is *universal*, i.e., every algebraic lattice with at most $2^{|X|}$ compact elements is a complete sublattice of Cl(X).

Theorem (MP + Shelah '11)

Mo(X) is universal.

Non-vegetarian lattices

Theorem (MP '06)

Cl(X) is *universal*, i.e., every algebraic lattice with at most $2^{|X|}$ compact elements is a complete sublattice of Cl(X).

Theorem (MP + Shelah '11)

Mo(X) is universal.

Theorem (MP + Shelah '12)

Gr(X) is universal.

Non-vegetarian lattices

Theorem (MP '06)

Cl(X) is *universal*, i.e., every algebraic lattice with at most $2^{|X|}$ compact elements is a complete sublattice of Cl(X).

Theorem (MP + Shelah '11)

Mo(X) is universal.

Theorem (MP + Shelah '12)

Gr(X) is universal.

So Cl(X) is a sublattice of Mo(X) is a sublattice of Gr(X).

Related work

Embeddings of algebraic lattices into subgroup lattices of (abstract) groups already known.

Related work

Embeddings of algebraic lattices into subgroup lattices of (abstract) groups already known.

E.g. Every algebraic lattice is an interval of the subgroup lattice of a group (Tuma '89).

Related work

Embeddings of algebraic lattices into subgroup lattices of (abstract) groups already known.

E.g. Every algebraic lattice is an interval of the subgroup lattice of a group (Tuma '89).

Size of group equals number of compact elements.

Related work

Embeddings of algebraic lattices into subgroup lattices of (abstract) groups already known.

E.g. Every algebraic lattice is an interval of the subgroup lattice of a group (Tuma '89).

Size of group equals number of compact elements.

Stone representation as permutation group acting on set of size $2^{|X|}$, and not $|X|$.

Local closure

Local closure

A permutation group $\mathcal{G}$ is called *(locally) closed* $\leftrightarrow$
for all $\alpha \in \mathrm{Sym}(X)$, if α can be interpolated by permutations from $\mathcal{G}$ on all finite sets, then $\alpha \in \mathcal{G}$.

Local closure

A permutation group $\mathcal{G}$ is called *(locally) closed* $\leftrightarrow$
for all $\alpha \in \mathrm{Sym}(X)$, if α can be interpolated by permutations from $\mathcal{G}$ on all finite sets, then $\alpha \in \mathcal{G}$.

Analogous definitions for transformation monoids, clones.

Local closure

A permutation group $\mathcal{G}$ is called *(locally) closed* $\leftrightarrow$
for all $\alpha \in \mathrm{Sym}(X)$, if α can be interpolated by permutations from $\mathcal{G}$ on all finite sets, then $\alpha \in \mathcal{G}$.

Analogous definitions for transformation monoids, clones.

- Group is closed $\leftrightarrow$ it is the *automorphism group* of a relational structure with domain X.
- Monoid is closed $\leftrightarrow$ it is the *endomorphism monoid* of a relational structure with domain X.
- Clone is closed $\leftrightarrow$ it is the *polymorphism clone* of a relational structure with domain X.

Local closure

A permutation group $\mathcal{G}$ is called *(locally) closed* $\leftrightarrow$
for all $\alpha \in \mathrm{Sym}(X)$, if α can be interpolated by permutations from $\mathcal{G}$ on all finite sets, then $\alpha \in \mathcal{G}$.

Analogous definitions for transformation monoids, clones.

- Group is closed $\leftrightarrow$ it is the *automorphism group* of a relational structure with domain X.
- Monoid is closed $\leftrightarrow$ it is the *endomorphism monoid* of a relational structure with domain X.
- Clone is closed $\leftrightarrow$ it is the *polymorphism clone* of a relational structure with domain X.

Topological structure of groups / monoids / clones important even for universal algebraists!

Topological Birkhoff (with M. Bodirsky)

Lattices of closed groups / monoids / clones

Closed groups / monoids / clones form complete lattices (meet = intersection): $\mathrm{Gr}_c(X)$ / $\mathrm{Mo}_c(X)$ / $\mathrm{Cl}_c(X)$.

Lattices of closed groups / monoids / clones

Closed groups / monoids / clones form complete lattices (meet = intersection): $\mathrm{Gr}_c(X)$ / $\mathrm{Mo}_c(X)$ / $\mathrm{Cl}_c(X)$.

Size: $2^{|X|}$. Non-algebraic!

Lattices of closed groups / monoids / clones

Closed groups / monoids / clones form complete lattices (meet = intersection): $\mathrm{Gr}_c(X)$ / $\mathrm{Mo}_c(X)$ / $\mathrm{Cl}_c(X)$.

Size: $2^{|X|}$. Non-algebraic!

Theorem (MP '09)

$M_{2^{\aleph_0}}$ embeds into $\mathrm{Cl}_c(\mathbb{N})$.

Lattices of closed groups / monoids / clones

Closed groups / monoids / clones form complete lattices (meet = intersection): $\mathrm{Gr}_c(X)$ / $\mathrm{Mo}_c(X)$ / $\mathrm{Cl}_c(X)$.

Size: $2^{|X|}$. Non-algebraic!

Theorem (MP '09)

$M_{2^{\aleph_0}}$ embeds into $\mathrm{Cl}_c(\mathbb{N})$.

Theorem (MP + Shelah '12)

$M_{2^{\aleph_0}}$ embeds into $\mathrm{Gr}(\mathbb{N})$ in such a way that the groups are F_σ (= unions of closed groups).

Lattices of closed groups / monoids / clones

Closed groups / monoids / clones form complete lattices (meet = intersection): $\mathrm{Gr}_c(X)$ / $\mathrm{Mo}_c(X)$ / $\mathrm{Cl}_c(X)$.

Size: $2^{|X|}$. Non-algebraic!

Theorem (MP '09)

$M_{2^{\aleph_0}}$ embeds into $\mathrm{Cl}_c(\mathbb{N})$.

Theorem (MP + Shelah '12)

$M_{2^{\aleph_0}}$ embeds into $\mathrm{Gr}(\mathbb{N})$ in such a way that the groups are F_σ (= unions of closed groups).

Theorem (MP + Shelah '12)

Assume that $\lambda^{<\lambda} = \lambda$ and *cofinality*$(\lambda) > \aleph_0$.

Then M_{2^λ} embeds into $\mathrm{Gr}_c(\lambda)$.

Open problems

Open problems

Problem

Does $M_{2^{\aleph_0}}$ embed into $\mathrm{Gr}_c(\mathbb{N})$?

Open problems

Problem

Does $M_{2^{\aleph_0}}$ embed into $\mathrm{Gr}_c(\mathbb{N})$?

Problem

Does $M_{2^{\aleph_0}}$ embed into $\mathrm{Mo}_c(\mathbb{N})$?

Open problems

Problem

Does $M_{2^{\aleph_0}}$ embed into $\mathrm{Gr}_c(\mathbb{N})$?

Problem

Does $M_{2^{\aleph_0}}$ embed into $\mathrm{Mo}_c(\mathbb{N})$?

Problem

Do all lattices (excluding impossible ones) embed into $\mathrm{Gr}_c(\mathbb{N})$ / $\mathrm{Mo}_c(\mathbb{N})$ / $\mathrm{Cl}_c(\mathbb{N})$?

Open problems

Problem

Does $M_{2^{\aleph_0}}$ embed into $\mathrm{Gr}_c(\mathbb{N})$?

Problem

Does $M_{2^{\aleph_0}}$ embed into $\mathrm{Mo}_c(\mathbb{N})$?

Problem

Do all lattices (excluding topologically impossible ones) embed into $\mathrm{Gr}_c(\mathbb{N})$ / $\mathrm{Mo}_c(\mathbb{N})$ / $\mathrm{Cl}_c(\mathbb{N})$?

Open problems

Problem

Does $M_{2^{\aleph_0}}$ embed into $\mathrm{Gr}_c(\mathbb{N})$?

Problem

Does $M_{2^{\aleph_0}}$ embed into $\mathrm{Mo}_c(\mathbb{N})$?

Problem

Do all lattices (excluding topologically impossible ones) embed into $\mathrm{Gr}_c(\mathbb{N})$ / $\mathrm{Mo}_c(\mathbb{N})$ / $\mathrm{Cl}_c(\mathbb{N})$?

Example of impossible: chain of length $2^{\aleph_0}$.