The algebraic dichotomy conjecture for infinite domain CSPs

Libor Barto [0]   **Michael Pinsker** [∞]

[0] Univerzita Karlova v Praze

[∞] Univerzita Karlova v Praze + Technische Universität Wien

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Constraint Satisfaction Problems (CSPs)

Let $\Gamma = (D; R_1, \ldots, R_n)$ be a relational structure (finite signature).

Definition CSP(Γ)

INPUT: A primitive positive sentence

$$\phi \equiv \exists x_1 \cdots \exists x_n \; R_{i_1}(\ldots) \wedge \cdots \wedge R_{i_m}(\ldots)$$

QUESTION: $\Gamma \models \phi$???

- Γ (i.e., its domain) can be finite or infinite.
- Γ finite $\Longrightarrow$ CSP(Γ) in NP.
- Γ infinite $\Longrightarrow$ CSP(Γ) can be anything.
- Γ finite $\Longrightarrow$ “algebraic approach".
- Γ infinite + ω-categorical $\Longrightarrow$ “algebraic-topological approach".

Γ is ω-categorical: countable and $\Gamma^n/\mathrm{Aut}(\Gamma)$ is finite for all $n \geq 1$.

Our result: **Topology is irrelevant.**

Thank you!

The algebraic-topological approach: clones

Computational problem

$\downarrow$

CSP(Γ)

$\downarrow$

$$\mathrm{Pol}(\Gamma) = \{h\colon \Gamma^n \to \Gamma \mid n \geq 1,\ h \text{ homomorphism}\}$$

The polymorphism clone Pol(Γ) has algebraic structure:

- composition of functions
- projections $\pi_i^n(x_1, \ldots, x_n) = x_i$

Pol(Γ) has topology: $(f_i)_{i\in\omega} \to f$ iff $f_i(\overline{c}) = f(\overline{c})$ for all $\overline{c}$, eventually.

Theorem (for ω-categorical Γ) (by Bodirsky + P)

$$\mathrm{Pol}(\Gamma) \cong \mathrm{Pol}(\Delta) \implies \mathrm{CSP}(\Gamma) \sim_{polytime} \mathrm{CSP}(\Delta).$$

Pol(Γ) $\cong$ Pol(Δ): there is a bijection preserving algebra and topology.

Stabilizers

Computational problem

$\downarrow$

$\mathrm{CSP}(\Gamma)$

$\downarrow$

$\mathrm{Pol}(\Gamma)$

Consider stabilizer of $\overline{c} = (c_1, \ldots, c_n) \in \Gamma^n$:

$$\mathrm{Pol}(\Gamma, \overline{c}) := \{ f \in \mathrm{Pol}(\Gamma) \mid f(c_i, \ldots, c_i) = c_i \text{ for all } i \}$$

Fact (for ω-categorical cores Γ, i.e., $\overline{\mathrm{Aut}(\Gamma)} = \mathrm{End}(\Gamma)$):

$$\mathrm{CSP}(\Gamma, \overline{c}) \sim_{polytime} \mathrm{CSP}(\Gamma) .$$

The projection clone **Proj**

Computational problem

$\downarrow$

$\mathrm{CSP}(\Gamma)$

$\downarrow$

$\mathrm{Pol}(\Gamma)$

$\downarrow$

$\mathrm{Pol}(\Gamma, c_1)$

$\downarrow$

$\mathrm{Pol}(\Gamma, c_1, c_2)$

$\ldots$

Let **Proj** be the clone of projections on any $\geq$ 2-element set.

$\exists \bar{c}\ (\mathrm{Pol}(\Gamma, \bar{c}) \to \mathbf{Proj}) \implies \mathrm{CSP}(\Gamma)$ is NP-hard (Bodirsky + P).

The algebraic-topological dichotomy conjecture

Conjecture (for reducts of finitely bounded homogeneous structures)

1. $\exists \bar{c}\ (\mathrm{Pol}(\Gamma, \bar{c}) \to \mathbf{Proj}) \implies \mathrm{CSP}(\Gamma)$ is NP-hard.
2. $\neg\exists \bar{c}\ (\mathrm{Pol}(\Gamma, \bar{c}) \to \mathbf{Proj}) \implies \mathrm{CSP}(\Gamma)$ is in P.

Problems

- Statements non-algebraic (unless Γ finite).
- Second statement negative.
  For finite Γ equivalent to positive statements:
  existence in $\mathrm{Pol}(\Gamma)$ of Taylor, weak nu, Siggers,
  cyclic function.

Topology is irrelevant

Computational problem

$\downarrow$

CSP(Γ)

$\downarrow$

Pol(Γ)

$\downarrow$

Pol($\Gamma, \bar{c}$)

Theorem (Barto + P)

Let Γ be an ω-categorical core. TFAE:

- $\neg\exists\bar{c}$ (Pol($\Gamma, \bar{c}$) $\rightarrow$ **Proj**)
- $\neg\exists\bar{c}$ (Pol($\Gamma, \bar{c}$) $\rightarrow$ **Proj**) even ignoring the topology
- Pol(Γ) is pseudo-Siggers, i.e., contains α, β, f satisfying $\alpha(f(x, y, x, z, y, z)) = \beta(f(y, x, z, x, z, y))$

The algebraic dichotomy conjecture

Remarks:

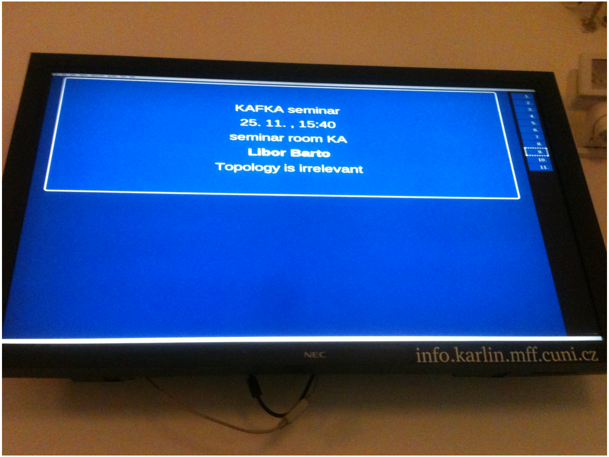
- Proof: "Pseudoloop lemma" via Bulatov's approach plus infinity.
- Non-trivial statement about all ω-categorical structures.

The Algebraic-Topological Dichotomy Conjecture
(for reducts of finitely bounded homogeneous structures)

1. $\exists \bar{c}$ $(\mathrm{Pol}(\Gamma, \bar{c}) \to \mathbf{Proj}) \implies \mathrm{CSP}(\Gamma)$ is NP-hard.
2. $\neg\exists \bar{c}$ $(\mathrm{Pol}(\Gamma, \bar{c}) \to \mathbf{Proj}) \implies \mathrm{CSP}(\Gamma)$ is in P.

The Algebraic Dichotomy Conjecture
(for reducts of finitely bounded homogeneous structures)

1. $\mathrm{Pol}(\Gamma)$ is not pseudo-Siggers $\implies \mathrm{CSP}(\Gamma)$ is NP-hard.
2. $\mathrm{Pol}(\Gamma)$ is pseudo-Siggers $\implies \mathrm{CSP}(\Gamma)$ is in P.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25. 11. , 15:40
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Libor Barto
Topology is irrelevant
NEC
info.karlin.mff.cuni.cz

Thank you!