Algebraic structure of polymorphism clones of infinite CSP templates

Michael Pinsker

joint work with Pierre Gillibert & Julius Jonušas

Technische Universität Wien /
Charles University Prague

Funded by Austrian Science Fund (FWF) grant P27600
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The Constraint Satisfaction Problem: Complexity and Approximability

Dagstuhl 2018

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- $\Leftrightarrow$ equations / identities satisfied by $\mathbf{A}$

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- $\Leftrightarrow$ structure of $\mathbf{A}$
- $\Leftrightarrow$ structure of the invariant relations of $\mathbf{A}$
- $\Leftrightarrow$ complexity of the corresponding CSP

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Fix propositional formula over language with one binary symbol $<$, e.g., $\phi(x, y, z) := ((x < y) \wedge (y < z)) \vee ((z < y) \wedge (y < x))$.

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- Number of values of solution not bounded - but for each instance do not need more than number of variables.
- CSP modeled by a single (countably) infinite template.

The conjecture and ω-categoricity

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Templates $\mathbb{A}$ definable in finitely bounded homogeneous structure.

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- Think: “fuzzy" finite algebra.
- However, simple factoring by $\mathcal{G}$ not possible (no congruence).

Single h1 equations

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Definition (Single h1 equation / identity)

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Theorem (Siggers '11, Kearnes + Marković + McKenzie '14)

$\mathbf{A}$ finite idempotent algebra, equationally non-trivial.
Then $\mathbf{A}$ has terms s, t satisfying the equations

$$s(a, r, e, a) = s(r, a, r, e)$$

and

$$t(x, y, x, z, y, z) = t(y, x, z, x, z, y).$$

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Definition (graph G_t of single h1 equation with function symbol t)

Vertices: all variables x_i, y_i of the equation.
Edges: From x_i to y_i, for all i.

Equations $\Leftrightarrow$ loop conditions

Equations ⇔ loop conditions

Fun fact

Let **A** be an algebra, and $t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n)$ be an equation.

TFAE:

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  Which does not matter.

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- For finite **A**: finite power $\mathbf{A}^n$ sufficient.
- Undirected loop lemma (Bulatov '05):
  finite graphs containing K_3 and invariant under an idempotent equationally non-trivial algebra have a loop.

The big loop condition collaps / idempotency

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Theorem (Olšák '18)

- All undirected non-bipartite loop conditions are equivalent.

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- Not equivalent to single h1 equation (Kazda '17).
- Absolute statement. Idempotency but no finiteness.

Higher dimensions

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Definition (*m*-dimensional h1 equation)

$$\begin{aligned} & t(x_1^1, \ldots, x_1^n) \\ = \; & t(x_2^1, \ldots, x_2^n) \\ & \cdots \\ = \; & t(x_m^1, \ldots, x_m^n) \end{aligned}$$

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Definition (m-dimensional h1 equation)

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Definition (relation R_t of m-dimensional h1 equations of t)

Domain: the variables x_i^j.
Tuples: columns.

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Definition (relation R_t of m-dimensional h1 equations of t)

Domain: the variables x_i^j.
Tuples: columns.

- **Examples:** weak near unanimity / Olšák term.
- Constant tuple in R_t means triviality of the equations.

Equations $\Rightarrow$ constant tuples

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Higher dimensional fun fact

Let **A** be an algebra. TFAE:

- **A** has a term satisfying $t(\ldots) = \cdots = t(\ldots)$ (with m occurrences).
- Whenever $H \leq (\mathbf{A}^\omega)^m$ and $R_t \to H$, then H has a loop (= a constant tuple).

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$K_n^m \ldots$ complete m-ary relation on n vertices without loops (NAE).

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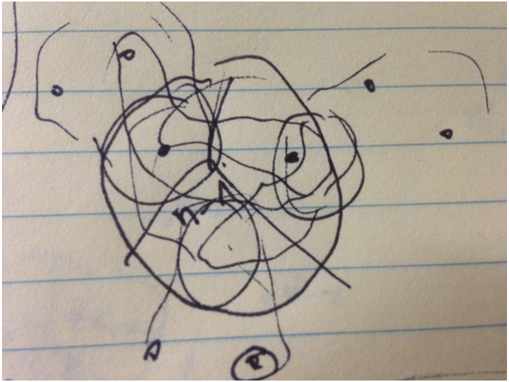

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Proof.

Let t be a k-ary Taylor term of **A**.
Take $t * t * t$ and distribute $n := 2k$ variables well among its variables.

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Corollary

K_4^3 holds in all idempotent equationally non-trivial algebras.
(along with Olšák's K_2^3).

From finite idempotent algebras to the ω-categorical

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Good news:

Marcin Kozik's talk on loop conditions in finite non-idempotent algebras.

ω-categoricity

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Evidence: Non-triviality does not imply satisfaction of loop conditions.

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Let **A** be the polymorphism algebra of an ω-categorical structure.

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If **A** satisfies non-trivial h1 identities locally (= on every finite set),

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$$e_1 \circ s(x, y, x, z, y, z) = e_2 \circ s(y, x, z, x, z, y) \; .$$

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- Above situation is the tractability condition of the infinite dichotomy conjecture (see Michael Kompatscher’s talk).

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- Proof is horrible.

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Let **A** be the polymorphism algebra of an ω-categorical structure which is a model-complete core ($\sim$ locally idempotent).

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Let **A** be the polymorphism algebra of an ω-categorical structure which is a model-complete core ($\sim$ locally idempotent).
TFAE:

- **A** has terms e_i and t satisfying $e_1 \circ t(\ldots) = \cdots = e_m \circ t(\ldots)$.

Pseudoloop conditions

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- **A** has terms e_i and t satisfying $e_1 \circ t(\ldots) = \cdots = e_m \circ t(\ldots)$.
- For all $n \geq 1$, whenever $H \leq (\mathbf{A}^n)^m$ and $R_t \to H$, then H has a pseudoloop. (= constant tuple modulo the large permutation group within **A**).

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- Call the second statement the R_t-pseudoloop condition.

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- **A** has terms e_i and t satisfying $e_1 \circ t(\dots) = \cdots = e_m \circ t(\dots)$.
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  then H has a pseudoloop.
  (= constant tuple modulo the large permutation group within **A**).

- Call the second statement the R_t-pseudoloop condition.
- **Example:** K_3^2-pseudoloop condition means satisfaction of
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Pseudoloop conditions

Higher dimensional pseudo fun fact

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- Barto - P. prove this condition holds assuming the tractability condition of the dichotomy conjecture.

The tiny pseudoloop condition collaps

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Sad proposition

For all $n \geq 4$ the K_n^2- and K_{n+1}^2-pseudoloop conditions are equivalent.

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The future

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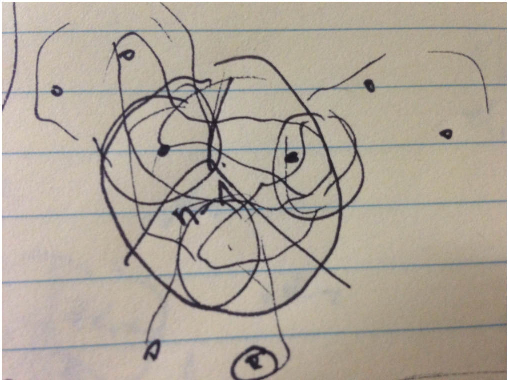
- Find methods for separating (pseudo-) loop conditions.



Thank you!