The 42 reducts of the random ordered graph

Michael Pinsker

Technische Universität Wien / Université Diderot - Paris 7

4th Novi Sad Algebraic Conference, 2013

Outline

- **Part I:** The setting of The Answer
- **Part II:** The 42 reducts of the random ordered graph
- **Part III:** The effect of The Answer
- **Part IV:** The question to The Answer

Part I: The setting of The Answer

Homogeneous structures

Let Δ be a countable structure.

Homogeneous structures

Let Δ be a countable structure.

Definition

Δ is homogeneous $:\leftrightarrow$
every isomorphism between finitely generated substructures of Δ extends to an automorphism of Δ.

Homogeneous structures

Let Δ be a countable structure.

Definition

Δ is homogeneous $:\leftrightarrow$
every isomorphism between finitely generated substructures of Δ extends to an automorphism of Δ.

Examples

Homogeneous structures

Let Δ be a countable structure.

Definition

Δ is homogeneous $:\leftrightarrow$
every isomorphism between finitely generated substructures of Δ extends to an automorphism of Δ.

Examples

- Order of the rationals $(\mathbb{Q}; <)$

Homogeneous structures

Let Δ be a countable structure.

Definition

Δ is homogeneous $:\leftrightarrow$
every isomorphism between finitely generated substructures of Δ extends to an automorphism of Δ.

Examples

- Order of the rationals $(\mathbb{Q}; <)$
- Random graph $(V; E)$

Homogeneous structures

Let Δ be a countable structure.

Definition

Δ is homogeneous $:\leftrightarrow$
every isomorphism between finitely generated substructures of Δ extends to an automorphism of Δ.

Examples

- Order of the rationals $(\mathbb{Q}; <)$
- Random graph $(V; E)$
- Free Boolean algebra with $\aleph_0$ generators

Fraïssé limits

Fraïssé limits

Let $\mathcal{C}$ be a class of finitely generated structures
in a countable language, closed under isomorphism.

Fraïssé limits

Let $\mathcal{C}$ be a class of finitely generated structures
in a countable language, closed under isomorphism.

Theorem (Fraïssé)

Assume $\mathcal{C}$

Fraïssé limits

Let $\mathcal{C}$ be a class of finitely generated structures
in a countable language, closed under isomorphism.

Theorem (Fraïssé)

Assume $\mathcal{C}$

- is closed under substructures

Fraïssé limits

Let $\mathcal{C}$ be a class of finitely generated structures
in a countable language, closed under isomorphism.

Theorem (Fraïssé)

Assume $\mathcal{C}$

- is closed under substructures
- has joint embeddings:
  for all $B, C \in \mathcal{C}$ there is $D \in \mathcal{C}$ containing isomorphic copies of B, C

Fraïssé limits

Let $\mathcal{C}$ be a class of finitely generated structures
in a countable language, closed under isomorphism.

Theorem (Fraïssé)

Assume $\mathcal{C}$

- is closed under substructures
- has joint embeddings:
  for all $B, C \in \mathcal{C}$ there is $D \in \mathcal{C}$ containing isomorphic copies of B, C
- has amalgamation:
  for all $A, B, C \in \mathcal{C}$ and embeddings $e_B : A \to B$ and $e_C : A \to C$
  there is $D \in \mathcal{C}$ and embeddings $f_B : B \to D$ and $f_C : C \to D$
  such that $f_B \circ e_B = f_C \circ e_C$.

Fraïssé limits

Let $\mathcal{C}$ be a class of finitely generated structures
in a countable language, closed under isomorphism.

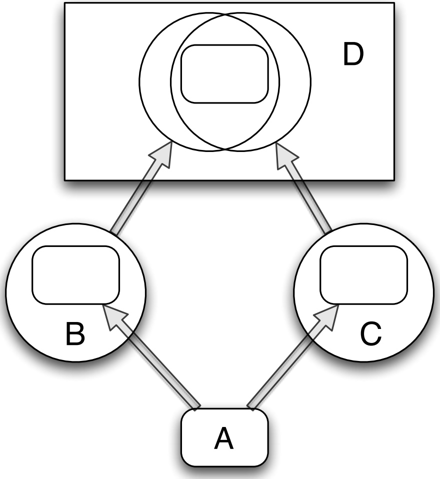
Theorem (Fraïssé)

Assume $\mathcal{C}$

- is closed under substructures
- has joint embeddings:
  for all $B, C \in \mathcal{C}$ there is $D \in \mathcal{C}$ containing isomorphic copies of B, C
- has amalgamation:
  for all $A, B, C \in \mathcal{C}$ and embeddings $e_B : A \to B$ and $e_C : A \to C$
  there is $D \in \mathcal{C}$ and embeddings $f_B : B \to D$ and $f_C : C \to D$
  such that $f_B \circ e_B = f_C \circ e_C$.

Then there exists a unique countable homogeneous structure Δ
whose age (=substructures up to iso) equals $\mathcal{C}$.

Amalgamation



Examples

Examples

- (Finite) linear orders $\leftrightarrow (\mathbb{Q}; <)$

Examples

- (Finite) linear orders $\leftrightarrow$ $(\mathbb{Q}; <)$
- Undirected graphs $\leftrightarrow$ random graph $(V; E)$

Examples

- (Finite) linear orders $\leftrightarrow$ $(\mathbb{Q}; <)$
- Undirected graphs $\leftrightarrow$ random graph $(V; E)$
- Boolean algebras $\leftrightarrow$ random (= free) Boolean algebra

Examples

- (Finite) linear orders $\leftrightarrow$ $(\mathbb{Q}; <)$
- Undirected graphs $\leftrightarrow$ random graph $(V; E)$
- Boolean algebras $\leftrightarrow$ random (= free) Boolean algebra
- Lattices $\leftrightarrow$ random lattice

Examples

- (Finite) linear orders $\leftrightarrow$ $(\mathbb{Q}; <)$
- Undirected graphs $\leftrightarrow$ random graph $(V; E)$
- Boolean algebras $\leftrightarrow$ random (= free) Boolean algebra
- Lattices $\leftrightarrow$ random lattice
- Distributive lattices $\leftrightarrow$ random distributive lattice

Examples

- (Finite) linear orders $\leftrightarrow$ $(\mathbb{Q}; <)$
- Undirected graphs $\leftrightarrow$ random graph $(V; E)$
- Boolean algebras $\leftrightarrow$ random (= free) Boolean algebra
- Lattices $\leftrightarrow$ random lattice
- Distributive lattices $\leftrightarrow$ random distributive lattice
- Partial orders $\leftrightarrow$ random partial order

Examples

- (Finite) linear orders $\leftrightarrow$ $(\mathbb{Q}; <)$
- Undirected graphs $\leftrightarrow$ random graph $(V; E)$
- Boolean algebras $\leftrightarrow$ random (= free) Boolean algebra
- Lattices $\leftrightarrow$ random lattice
- Distributive lattices $\leftrightarrow$ random distributive lattice
- Partial orders $\leftrightarrow$ random partial order
- Tournaments $\leftrightarrow$ random tournament

Examples

- (Finite) linear orders $\leftrightarrow$ $(\mathbb{Q}; <)$
- Undirected graphs $\leftrightarrow$ random graph $(V; E)$
- Boolean algebras $\leftrightarrow$ random (= free) Boolean algebra
- Lattices $\leftrightarrow$ random lattice
- Distributive lattices $\leftrightarrow$ random distributive lattice
- Partial orders $\leftrightarrow$ random partial order
- Tournaments $\leftrightarrow$ random tournament
- Linearly ordered graphs $\leftrightarrow$ random ordered graph $(D; <, E)$

Reducts

Reducts

Let Δ be a structure.

Reducts

Let Δ be a structure.

Definition

A reduct of Δ is a structure on the same domain whose relations and functions are first-order definable in Δ (without parameters).

Reducts

Let Δ be a structure.

Definition

A reduct of Δ is a structure on the same domain
whose relations and functions are first-order definable in Δ
(without parameters).

Examples

Reducts

Let Δ be a structure.

Definition

A reduct of Δ is a structure on the same domain whose relations and functions are first-order definable in Δ (without parameters).

Examples

- $(\mathbb{Q}; <)$: reduct $(\mathbb{Q}; \text{Between}(x, y, z))$

Reducts

Let Δ be a structure.

Definition

A reduct of Δ is a structure on the same domain whose relations and functions are first-order definable in Δ (without parameters).

Examples

- $(\mathbb{Q}; <)$: reduct $(\mathbb{Q}; \text{Between}(x, y, z))$
- $(\mathbb{Q}; <)$: reduct $(\mathbb{Q}; >)$

Reducts

Let Δ be a structure.

Definition

A reduct of Δ is a structure on the same domain whose relations and functions are first-order definable in Δ (without parameters).

Examples

- $(\mathbb{Q}; <)$: reduct $(\mathbb{Q}; \text{Between}(x, y, z))$
- $(\mathbb{Q}; <)$: reduct $(\mathbb{Q}; >)$
- random graph $(V; E)$: reduct $(V; K_3(x, y, z))$

Reducts

Let Δ be a structure.

Definition

A reduct of Δ is a structure on the same domain whose relations and functions are first-order definable in Δ (without parameters).

Examples

- $(\mathbb{Q}; <)$: reduct $(\mathbb{Q}; \text{Between}(x, y, z))$
- $(\mathbb{Q}; <)$: reduct $(\mathbb{Q}; >)$
- random graph $(V; E)$: reduct $(V; K_3(x, y, z))$
- random poset $(P; \leq)$: reduct $(P; \perp(x, y))$

Reducts

Let Δ be a structure.

Definition

A reduct of Δ is a structure on the same domain whose relations and functions are first-order definable in Δ (without parameters).

Examples

- $(\mathbb{Q}; <)$: reduct $(\mathbb{Q}; \text{Between}(x, y, z))$
- $(\mathbb{Q}; <)$: reduct $(\mathbb{Q}; >)$
- random graph $(V; E)$: reduct $(V; K_3(x, y, z))$
- random poset $(P; \leq)$: reduct $(P; \perp(x, y))$

Problem

Understand the reducts of homogeneous structures.

Motivation

Motivation

Why reducts?

Why reducts?

- *Understand* Δ *itself:*

Why reducts?

- *Understand Δ itself:*
  – its first-order theory

Why reducts?

- *Understand Δ itself:*

  – its first-order theory

  – its symmetries (via connection with permutation groups)

Motivation

Why reducts?

- *Understand Δ itself:*

  – its first-order theory

  – its symmetries (via connection with permutation groups)

- *Understand the age $\mathcal{C}$ of Δ:*

Motivation

Why reducts?

- *Understand Δ itself:*

  – its first-order theory

  – its symmetries (via connection with permutation groups)

- *Understand the age $\mathcal{C}$ of Δ:*

  – uniform group actions on $\mathcal{C}$
  (via permutation groups - combinatorics of $\mathcal{C}$)

Why reducts?

- *Understand Δ itself:*

  – its first-order theory

  – its symmetries (via connection with permutation groups)

- *Understand the age $\mathcal{C}$ of Δ:*

  – uniform group actions on $\mathcal{C}$
  (via permutation groups - combinatorics of $\mathcal{C}$)

  – Constraint Satisfaction Problems related to $\mathcal{C}$:
  Graph-SAT, Poset-SAT,...

Reducts up to first-order equivalence

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Quasiorder.

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Quasiorder.

Consider reducts Γ, Γ' equivalent iff $\Gamma \leq \Gamma'$ and $\Gamma' \leq \Gamma$.

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Quasiorder.

Consider reducts Γ, Γ' equivalent iff $\Gamma \leq \Gamma'$ and $\Gamma' \leq \Gamma$.

Factoring out we get a complete lattice.

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Quasiorder.

Consider reducts Γ, Γ' equivalent iff $\Gamma \leq \Gamma'$ and $\Gamma' \leq \Gamma$.

Factoring out we get a complete lattice.

Multiple choice: Equivalent or not?

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Quasiorder.

Consider reducts Γ, Γ' equivalent iff $\Gamma \leq \Gamma'$ and $\Gamma' \leq \Gamma$.

Factoring out we get a complete lattice.

Multiple choice: Equivalent or not?

- $(\mathbb{Q}; <)$ and $(\mathbb{Q}; >)$

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Quasiorder.

Consider reducts Γ, Γ' equivalent iff $\Gamma \leq \Gamma'$ and $\Gamma' \leq \Gamma$.

Factoring out we get a complete lattice.

Multiple choice: Equivalent or not?

- $(\mathbb{Q}; <)$ and $(\mathbb{Q}; >)$
- $(\mathbb{Q}; <)$ and $(\mathbb{Q}; \text{Between}(x, y, z))$

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Quasiorder.

Consider reducts Γ, Γ' equivalent iff $\Gamma \leq \Gamma'$ and $\Gamma' \leq \Gamma$.

Factoring out we get a complete lattice.

Multiple choice: Equivalent or not?

- $(\mathbb{Q}; <)$ and $(\mathbb{Q}; >)$
- $(\mathbb{Q}; <)$ and $(\mathbb{Q}; \text{Between}(x, y, z))$
- random poset $(P; \leq)$ and $(P; \perp(x, y))$

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Quasiorder.

Consider reducts Γ, Γ' equivalent iff $\Gamma \leq \Gamma'$ and $\Gamma' \leq \Gamma$.

Factoring out we get a complete lattice.

Multiple choice: Equivalent or not?

- $(\mathbb{Q}; <)$ and $(\mathbb{Q}; >)$
- $(\mathbb{Q}; <)$ and $(\mathbb{Q}; \text{Between}(x, y, z))$
- random poset $(P; \leq)$ and $(P; \perp(x, y))$
- random graph $(V; E)$ and $(V; K_3(x, y, z))$

Reducts up to first-order equivalence

For reducts Γ, Γ' of Δ set $\Gamma \leq \Gamma'$ iff Γ is a reduct of Γ'.

Quasiorder.

Consider reducts Γ, Γ' equivalent iff $\Gamma \leq \Gamma'$ and $\Gamma' \leq \Gamma$.

Factoring out we get a complete lattice.

Multiple choice: Equivalent or not?

- $(\mathbb{Q}; <)$ and $(\mathbb{Q}; >)$
- $(\mathbb{Q}; <)$ and $(\mathbb{Q}; \text{Between}(x, y, z))$
- random poset $(P; \leq)$ and $(P; \perp(x, y))$
- random graph $(V; E)$ and $(V; K_3(x, y, z))$

Question

How many inequivalent reducts?

Examples

Examples

- $(\mathbb{Q}; <)$: 5 (Cameron '76)

Examples

- $(\mathbb{Q};<)$: 5 (Cameron '76)
- random graph $(V;E)$: 5 (Thomas '91)

Examples

- $(\mathbb{Q}; <)$: 5 (Cameron ’76)
- random graph $(V; E)$: 5 (Thomas ’91)
- random k-hypergraph: $2^k + 1$ (Thomas ’96)

Examples

- $(\mathbb{Q};<)$: 5 (Cameron '76)
- random graph $(V;E)$: 5 (Thomas '91)
- random k-hypergraph: 2^k+1 (Thomas '96)
- random tournament: 5 (Bennett '97)

Examples

- $(\mathbb{Q}; <)$: 5 (Cameron '76)
- random graph $(V; E)$: 5 (Thomas '91)
- random k-hypergraph: $2^k + 1$ (Thomas '96)
- random tournament: 5 (Bennett '97)
- $(\mathbb{Q}; <, 0)$: 116 (Junker+Ziegler '08)

Examples

- $(\mathbb{Q};<)$: 5 (Cameron '76)
- random graph $(V;E)$: 5 (Thomas '91)
- random k-hypergraph: $2^k + 1$ (Thomas '96)
- random tournament: 5 (Bennett '97)
- $(\mathbb{Q};<,0)$: 116 (Junker+Ziegler '08)
- random partial order: 5 (Pach+MP+Pongrácz+Szabó '11)

Examples

- $(\mathbb{Q};<)$: 5 (Cameron '76)
- random graph $(V;E)$: 5 (Thomas '91)
- random k-hypergraph: 2^k+1 (Thomas '96)
- random tournament: 5 (Bennett '97)
- $(\mathbb{Q};<,0)$: 116 (Junker+Ziegler '08)
- random partial order: 5 (Pach+MP+Pongrácz+Szabó '11)

Conjecture (Thomas '91)

Homogeneous structures in finite relational language have finitely many reducts.

Permutation groups

A permutation group is closed $:\leftrightarrow$
it contains all permutations which it can interpolate on finite subsets.

Permutation groups

A permutation group is closed $:\leftrightarrow$
it contains all permutations which it can interpolate on finite subsets.

Theorem (Corollary of Ryll-Nardzewski, Engeler, Svenonius)

Let Δ be homogeneous in a finite relational language.

Then the mapping

$$\Gamma \mapsto \mathrm{Aut}(\Gamma)$$

Permutation groups

A permutation group is closed $:\leftrightarrow$
it contains all permutations which it can interpolate on finite subsets.

Theorem (Corollary of Ryll-Nardzewski, Engeler, Svenonius)

Let Δ be homogeneous in a finite relational language.

Then the mapping

$$\Gamma \mapsto \mathrm{Aut}(\Gamma)$$

is an anti-isomorphism
from the lattice of reducts
to the lattice of closed supergroups of $\mathrm{Aut}(\Delta)$.

The rationals $(\mathbb{Q}; <)$

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

Theorem (Cameron '76)

The closed supergroups of $\mathrm{Aut}(\mathbb{Q}; <)$ are precisely:

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

Theorem (Cameron '76)

The closed supergroups of $\mathrm{Aut}(\mathbb{Q}; <)$ are precisely:

- $\mathrm{Aut}(\mathbb{Q}; <)$

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

Theorem (Cameron '76)

The closed supergroups of $\text{Aut}(\mathbb{Q}; <)$ are precisely:

- $\text{Aut}(\mathbb{Q}; <)$
- $\langle \{\leftrightarrow\} \cup \text{Aut}(\mathbb{Q}; <) \rangle$

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

Theorem (Cameron '76)

The closed supergroups of Aut$(\mathbb{Q}; <)$ are precisely:

- Aut$(\mathbb{Q}; <)$
- $\langle \{\leftrightarrow\} \cup \text{Aut}(\mathbb{Q}; <) \rangle$
- $\langle \{\circlearrowleft\} \cup \text{Aut}(\mathbb{Q}; <) \rangle$

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

Theorem (Cameron '76)

The closed supergroups of Aut$(\mathbb{Q}; <)$ are precisely:

- Aut$(\mathbb{Q}; <)$
- $\langle \{\leftrightarrow\} \cup \text{Aut}(\mathbb{Q}; <)\rangle$
- $\langle \{\circlearrowleft\} \cup \text{Aut}(\mathbb{Q}; <)\rangle$
- $\langle \{\leftrightarrow, \circlearrowleft\} \cup \text{Aut}(\mathbb{Q}; <)\rangle$

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

Theorem (Cameron '76)

The closed supergroups of $\mathrm{Aut}(\mathbb{Q}; <)$ are precisely:

- $\mathrm{Aut}(\mathbb{Q}; <)$
- $\langle \{\leftrightarrow\} \cup \mathrm{Aut}(\mathbb{Q}; <) \rangle$
- $\langle \{\circlearrowleft\} \cup \mathrm{Aut}(\mathbb{Q}; <) \rangle$
- $\langle \{\leftrightarrow, \circlearrowleft\} \cup \mathrm{Aut}(\mathbb{Q}; <) \rangle$
- $\mathrm{Sym}(\mathbb{Q})$

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

Theorem (Cameron '76)

The closed supergroups of $\mathrm{Aut}(\mathbb{Q}; <)$ are precisely:

- $\mathrm{Aut}(\mathbb{Q}; <)$
- $\langle \{\leftrightarrow\} \cup \mathrm{Aut}(\mathbb{Q}; <) \rangle$ = $\mathrm{Aut}(\mathbb{Q}; \mathrm{Between}(x, y, z))$
- $\langle \{\circlearrowleft\} \cup \mathrm{Aut}(\mathbb{Q}; <) \rangle$
- $\langle \{\leftrightarrow, \circlearrowleft\} \cup \mathrm{Aut}(\mathbb{Q}; <) \rangle$
- $\mathrm{Sym}(\mathbb{Q})$

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

Theorem (Cameron '76)

The closed supergroups of $\mathrm{Aut}(\mathbb{Q}; <)$ are precisely:

- $\mathrm{Aut}(\mathbb{Q}; <)$
- $\langle\{\leftrightarrow\} \cup \mathrm{Aut}(\mathbb{Q}; <)\rangle = \mathrm{Aut}(\mathbb{Q}; \mathrm{Between}(x, y, z))$
- $\langle\{\circlearrowleft\} \cup \mathrm{Aut}(\mathbb{Q}; <)\rangle = \mathrm{Aut}(\mathbb{Q}; \mathrm{Cyclic}(x, y, z))$
- $\langle\{\leftrightarrow, \circlearrowleft\} \cup \mathrm{Aut}(\mathbb{Q}; <)\rangle$
- $\mathrm{Sym}(\mathbb{Q})$

The rationals $(\mathbb{Q}; <)$

Let $\leftrightarrow$ be any permutation of $\mathbb{Q}$ which reverses the order.

Let $\circlearrowleft$ be any permutation of $\mathbb{Q}$ which for some irrational π puts $(-\infty; \pi)$ behind $(\pi; \infty)$ and preserves the order otherwise.

Theorem (Cameron '76)

The closed supergroups of $\mathrm{Aut}(\mathbb{Q}; <)$ are precisely:

- $\mathrm{Aut}(\mathbb{Q}; <)$
- $\langle\{\leftrightarrow\} \cup \mathrm{Aut}(\mathbb{Q}; <)\rangle = \mathrm{Aut}(\mathbb{Q}; \mathrm{Between}(x, y, z))$
- $\langle\{\circlearrowleft\} \cup \mathrm{Aut}(\mathbb{Q}; <)\rangle = \mathrm{Aut}(\mathbb{Q}; \mathrm{Cyclic}(x, y, z))$
- $\langle\{\leftrightarrow, \circlearrowleft\} \cup \mathrm{Aut}(\mathbb{Q}; <)\rangle = \mathrm{Aut}(\mathbb{Q}; \mathrm{Separate}(x, y, u, v))$
- $\mathrm{Sym}(\mathbb{Q})$

The random graph $(V; E)$

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$
switches edges and non-edges between A and $V \setminus A$
and preserves the graph relation on A and $V \setminus A$.

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$
switches edges and non-edges between A and $V \setminus A$
and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$ switches edges and non-edges between A and $V \setminus A$ and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

- $\mathrm{Aut}(V; E)$

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$
switches edges and non-edges between A and $V \setminus A$
and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

- $\mathrm{Aut}(V; E)$
- $\langle \{\mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$
switches edges and non-edges between A and $V \setminus A$
and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

- $\mathrm{Aut}(V; E)$
- $\langle \{\mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$
- $\langle \{-\} \cup \mathrm{Aut}(V; E) \rangle$

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$
switches edges and non-edges between A and $V \setminus A$
and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

- $\mathrm{Aut}(V; E)$
- $\langle \{\mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$
- $\langle \{-\} \cup \mathrm{Aut}(V; E) \rangle$
- $\langle \{-, \mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$
switches edges and non-edges between A and $V \setminus A$
and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

- $\mathrm{Aut}(V; E)$
- $\langle \{\mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$
- $\langle \{-\} \cup \mathrm{Aut}(V; E) \rangle$
- $\langle \{-, \mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$
- $\mathrm{Sym}(V)$

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$ switches edges and non-edges between A and $V \setminus A$ and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

- $\mathrm{Aut}(V; E)$
- $\langle \{\mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$
- $\langle \{-\} \cup \mathrm{Aut}(V; E) \rangle$
- $\langle \{-, \mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$
- $\mathrm{Sym}(V)$

For $k \geq 1$, let $R^{(k)}$ consist of the k-tuples of distinct elements of V which induce an odd number of edges.

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$
switches edges and non-edges between A and $V \setminus A$
and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

- $\mathrm{Aut}(V; E)$
- $\langle \{\mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle = \mathrm{Aut}(V; R^{(3)})$
- $\langle \{-\} \cup \mathrm{Aut}(V; E) \rangle$
- $\langle \{-, \mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$
- $\mathrm{Sym}(V)$

For $k \geq 1$, let $R^{(k)}$ consist of the k-tuples of distinct elements of V which induce an odd number of edges.

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$ switches edges and non-edges between A and $V \setminus A$ and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

- $\mathrm{Aut}(V; E)$
- $\langle \{\mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle \;\; = \mathrm{Aut}(V; R^{(3)})$
- $\langle \{-\} \cup \mathrm{Aut}(V; E) \rangle \;\; = \mathrm{Aut}(V; R^{(4)})$
- $\langle \{-, \mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle$
- $\mathrm{Sym}(V)$

For $k \geq 1$, let $R^{(k)}$ consist of the k-tuples of distinct elements of V which induce an odd number of edges.

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

Let sw be any permutation which for some finite $A \subseteq V$ switches edges and non-edges between A and $V \setminus A$ and preserves the graph relation on A and $V \setminus A$.

Theorem (Thomas '91)

The closed supergroups of $\mathrm{Aut}(V; E)$ are precisely:

- $\mathrm{Aut}(V; E)$
- $\langle \{\mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle = \mathrm{Aut}(V; R^{(3)})$
- $\langle \{-\} \cup \mathrm{Aut}(V; E) \rangle = \mathrm{Aut}(V; R^{(4)})$
- $\langle \{-, \mathrm{sw}\} \cup \mathrm{Aut}(V; E) \rangle = \mathrm{Aut}(V; R^{(5)})$
- $\mathrm{Sym}(V)$

For $k \geq 1$, let $R^{(k)}$ consist of the k-tuples of distinct elements of V which induce an odd number of edges.

Part II: The 42 reducts of the random ordered graph

The random ordered graph

Definition

The random ordered graph $(D; <, E)$ is
the unique countable linearly ordered graph which

- contains all finite linearly ordered graphs
- is homogeneous.

The random ordered graph

Definition

The random ordered graph $(D; <, E)$ is
the unique countable linearly ordered graph which

- contains all finite linearly ordered graphs
- is homogeneous.

Observation

- $(D; <)$ is the order of the rationals
- $(D; E)$ is the random graph

The random ordered graph

Definition

The random ordered graph $(D; <, E)$ is
the unique countable linearly ordered graph which

- contains all finite linearly ordered graphs
- is homogeneous.

Observation

- $(D; <)$ is the order of the rationals
- $(D; E)$ is the random graph

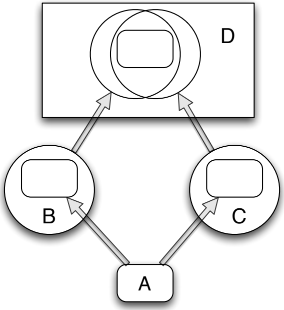
This is because the two structures are superposed freely, i.e.,
in all possible ways.

Strong amalgamation

Strong amalgamation

Definition

A class $\mathcal{C}$ has strong amalgamation $:\leftrightarrow$
for all $A, B, C \in \mathcal{C}$ and embeddings $e_B : A \to B$ and $e_C : A \to C$
there is $D \in \mathcal{C}$ and embeddings $f_B : B \to D$ and $f_C : C \to D$
such that $f_B \circ e_B = f_C \circ e_C$
and $f_B[B] \cap f_C[C] = f_B \circ e_B[A]$.



Mixing

Mixing

Let τ_1, τ_2 be disjoint languages.

Let $\mathcal{C}_1, \mathcal{C}_2$ Fraïssé classes in those languages, Δ_1, Δ_2 be their limits.

Mixing

Let τ_1, τ_2 be disjoint languages.

Let $\mathcal{C}_1, \mathcal{C}_2$ Fraïssé classes in those languages, Δ_1, Δ_2 be their limits.

Free superposition

Assume that $\mathcal{C}_1, \mathcal{C}_2$ have strong amalgamation.

Mixing

Let τ_1, τ_2 be disjoint languages.

Let $\mathcal{C}_1, \mathcal{C}_2$ Fraïssé classes in those languages, Δ_1, Δ_2 be their limits.

Free superposition

Assume that $\mathcal{C}_1, \mathcal{C}_2$ have strong amalgamation.

Then the class $\mathcal{C}$ of $\tau_1 \cup \tau_2$-structures whose τ_i-reduct is in $\mathcal{C}_i$

Mixing

Let τ_1, τ_2 be disjoint languages.

Let $\mathcal{C}_1, \mathcal{C}_2$ Fraïssé classes in those languages, Δ_1, Δ_2 be their limits.

Free superposition

Assume that $\mathcal{C}_1, \mathcal{C}_2$ have strong amalgamation.

Then the class $\mathcal{C}$ of $\tau_1 \cup \tau_2$-structures whose τ_i-reduct is in $\mathcal{C}_i$

- is a Fraïssé class and
- the τ_i-reduct of its limit is isomorphic to Δ_i.

Trivial reducts of the random ordered graph

Trivial reducts of the random ordered graph

- Every reduct of $(D; <)$ is a reduct of the random ordered graph.

Trivial reducts of the random ordered graph

- Every reduct of $(D; <)$ is a reduct of the random ordered graph.
- Every reduct of $(D; E)$ is a reduct of the random ordered graph.

Trivial reducts of the random ordered graph

- Every reduct of $(D; <)$ is a reduct of the random ordered graph.
- Every reduct of $(D; E)$ is a reduct of the random ordered graph.
- If $(D; R)$ is a reduct of $(D; <)$
  and $(D; S)$ is a reduct of $(D; E)$
  then $(D; R, S)$ is a reduct of the random ordered graph.

Trivial reducts of the random ordered graph

- Every reduct of $(D; <)$ is a reduct of the random ordered graph.
- Every reduct of $(D; E)$ is a reduct of the random ordered graph.
- If $(D; R)$ is a reduct of $(D; <)$
  and $(D; S)$ is a reduct of $(D; E)$
  then $(D; R, S)$ is a reduct of the random ordered graph.

Corresponds to intersecting the groups $\mathrm{Aut}(D; R)$ and $\mathrm{Aut}(D; S)$.

Trivial reducts of the random ordered graph

- Every reduct of $(D; <)$ is a reduct of the random ordered graph.
- Every reduct of $(D; E)$ is a reduct of the random ordered graph.
- If $(D; R)$ is a reduct of $(D; <)$
  and $(D; S)$ is a reduct of $(D; E)$
  then $(D; R, S)$ is a reduct of the random ordered graph.

Corresponds to intersecting the groups $\mathrm{Aut}(D; R)$ and $\mathrm{Aut}(D; S)$.

Yields distinct reducts because of free superposition.

Trivial reducts of the random ordered graph

- Every reduct of $(D; <)$ is a reduct of the random ordered graph.
- Every reduct of $(D; E)$ is a reduct of the random ordered graph.
- If $(D; R)$ is a reduct of $(D; <)$
  and $(D; S)$ is a reduct of $(D; E)$
  then $(D; R, S)$ is a reduct of the random ordered graph.

Corresponds to intersecting the groups $\mathrm{Aut}(D; R)$ and $\mathrm{Aut}(D; S)$.

Yields distinct reducts because of free superposition.

Examples

Trivial reducts of the random ordered graph

- Every reduct of $(D; <)$ is a reduct of the random ordered graph.
- Every reduct of $(D; E)$ is a reduct of the random ordered graph.
- If $(D; R)$ is a reduct of $(D; <)$
  and $(D; S)$ is a reduct of $(D; E)$
  then $(D; R, S)$ is a reduct of the random ordered graph.

Corresponds to intersecting the groups $\mathrm{Aut}(D; R)$ and $\mathrm{Aut}(D; S)$.

Yields distinct reducts because of free superposition.

Examples

- Keeping the order while flipping the graph relation.

Trivial reducts of the random ordered graph

- Every reduct of $(D; <)$ is a reduct of the random ordered graph.
- Every reduct of $(D; E)$ is a reduct of the random ordered graph.
- If $(D; R)$ is a reduct of $(D; <)$
  and $(D; S)$ is a reduct of $(D; E)$
  then $(D; R, S)$ is a reduct of the random ordered graph.

Corresponds to intersecting the groups $\mathrm{Aut}(D; R)$ and $\mathrm{Aut}(D; S)$.

Yields distinct reducts because of free superposition.

Examples

- Keeping the order while flipping the graph relation.
- Reversing the order while keeping the graph relation.

Trivial reducts of the random ordered graph

- Every reduct of $(D; <)$ is a reduct of the random ordered graph.
- Every reduct of $(D; E)$ is a reduct of the random ordered graph.
- If $(D; R)$ is a reduct of $(D; <)$
  and $(D; S)$ is a reduct of $(D; E)$
  then $(D; R, S)$ is a reduct of the random ordered graph.

Corresponds to intersecting the groups $\mathrm{Aut}(D; R)$ and $\mathrm{Aut}(D; S)$.

Yields distinct reducts because of free superposition.

Examples

- Keeping the order while flipping the graph relation.
- Reversing the order while keeping the graph relation.

Lemma

The random ordered graph has at least 25 reducts.

Similarities between the order and the graph reducts

Similarities between the order and the graph reducts

The following permutations yield new non-trivial reducts.

Similarities between the order and the graph reducts

The following permutations yield new non-trivial reducts.

- reversing the order and simultaneously flipping the graph relation

Similarities between the order and the graph reducts

The following permutations yield new non-trivial reducts.

- reversing the order and simultaneously flipping the graph relation
- for an irrational π, put $(-\infty, \pi)$ behind (π, ∞) whilst flipping the graph relation between these parts.

Similarities between the order and the graph reducts

The following permutations yield new non-trivial reducts.

- reversing the order and simultaneously flipping the graph relation
- for an irrational π, put $(-\infty, \pi)$ behind (π, ∞) whilst flipping the graph relation between these parts.

No other combination of this kind!

Similarities between the order and the graph reducts

The following permutations yield new non-trivial reducts.

- reversing the order and simultaneously flipping the graph relation
- for an irrational π, put $(-\infty, \pi)$ behind (π, ∞) whilst flipping the graph relation between these parts.

No other combination of this kind!

Lemma

The random ordered graph has at least 27 reducts.

Hello random tournament!

Hello random tournament!

Definition

A tournament is a digraph with precisely one edge between any two vertices.

Hello random tournament!

Definition

A tournament is a digraph with precisely one edge between any two vertices.

Theorem (Bennett '97)

The random tournament has 5 reducts.

Hello random tournament!

Definition

A tournament is a digraph with precisely one edge between any two vertices.

Theorem (Bennett '97)

The random tournament has 5 reducts.

Observation

Set $T(x,y)$ iff $x < y \wedge E(x,y)$ or $x > y \wedge N(x,y)$.

Then $(D; T)$ is the random tournament.

Hello random tournament!

Definition

A tournament is a digraph with precisely one edge between any two vertices.

Theorem (Bennett '97)

The random tournament has 5 reducts.

Observation

Set $T(x, y)$ iff $x < y \wedge E(x, y)$ or $x > y \wedge N(x, y)$.

Then $(D; T)$ is the random tournament.

Lemma

The random ordered graph has at least 32 reducts.

Finally, some asymmetry

Finally, some asymmetry

The following permutations yield new non-trivial reducts.

Finally, some asymmetry

The following permutations yield new non-trivial reducts.

- preserving the order
  whilst flipping the graph relation below some irrational.

Finally, some asymmetry

The following permutations yield new non-trivial reducts.

- preserving the order
  whilst flipping the graph relation below some irrational.
- preserving the order
  whilst flipping the graph relation above some irrational.

Finally, some asymmetry

The following permutations yield new non-trivial reducts.

- preserving the order
  whilst flipping the graph relation below some irrational.
- preserving the order
  whilst flipping the graph relation above some irrational.

There are no "dual" permutations of these.

Finally, some asymmetry

The following permutations yield new non-trivial reducts.

- preserving the order
  whilst flipping the graph relation below some irrational.
- preserving the order
  whilst flipping the graph relation above some irrational.

There are no “dual” permutations of these.

Lemma

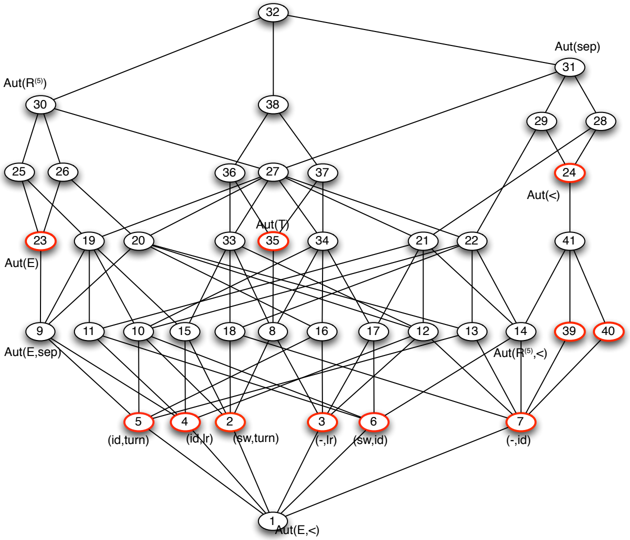
The random ordered graph has at least 32 reducts.

Theorem (Bodirsky+MP+Pongrácz '13)

The random ordered graph has 41 reducts.

Theorem (Bodirsky+MP+Pongrácz '13)

The random ordered graph has 41 reducts.



Part III: The effect of The Answer

Discussion

Discussion

We have learnt from the result:

We have learnt from the result:

- similarities between the symmetries of the order and the graph

Discussion

We have learnt from the result:

- similarities between the symmetries of the order and the graph
- nonetheless their combination yields an asymmetry

Discussion

We have learnt from the result:

- similarities between the symmetries of the order and the graph
- nonetheless their combination yields an asymmetry
- we cannot calculate the reducts of a superposed structure from its factors

Discussion

We have learnt from the result:

- similarities between the symmetries of the order and the graph
- nonetheless their combination yields an asymmetry
- we cannot calculate the reducts of a superposed structure from its factors

On a technical level:

Discussion

We have learnt from the result:

- similarities between the symmetries of the order and the graph
- nonetheless their combination yields an asymmetry
- we cannot calculate the reducts of a superposed structure from its factors

On a technical level:

- our Ramsey-theoretic method is quite efficient (first classification of free superposition)

We have learnt from the result:

- similarities between the symmetries of the order and the graph
- nonetheless their combination yields an asymmetry
- we cannot calculate the reducts of a superposed structure from its factors

On a technical level:

- our Ramsey-theoretic method is quite efficient (first classification of free superposition)
- improved it to reduce work to the join irreducible elements

Discussion

We have learnt from the result:

- similarities between the symmetries of the order and the graph
- nonetheless their combination yields an asymmetry
- we cannot calculate the reducts of a superposed structure from its factors

On a technical level:

- our Ramsey-theoretic method is quite efficient (first classification of free superposition)
- improved it to reduce work to the join irreducible elements
- our method is not sporadic (same for order, graph, tournament)

Ramsey structures

Ramsey structures

Definition (Ramsey structure Δ)

Ramsey structures

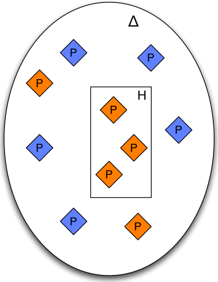
Definition (Ramsey structure Δ)

For all finite substructures P, H of Δ:
Whenever we color the copies of P in Δ with 2 colors
then there is a monochromatic copy of H in Δ.

Ramsey structures

Definition (Ramsey structure Δ)

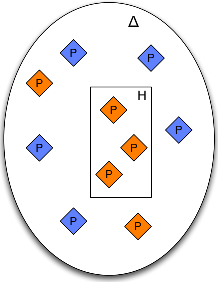
For all finite substructures P, H of Δ:
Whenever we color the copies of P in Δ with 2 colors
then there is a monochromatic copy of H in Δ.



Ramsey structures

Definition (Ramsey structure Δ)

For all finite substructures P, H of Δ:
Whenever we color the copies of P in Δ with 2 colors
then there is a monochromatic copy of H in Δ.



Theorem (Nešetřil-Rödl)

The random ordered graph is Ramsey.

Canonical functions

Canonical functions

Definition

Let Δ, Λ be structures.

$f : \Delta \to \Lambda$ is canonical iff

for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type in Δ

$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type in Λ.

Canonical functions

Definition

Let Δ, Λ be structures.

$f : \Delta \to \Lambda$ is canonical iff

for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type in Δ

$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type in Λ.

Examples on $(D; <, E)$

Canonical functions

Definition

Let Δ, Λ be structures.

$f : \Delta \to \Lambda$ is canonical iff

for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type in Δ

$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type in Λ.

Examples on $(D; <, E)$

- self-embeddings

Canonical functions

Definition

Let Δ, Λ be structures.

$f : \Delta \to \Lambda$ is canonical iff

for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type in Δ

$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type in Λ.

Examples on $(D; <, E)$

- self-embeddings
- reversing $<$, preserving edges and non-edges

Canonical functions

Definition

Let Δ, Λ be structures.

$f : \Delta \to \Lambda$ is canonical iff

for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type in Δ

$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type in Λ.

Examples on $(D; <, E)$

- self-embeddings
- reversing $<$, preserving edges and non-edges
- preserving $<$, flipping edges and non-edges

Canonical functions

Definition

Let Δ, Λ be structures.

$f : \Delta \to \Lambda$ is canonical iff

for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type in Δ

$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type in Λ.

Examples on $(D; <, E)$

- self-embeddings
- reversing $<$, preserving edges and non-edges
- preserving $<$, flipping edges and non-edges
- preserving $<$, send to clique

Canonizing functions on Ramsey structures

Canonizing functions on Ramsey structures

Magical proposition (Bodirsky+MP+Tsankov '11)

Let

- Δ is ordered Ramsey homogeneous finite language
- $f : \Delta \to \Delta$
- $c_1, \ldots, c_n \in \Delta$.

Canonizing functions on Ramsey structures

Magical proposition (Bodirsky+MP+Tsankov '11)

Let

- Δ is ordered Ramsey homogeneous finite language
- $f : \Delta \to \Delta$
- $c_1, \ldots, c_n \in \Delta$.

Then the closed monoid generated by $\{f\} \cup \mathrm{Aut}(\Delta)$ contains a function g which

Canonizing functions on Ramsey structures

Magical proposition (Bodirsky+MP+Tsankov '11)

Let

- Δ is ordered Ramsey homogeneous finite language
- $f : \Delta \to \Delta$
- $c_1, \ldots, c_n \in \Delta$.

Then the closed monoid generated by $\{f\} \cup \mathrm{Aut}(\Delta)$ contains a function g which

- is canonical as a function from $(\Delta, c_1, \ldots, c_n)$ to Δ
- is identical with f on $\{c_1, \ldots, c_n\}$.

Canonizing functions on Ramsey structures

Magical proposition (Bodirsky+MP+Tsankov '11)

Let

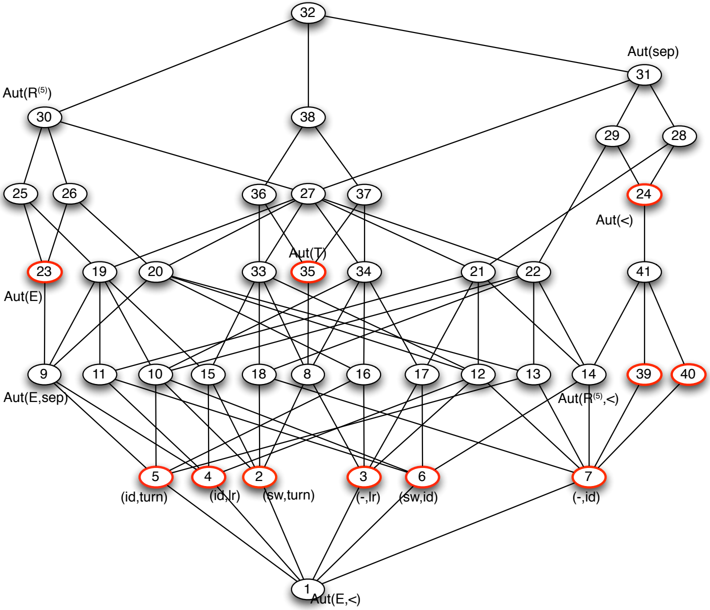
- Δ is ordered Ramsey homogeneous finite language
- $f : \Delta \to \Delta$
- $c_1, \ldots, c_n \in \Delta$.

Then the closed monoid generated by $\{f\} \cup \mathrm{Aut}(\Delta)$ contains a function g which

- is canonical as a function from $(\Delta, c_1, \ldots, c_n)$ to Δ
- is identical with f on $\{c_1, \ldots, c_n\}$.

Note:

- only finitely many different behaviors of canonical functions.
- g, g' same behavior $\to$ generate one another (with $\mathrm{Aut}(\Delta)$).


32
Aut(sep)
31
Aut(R(5))
30
38
29
28
25
26
36
27
37
24
Aut(<)
Aut(T)
23
19
20
33
35
34
21
22
41
Aut(E)
9
11
10
15
18
8
16
17
12
13
14
39
40
Aut(E,sep)
Aut(R(5),<)
5
4
2
3
6
7
(id,turn)
(id,lr)
(sw,turn)
(-,lr)
(sw,id)
(-,id)
1
Aut(E,<)

Part IV: The Question to The Answer

The Question

The Question

Problem

Suppose that Δ_1, Δ_2 have finitely many reducts.

Does their free superposition have finitely many reducts?

The Question

Problem

Suppose that Δ_1, Δ_2 have finitely many reducts.

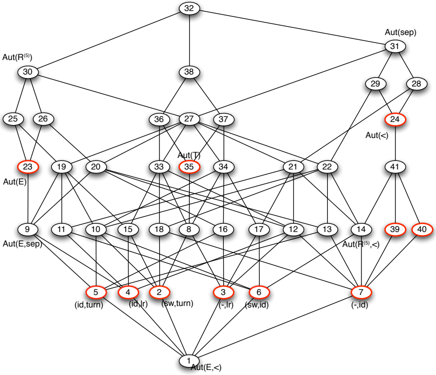
Does their free superposition have finitely many reducts?

Problem

Suppose that Δ is homogeneous in a finite relational language.

Does it have a finite homogeneous extension which is Ramsey?

Thank you!



"The Answer to the Great Question. . .
Of Life, the Universe and Everything. . . Is. . . Forty-two,"
said Deep Thought, with infinite majesty and calm.

Douglas Adams