Reconstructing the topology of clones

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Joint work with Manuel Bodirsky and András Pongrácz

Workshop on Homogeneous Structures

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Outline

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- **Part I:** Reconstructing structures from their automorphism groups and polymorphism clones

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- **Part II:** The topology of algebras

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- **Part III:** Reconstruction notions

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- **Part IV:** Negative results

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- **Part IV:** Negative results
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- **Part VI:** Perspectives & Open problems

Part I

Reconstructing structures from their automorphism groups and polymorphism clones

Reconstructing structures up to first-order . . .

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countable

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countable, ω-categorical

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Aut()

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Aut() $\to$

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Let Δ, Γ be ω-categorical structures on the same domain. Then Aut(Δ) $=$ Aut(Γ) iff Δ, Γ are first-order interdefinable.

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Theorem (Ahlbrandt + Ziegler '86)

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- Can we reconstruct the topological structure of closed oligomorphic permutation groups from their algebraic structure?

Better reconstruction plans

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Observe: $\mathrm{Pol}(\Delta) \supseteq \mathrm{End}(\Delta) \supseteq \mathrm{Aut}(\Delta)$.

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Let Δ, Γ be ω-categorical structures on the same domain. Then:
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Confer Manuel Bodirsky's talk.

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Part II

The topology of algebras

Clones from algebras

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Structural conclusions about *finite* $\mathfrak{A}$ from variety of $\mathfrak{A}$
(i.e., from abstract clone $\mathrm{Clo}(\mathfrak{A})$).

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Theorem (Birkhoff 1935)

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite.

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Part III

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Similarly for closed subgroups of $\mathbf{S}_\infty$ and closed submonoids of $\mathbf{O}^{(1)}$.

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Unclear for monoids and clones.

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- $(\mathbb{N}; =)$ (Dixon+Neumann+Thomas'86)

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  various ω-categorical semilinear orders
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- the random *k*-hypergraphs
  the random K_n-free graphs
  the Henson digraphs

  (Barbina+MacPherson '07).

Part IV

Negative results

Automatic continuity for monoids / clones

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If Δ is ω-categorical and has no algebraicity, then $\mathrm{End}(\Delta)$ does not have automatic continuity.

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Recall: $\mathbf{1}$ is the clone of projections on a set of at least two elements.

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Recall: **1** is the clone of projections on a set of at least two elements.

Important in constraint satisfaction: "main reason" for NP-hardness of the CSP of a structure.

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Theorem (Bodirsky + MP + Pongrácz '13)

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Moreover, this clone also has a continuous homomorphism to **1**.

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Theorem (Bodirsky + MP + Pongrácz '13)

There exists an oligomorphic closed submonoid $\mathbf{M}$ of $\mathbf{O}^{(1)}$ and $\xi\colon \mathbf{M} \to \mathbf{M}$ such that:

- ξ is an isomorphism;
- ξ fixes the invertibles of $\mathbf{M}$ pointwise;
- ξ is not continuous.

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Theorem (Evans + Hewitt '90)

There exists an oligomorphic closed subgroup $\mathbf{G}$ of $\mathbf{S}_\infty$ which does not have reconstruction.

Reconstruction

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Problem

Find an oligomorphic closed subclone of $\mathbf{O}$ without reconstruction.

Part V

Positive results

Strategy

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For constraint satisfaction:

Can assume that $\mathbf{C}$ is a model-complete core: $\mathbf{G}_C$ is dense in $\mathbf{C}^{(1)}$.

From groups to their closure

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Theorem (Bodirsky + MP + Pongrácz ’13)

Let Δ be homogeneous in a finite relational language without algebraicity ($\leftrightarrow$ strong amalgamation).

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Fact

There are closed oligomorphic subclones of **O** without transitive action.

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there exist unary $(\alpha^i)_{i\in\omega}$ and $(\beta_1^i)_{i\in I}, \ldots, (\beta_n^i)_{i\in\omega}$ in **C** with

- $g^i(x_1, \ldots, x_n) = \alpha^i(f_U(\beta^i(x_1), \ldots, \beta^i(x_n)))$ and
- $(\alpha^i)_{i\in\omega}$ and $(\beta_1^i)_{i\in I}, \ldots, (\beta_n^i)_{i\in\omega}$ converge.

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Many more. . . ?

Part VI

Perspectives & Open problems

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- Is there a model of ZF where all homomorphisms from oligomorphic closed subclones of **O** to the projection clone **1** are continuous?

Thank you!