Equations in algebras induced by beautiful first-order structures

Michael Pinsker

Technische Universität Wien

and

Univerzita Karlova v Praze

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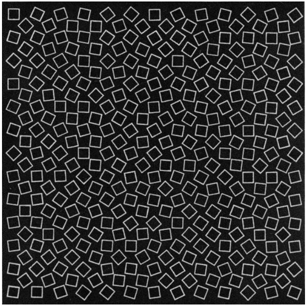
Outline

I: Equations in algebras

II: Algebras as invariants of first-order structures

III: Compact is the new beautiful?

IV: The future



I: Equations in finite algebras

Equations / identities

Algebra: $\mathbf{A} = (A, (f_i)_{i \in I})$; each $f_i \colon A^{n_i} \to A$.

Equation / identity:

Formal expression $s \approx t$, for abstract *terms* s, t over some functional language τ, e.g.

$$g(g(x,y),z) \approx g(x, g(y,z)) \; .$$

Equation *is satisfied* in $\mathbf{A}$ if g can be assigned a *term function* $\mathbf{g}$ over $\mathbf{A}$ such that

$$\mathbf{A} \models \; \forall x \, \forall y \, \forall z \;\; \mathbf{g}(\mathbf{g}(x,y),z) = \mathbf{g}(x, \mathbf{g}(y,z)) \; .$$

Independent of language of $\mathbf{A}$.

Examples.

- The above equation satisfied in any $\mathbf{A}$ by a projection.
- The equations $s(y,y,x) = s(x,y,y) = x$ are satisfied in any group $(A, +, -)$. [Set $\mathbf{s}(x,y,z) := x - y + z$].

Clones

Universal Algebra: Equations satisfied in **A** $\iff$ *structure* of **A**.

Examples: Congruences of **A**.

- **A** satisfies $s(x,y,y) \approx s(y,y,x) \approx x \quad \Longrightarrow$
  A has permuting congruences (CP)(+ all algebras in *variety* of **A**).
  Example: groups $(A,+,-)$. *(Mal'cev '54)*
- **A** satisfies *near unanimity (nu)* equations

$$n(x,\dots,x,y) \approx n(x,\dots,x,y,x) \approx \dots \approx n(y,x,\dots,x) = x$$

$\Longrightarrow$ **A** (+ its variety) is congruence distributive (CD).

Example: lattices. Equivalent to CD: *Jónsson '68 equations*.

Term clone Clo(**A**) of algebra **A**: smallest set of functions on A

- containing fundamental operations, projections
- closed under composition.

Goal: Structure of **A** $\Leftrightarrow$ composition structure of Clo(**A**).

Clone homomorphisms

$\mathrm{Clo}(\mathbf{A}) \subseteq \mathrm{Clo}(\mathbf{B}) \implies$ **B** has more structure.

More generally:

Definition

$\xi\colon \mathrm{Clo}(\mathbf{A}) \to \mathrm{Clo}(\mathbf{B})$ is a *clone homomorphism* if it preserves

- arities;
- projections;
- composition.

Quasiorder on algebras. The larger **A**, the more structure.

A smallest element: *clone* **0** *of projections* on set of ≥ 2 elements.

Important question

Consequences for **A** when $\mathrm{Clo}(\mathbf{A}) \not\to \mathbf{0}$? Draw picture. Say why.

Non-triviality + Idempotency

Definition

$\mathbf{A}$ is *idempotent* if $t(x, \ldots, x) \approx x$ holds for all $t \in \mathrm{Clo}(\mathbf{A})$.

Why idempotent? Practical + Philosophical reasons + Evidence.

Theorem (Olšák '16)

Let $\mathbf{A}$ be idempotent, and $\mathrm{Clo}(\mathbf{A}) \not\to \mathbf{0}$. Then $\mathbf{A}$ satisfies

$$t(xyy, yxx) \approx t(yxy, xyx) \approx t(yyx, xxy).$$

Theorem (Maróti + McKenzie '08)

Let $\mathbf{A}$ be idempotent, in *locally finite variety*, $\mathrm{Clo}(\mathbf{A}) \not\to \mathbf{0}$.
Then $\mathbf{A}$ satisfies for some arity $n \geq 2$

$$w(x, \ldots, x, y) \approx w(x, \ldots, x, y, x) \approx \cdots \approx w(y, x, \ldots, x)\ .$$

(*weak near unanimity term*)

Non-triviality + Idempotency + finiteness

Theorem (Siggers '11; Kearnes + Marković + McKenzie '14)

Let **A** be idempotent, in locally finite variety, and Clo(**A**) $\not\to$ **0**.
Then **A** satisfies

$$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y) \; .$$

and

$$q(a, r, e, a) \approx q(r, a, r, e) \; .$$

(6-ary and 4-ary Siggers terms)

Theorem (Barto + Kozik '11)

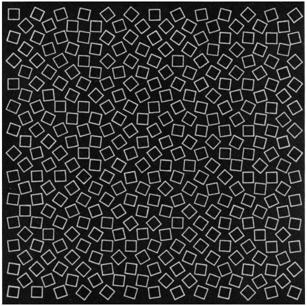
Let **A** be idempotent, *finite*, and Clo(**A**) $\not\to$ **0**. Then **A** satisfies

$$c(x_1, \ldots, x_n) \approx c(x_2, \ldots, x_n, x_1) \quad \text{for some } n \geq 2 \; .$$

(*Cyclic term*)

Summary of Part I

- Equations of $\mathrm{Clo}(\mathbf{A}) \Leftrightarrow$ Structure of $\mathbf{A}$ (e.g. congrucences)
- Idempotency + any equations ($\mathrm{Clo}(\mathbf{A}) \not\to \mathbf{0}$) $\Rightarrow$ Structure
- Add finiteness conditions $\Rightarrow$ More structure



II: Algebras from first-order structures

Clones as symmetries

Let $\mathbb{A} = (A, (R_i)_{i \in I})$ be a first-order structure with relations R_i (wlog).

Symmetries:

- $\mathrm{Aut}(\mathbb{A}) \subseteq \mathrm{Sym}(A)$...automorphism group of $\mathbb{A}$;
- $\mathrm{End}(\mathbb{A}) \subseteq A^A$...endomorphism monoid of $\mathbb{A}$;
- $\mathrm{Pol}(\mathbb{A}) \subseteq \bigcup_{n \geq 1} A^{A^n}$...polymorphism clone of $\mathbb{A}$.

$$\mathrm{Pol}(\mathbb{A}) := \{f \colon \mathbb{A}^n \to \mathbb{A} \mid f \text{ homomorphism}\}.$$

Topology:

A...discrete, A^{A^n}...product topology (pointwise convergence),
$\bigcup_{n \geq 1} A^{A^n}$...sum space.

A permutation group / transformation monoid / term clone
is of the form $\mathrm{Aut}(\mathbb{A})$ / $\mathrm{End}(\mathbb{A})$ / $\mathrm{Pol}(\mathbb{A})$ $\Leftrightarrow$ it is *topologically closed*.

Galois-correspondence

The assignments

$$\mathbb{A} \mapsto \mathrm{Pol}(\mathbb{A})$$
$$\mathbf{A} \mapsto \mathrm{Inv}(\mathbf{A})$$

define a Galois-correspondence (Inv($\mathbf{A}$)... invariant relations).

First-order structures $\leftrightarrow$ algebras.

Roughly:

- Aut($\mathbb{A}$)... first-order structure of $\mathbb{A}$.
- Pol($\mathbb{A}$)... primitive positive structure of $\mathbb{A}$ (finer).

More conditions of beauty for $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$ via correspondence:

- Finitely related, i.e., $\mathbb{A}$ finite + finite language.
- $\mathbb{A}$ ω-categorical.
- $\mathbb{A}$ Ramsey.
- $\mathbb{A}$ definable in finitely bounded homogeneous structure...

Finitely related algebras (classical)

Conjecture (Valeriote)

Let **A** be in *congruence modular variety* + finitely related.
Then the number of subalgebras of $\mathbf{A}^n$ grows only exponentially with n.

Conjecture (Zádori)

Let **A** be in *congruence distributive variety* + finitely related.
Then **A** satisfies near unanimity equations.

CSP($\mathbb{A}$): computational problem of deciding primitive positive theory of $\mathbb{A}$.

For $\mathbb{A}$ finite: in NP.

Conjecture (Feder + Vardi '92)

Let $\mathbb{A} = (A, R_1, \ldots, R_n)$ be finite, with (wlog) idempotent Pol($\mathbb{A}$).
If Pol($\mathbb{A}$) $\not\to$ **0**, then CSP($\mathbb{A}$) is polynomial-time solvable.

Finitely related algebras

Theorem (Barto '12)

Let **A** be in *congruence modular variety* + finitely related.
Then the number of subalgebras of $\mathbf{A}^n$ grows only exponentially with n.

Corollary (Barto '10)

Let **A** be in *congruence distributive variety* + finitely related.
Then **A** satisfies near unanimity equations.

CSP($\mathbb{A}$): computational problem of
deciding primitive positive theory of $\mathbb{A}$.

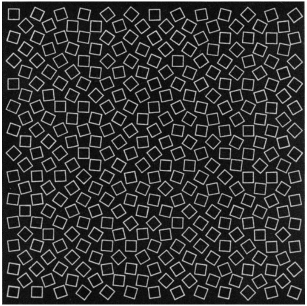
For $\mathbb{A}$ finite: in NP.

Theorem (Bulatov '17, Zhuk '17)

Let $\mathbb{A} = (A, R_1, \ldots, R_n)$ be finite, with (wlog) idempotent Pol($\mathbb{A}$).
If Pol($\mathbb{A}$) $\not\to$ **0**, then CSP($\mathbb{A}$) is polynomial-time solvable.

Summary of Part II

- Algebras $\Leftrightarrow$ First-order structures
- Provides more notions of beauty
- Finitely related: works



III: Compact is the new finite

Beautiful structures: compactness

Let $\mathbb{A}$ be countable *ω-categorical*.

$\Leftrightarrow$ $\mathbb{A}^n$ has finitely many first-order definable subsets for every $n \geq 1$

$\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ has finitely many orbits for every $n \geq 1$.

$\Leftrightarrow$: $\mathrm{Aut}(\mathbb{A})$ is *oligomorphic*.

$\mathbb{A}$ “small", so $\mathrm{Pol}(\mathbb{A})$ “large" (not locally finite . . .). But:

Observation

For $f, g \in \mathrm{Pol}(\mathbb{A})$, set

$$f \sim g :\leftrightarrow f \in \overline{\{\alpha \circ g \mid \alpha \in \mathrm{Aut}(\mathbb{A})\}}$$

If $\mathbb{A}$ is ω-categorical, then $\mathrm{Pol}(\mathbb{A})/_{\sim}$ is compact.

Factoring compatible with continuous clone homomorphisms $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$.

Topological Birkhoff

$\mathrm{HSP}^{\mathrm{fin}}(\mathrm{Clo}(\mathbf{A}))\ldots$ all clones ("actions") obtained from $\mathrm{Clo}(\mathbf{A})$ by

- factoring the domain: $\mathrm{Clo}(\mathbf{A}) \curvearrowright A/\sim$
- restricting the domain: $\mathrm{Clo}(\mathbf{A}) \curvearrowright A' \subseteq A$
- taking finite powers of the domain: $\mathrm{Clo}(\mathbf{A}) \curvearrowright A^n$

Theorem (Birkhoff '35)

Let $\mathbf{A}$ be finite.

$\mathrm{Clo}(\mathbf{A}) \to \mathbf{0} \quad \Leftrightarrow \quad \mathbf{0} \in \mathrm{HSP}^{\mathrm{fin}}(\mathrm{Clo}(\mathbf{A}))$.

Theorem (Bodirsky + P. '11)

Let $\mathbb{A}$ be ω-categorical.

$\mathrm{Pol}(\mathbb{A}) \to \mathbf{0}$ continuously $\quad \Leftrightarrow \quad \mathbf{0} \in \mathrm{HSP}^{\mathrm{fin}}(\mathrm{Pol}(\mathbb{A}))$.

In this situation, $\mathbb{A}$ is rich:
All finite structures have a *primitive positive interpretation* in $\mathbb{A}$.
$\mathrm{CSP}(\mathbb{A})$ is NP-hard.

Topological non-triviality

Let $\mathbb{A}$ be ω-categorical, $\mathrm{Pol}(\mathbb{A}) \not\to \mathbf{0}$ continuously.

Wish to derive equations satisfied in $\mathrm{Pol}(\mathbb{A})$.

Problem: anti-idempotency of $\mathrm{Pol}(\mathbb{A})$ implied by ω-categoricity.

Approach I: **Approximate idempotency.**

Assume $\overline{\mathrm{Aut}(\mathbb{A})} = \mathrm{End}(\mathbb{A})$, consider *stabilizers* $\mathrm{Pol}(\mathbb{A}, a_0, \ldots, a_n)$.

Theorem (Barto + P. '16)

Let $\mathbb{A}$ be ω-categorical, $\overline{\mathrm{Aut}(\mathbb{A})} = \mathrm{End}(\mathbb{A})$. TFAE:

- No stabilizer of $\mathrm{Pol}(\mathbb{A})$ maps continuously to $\mathbf{0}$
- No stabilizer of $\mathrm{Pol}(\mathbb{A})$ maps to $\mathbf{0}$
- $\mathrm{Pol}(\mathbb{A})$ satisfies

$$e \circ s(x, y, x, z, y, z) \approx e' \circ s(y, x, z, x, z, y)$$

Remark. Compatible with CSP; interpretations with parameters.

Linear non-triviality

Approach I: **Approximate idempotency.**

Approach II: **Ignore idempotency.**

Based on observation that most interesting equations are *linear* (non-nested).

Consider *clonoid homomorphisms* $\mathrm{Clo}(\mathbf{A}) \rightsquigarrow \mathrm{Clo}(\mathbf{B})$ preserving only linear equations. Ignore unary functions!

What if $\mathrm{Pol}(\mathbb{A}) \not\rightsquigarrow \mathbf{0}$?

Theorem (Barto + Opršal + P. '16; Barto + P. '16; Pham '17)

Let $\mathbb{A}$ be ω-categorical.

If $\mathrm{Pol}(\mathbb{A}) \rightsquigarrow \mathbf{0}$ unif. continuous, then $\mathbb{A}$ rich (e.g. $\mathrm{CSP}(\mathbb{A})$ is NP-hard).
If $\mathrm{Pol}(\mathbb{A}) \not\rightsquigarrow \mathbf{0}$ unif. continuous, then $\mathrm{Pol}(\mathbb{A})$ satisfies

$$e \circ s(x,y,x,z,y,z) \approx e' \circ s(y,x,z,x,z,y) \; .$$

Orbit growth: extremely beautiful structures

Converse?

Theorem (Olšák + Barto + Kompatscher + Pham + P. '17)

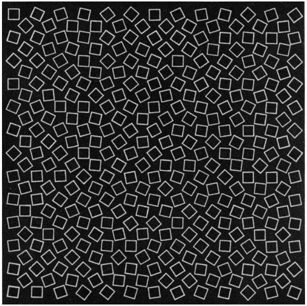
There exists ω-categorical $\mathbb{A}$ with $\overline{\mathrm{Aut}(\mathbb{A})} = \mathrm{End}(\mathbb{A})$ such that:

- $\mathrm{Pol}(\mathbb{A})$ satisfies $e \circ s(x, y, x, z, y, z) \approx e' \circ s(y, x, z, x, z, y)$
- $\mathrm{Pol}(\mathbb{A}) \rightsquigarrow \mathbf{0}$ uniformly continuous.

Theorem (Olšák + Barto + Kompatscher + Pham + P. '17)

The above imply that $\mathrm{Aut}(\mathbb{A})$ has doubly exponential orbit growth.

Hence, for $\mathbb{A}$ with slower orbit growth:
$\mathrm{Pol}(\mathbb{A}) \rightsquigarrow \mathbf{0}$ uniformly cont. $\Leftrightarrow$ no pseudo-Siggers equation.



IV: The future

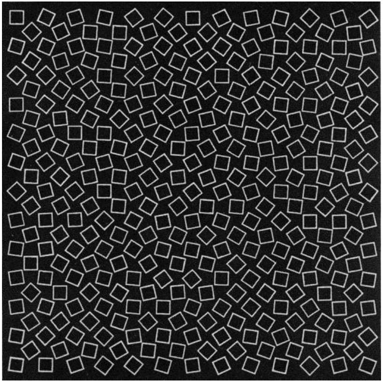
Summary

Summary:

- Idempotent / locally finite / finite / finitely related:
  abundance of recent deep *"equations $\Leftrightarrow$ structure"* theorems
- ω-categoricity yields compactness:
  Work "modulo group" and obtain *"equations $\Leftrightarrow$ structure"*.

Future:

- Role of topology: which structural properties are algebraic?
  $\mathrm{Pol}(\mathbb{A}) \rightsquigarrow \mathbf{0} \;\Leftrightarrow\; \mathrm{Pol}(\mathbb{A}) \rightsquigarrow \mathbf{0}$ unif. continuously?
- How far does compactness approach go?
  6-ary pseudo-Siggers $\implies$ 4-ary pseudo-Siggers?
- Equations are "fixed points" of clone actions in lack of space.
  General dynamical / Ramsey theoretic principle behind this?
- News: 6-ary Siggers $\implies$ 4-ary Siggers in general! (Olšák '18)



Gerhard von Graevenitz *Regularity - Irregularity V*