Clones on Ramsey structures

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The Clowns Day

ALC Fest 2014

Clowns in the ’00s

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The clown lattice

$X\dots$ infinite base set of size $\aleph_\alpha$.
$\mathcal{O}^{(n)} = X^{X^n} = \{f : X^n \to X\}\dots$ n-ary functions on X.
$\mathcal{O} = \bigcup_{n\geq 1} \mathcal{O}^{(n)}\dots$ finitary operations on X.

Outline

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3 Conjectures

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- Homogeneous structures and Ramsey structures

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- Function clones / CSPs on Ramsey structures

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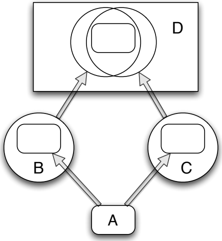
Theorem (Fraïssé)

Homogeneous relational structures $\leftrightarrow$
Fraïssé classes of finite structures.



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- Binary trees $(T; \leq, xy|z)$ $\leftrightarrow$ random tree

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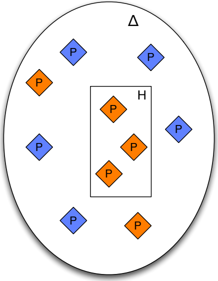

Ramsey expansions

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Example: Graphs $\rightarrow$ Ordered graphs

Variant: Finitely bounded Fraïssé classes
(given by finitely many forbidden substructures)

Clones on Ramsey structures

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- Automorphism groups $\supseteq \mathrm{Aut}(\Delta)$: Symmetries of the Fraïssé class
- Polymorphism clones $\supseteq \mathrm{Aut}(\Delta)$: CSPs over the Fraïssé class ("Is there a graph such that...?")

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Conjecture (Thomas '91)

Fraïssé classes in finite language have finitely many symmetries.

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- re-rooting binary trees with a distinguished branch

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Let $\mathrm{Pol}(\Gamma) \supseteq \mathrm{Aut}(\Delta)$, where Γ has finite language.

Then there exist canonical functions $f_1, \ldots, f_m$ outside $\mathrm{Pol}(\Gamma)$ such that any polymorphism clone on Δ not contained in $\mathrm{Pol}(\Gamma)$ contains an f_i.

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Kerrie Warren, *Composing chaos* (*Patterns in chaos*)