Constraint Satisfaction on Infinite Domains

1st session

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Technische Universität Wien / Université Diderot - Paris 7

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Algebraic and Model Theoretical Methods in Constraint Satisfaction
Banff International Research Station

2014

Outline

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Part I: CSPs / dividing the world /
pp definitions, polymorphism clones, ω-categoricity

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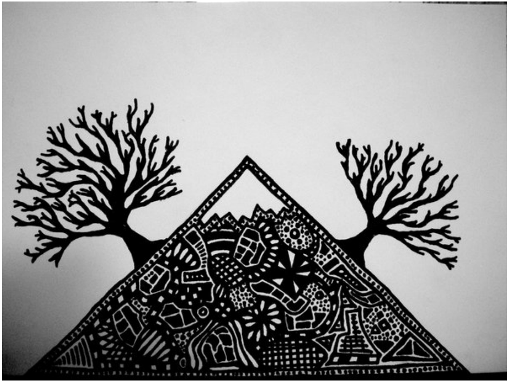
Building new dimension out of two smaller



Part I:

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Irrelevant whether Γ is finite or infinite. But language finite.

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HOM(Γ) and CSP(Γ) are equivalent.

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Question: Is $(D; E)$ acyclic?

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Question: Is there a linear order on the variables such that for each triple (x, y, z) either $x < y < z$ or $z < y < x$?

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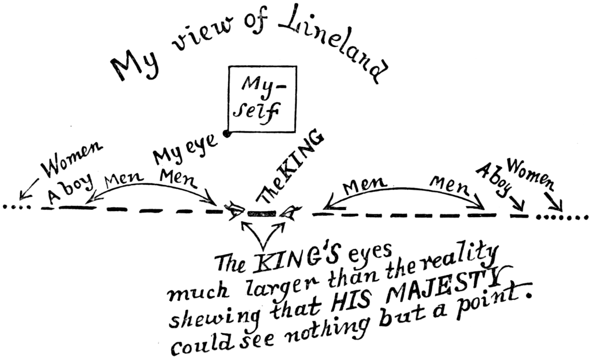

Dividing the world

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Computational problem: Graph-SAT(Ψ)

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- A finite set W of variables (vertices), and
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Γ_Ψ is a reduct of the random graph, i.e.,
a structure with a first-order definition in G.

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An instance

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Graph-SAT problems $\leftrightarrow$ CSPs of reducts of the random graph.

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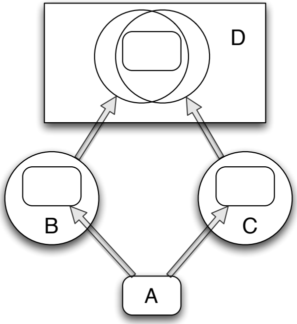
The class of finite graphs has amalgamation.



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- Henson digraphs (forbidden tournaments)

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Boolean-SAT problems $\leftrightarrow$ CSPs of structures on $\{0,1\}$.

Dividing the world

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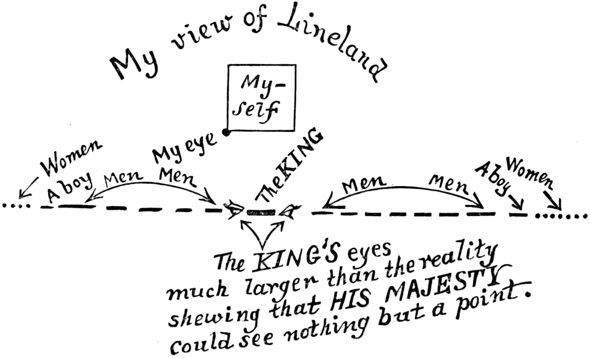
- $\mathcal{C}$-SAT: If Fraïssé class $\mathcal{C}$ is decidable (in NP), then $\mathcal{C}$-SAT is decidable (in NP)
- CSPs are classes of finite τ-structures closed under inverse homomorphic images and unions



pp definitions, polymorphism clones, ω-categoricity

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Observation (Bulatov + Krokhin + Jeavons '00)

Expanding Γ by pp definable relations
increases the complexity of the CSP by at most polynomial-time.

Polymorphism clones

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Observe: $\text{Pol}(\Gamma) \supseteq \text{End}(\Gamma) \supseteq \text{Aut}(\Gamma)$.

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Let Γ be a countable ω-categorical structure.

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Corollary

Let Γ be ω-categorical.

If Pol(Γ) $\subseteq$ Pol(Γ'),
then CSP(Γ') is polynomial-time reducible to CSP(Γ).

ω-categoricity

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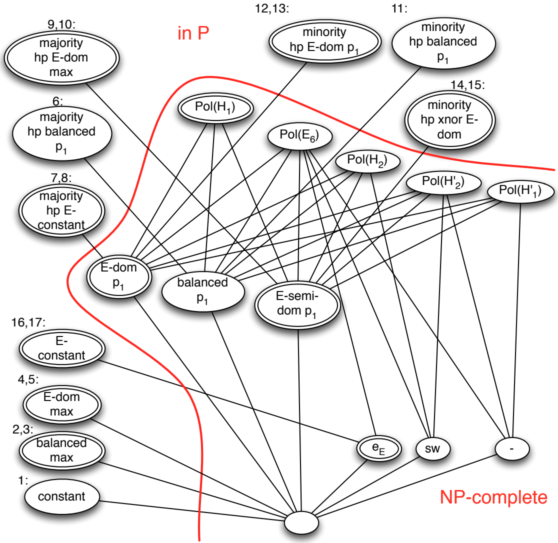
Theorem (Ryll-Nardzewski, Engeler, Svenonius)

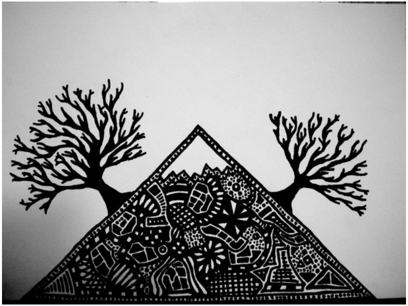
Let Γ be countable. TFAE:

- Aut(Γ) is oligomorphic;
- Γ is ω-categorical: the only countable model of its theory.

Graph-SAT classification



I call our world Flatland,
not because we call it so,
but to make its nature clearer to you, my happy readers,
who are privileged to live in Space.



2nd session: 14:00